ON ORTHOGONAL POLYNOMIALS

BY
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1. Let \( f(z) \) be a function which is analytic inside and on the ellipse \( C \), having the points \( \pm 1 \) for its foci. We suppose also that, for real \( x \) in the interval \((-1, +1)\), \( f(x) \) is real. We have

\[
f(y) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - y}.
\]

Consider now the function \( p(x) \), summable and not negative in the interval \((-1, +1)\) and satisfying the condition that

\[
\int_{-1}^{1} \log p(x)dx
\]

exists.

Consider also the normal orthogonal polynomials \( P_0, P_1(x), P_2(x), \ldots, \)

\[
P_k(x) = d_0^{(k)} x^k + d_1^{(k)} x^{k-1} + \cdots + d_k^{(k)} \quad [d_0^{(k)} > 0],
\]

corresponding to the characteristic function \( p(x) \), i.e.

\[
\int_{-1}^{1} p(x)P_k(x)P_s(x)dx = \begin{cases} 0, & k \neq s, \\ 1, & k = s. \end{cases}
\]

Then the series

\[
\sum_{k=0}^{\infty} a_k P_k(y) \quad \left( a_k = \int_{-1}^{1} \frac{p(y)P_k(y)}{z - y}dy \right)
\]

converges absolutely and uniformly with respect to \( y \), if \( y \) lies in any domain lying wholly inside the ellipse \( C \), which passes through the point \( z \) and has the points \( \pm 1 \) for its foci, \( \dagger \) i.e.,

\[
\left| \frac{y + (y^2 - 1)^{1/2}}{z + (z^2 - 1)^{1/2}} \right| < 1 - \epsilon \quad (\epsilon > 0 \text{ arbitrarily small}),
\]

\[
\left| z + (z^2 - 1)^{1/2} \right| > 1, \quad \left| y + (y^2 - 1)^{1/2} \right| \geq 1.
\]

\* Presented to the Society, December 30, 1930; received by the editors in June, 1929.
Hence, introducing the functions $Q_k(z)$ of the second kind\(^*\)
\begin{equation}
    a_k = Q_k(z) \quad (k = 0, 1, 2, 3, \cdots),
\end{equation}
\begin{equation}
    \frac{1}{z - y} = \sum_{k=0}^{\infty} P_k(y)Q_k(z).
\end{equation}
Inserting this value of $1/(z - y)$ in (1) we get
\begin{equation}
    f(y) = \frac{1}{2\pi i} \int_C \left\{ \sum_{k=0}^{\infty} P_k(y)Q_k(z) \right\} f(z)dz = \sum_{k=0}^{\infty} C_k P_k(y),
\end{equation}
\begin{equation}
    C_k = \frac{1}{2\pi i} \int_C f(z)Q_k(z)dz \quad (k = 0, 1, 2, 3, \cdots).
\end{equation}
The coefficients $C_k$ may be found independently:
\begin{equation}
    C_k = \int_{-1}^{1} p(x)f(x)P_k(x)dx \quad (k = 0, 1, 2, 3, \cdots),
\end{equation}
whence we find the relation
\begin{equation}
    \frac{1}{2\pi i} \int_C f(z)Q_k(z)dz = \int_{-1}^{1} p(x)f(x)P_k(x)dx \quad (k = 0, 1, 2, 3, \cdots).
\end{equation}
\iffalse
\begin{equation}
    \int_{-1}^{1} p(x)f(x)P_k(x)dx = \int_{-1}^{1} f(x)P_k(x)dx \quad (k = 0, 1, 2, 3, \cdots).
\end{equation}
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\begin{equation}
    \frac{1}{2\pi i} \int_C f(z)Q_k(z)dz = \int_{-1}^{1} f(x)p(x)P_k(x)dx = \int_{-1}^{1} f(x)P_k(x)dx \quad (k = 0, 1, 2, 3, \cdots).
\end{equation}
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\begin{equation}
    \int_{-1}^{1} f(x)p(x)P_k(x)dx = \int_{-1}^{1} f(x)P_k(x)dx \quad (k = 0, 1, 2, 3, \cdots).
\end{equation}
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Put $k = 0$ and $f(x) = P_s(x)P_r(x)$. Then
\begin{equation}
    \frac{1}{2\pi i} \int_C Q_0(z)P_s(z)P_r(z)dz = \int_{-1}^{1} p(x)P_s(x)P_r(x)dx = \begin{cases} 0, & s \neq r, \\ 1, & s = r. \end{cases}
\end{equation}
This formula shows that the polynomials $P_s(x)$, which are orthogonal and normal in the interval $(-1, +1)$ with the characteristic function $p(x)$, have the same property ($x$ being replaced by the complex variable $z$) on the contour $C$ with the characteristic function
\begin{equation}
    \frac{Q_0(z)}{P_0} = \int_{-1}^{1} \frac{p(x)dx}{z - x}. \dagger
\end{equation}
In particular, we find that the normalized trigonometric polynomials
\begin{itemize}
  \item J. Sokhotzki, The Theory of Integral Residues with Applications (Thesis in Russian), St. Petersburg, 1868, p. 59, where formula (7) was established in a different way.
\end{itemize}
$$Tk(z) = \left(\frac{2}{\pi}\right)^{1/2} \cos k \arccos z$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{\left(\frac{z + (z^2 - 1)^{1/2}}{2} + \frac{(z - (z^2 - 1)^{1/2})^k}{2}\right)}{2}$$

$$\left( k = 1, 2, \ldots ; T_0 = \left(\frac{1}{\pi}\right)^{1/2}\right),$$

orthogonal on $(-1, 1)$ with the characteristic function $\rho(z) = 1/(1 - z^2)^{1/2}$, are orthogonal on the contour $C$ with the characteristic function $\pi/(z^2 - 1)^{1/2}$.

In fact, according to S. Bernstein,* we have

$$Q_0(z) = \left(\frac{\pi}{z^2 - 1}\right)^{1/2}, \quad Q_k(z) = \frac{(2\pi)^{1/2}}{(z^2 - 1)^{1/2}} \frac{\{z + (z^2 - 1)^{1/2}\}^k}{k!}$$

($k = 1, 2, \ldots$).

2. A well known property of the polynomials (9) is the following. The formal developments

$$f(x) \sim \sum_{k=0}^{\infty} A_k T_k(x), \quad \phi(x) \sim \sum_{k=0}^{\infty} B_k T_k(x)$$

$$A_k = \int_{-1}^{1} \frac{f(x) T_k(x)}{(1 - x^2)^{1/2}} dx, \quad B_k = \int_{-1}^{1} \frac{\phi(x) T_k(x)}{(1 - x^2)^{1/2}} dx$$

imply, provided the integrals $\int_{-1}^{1} (f^2(x)/(1 - x^2)^{1/2}) dx$ and $\int_{-1}^{1} (\phi^2(x)/(1 - x^2)^{1/2}) dx$ exist,

$$\int_{-1}^{1} f(x)\phi(x) \mathrm{d}x = \sum_{k=0}^{\infty} A_k B_k.$$

Apply (11) to $f(x) = \phi(x)(1 - x^2)^{1/2}$, $\phi(x) = 1/(z - x)$, assuming that

$$\int_{-1}^{1} \rho^2(x)(1 - x^2)^{1/2} dx = \int_{0}^{\pi} \rho^2(\cos \phi) \sin^2 \phi d\phi$$

exists. Thus we get, writing

$$\rho(x)(1 - x^2)^{1/2} \sim \sum_{k=0}^{\infty} \rho_k T_k(x)$$

$$\left( \rho_k = \int_{-1}^{1} \rho(x) T_k(x) dx \right),$$

$$\frac{1}{z - x} = \sum_{k=0}^{\infty} T_k(x) Q_k(z),$$

ON ORTHOGONAL POLYNOMIALS

(15) \[
\frac{Q_0(x)}{P_0(x)} = \int_{-1}^{1} \frac{p(x)(1 - x^2)^{1/2}}{z - x} \frac{dx}{(1 - x^2)^{1/2}} = \sum_{k=0}^{\infty} c_k Q_k(z)
\]

(see (3), \(Q_k(z)\) given by (10)).

Hence, the formal trigonometric expansion

(16) \[
\phi(x) \sim \left(\frac{2}{\pi}\right)^{1/2} \left\{ \frac{a_0}{2^{1/2}} + \sum_{k=1}^{\infty} a_k \cos k\phi \right\}
\]

\[
\left( a_k = \int_{0}^{\pi} \phi(x) \sin \phi \cos k\phi d\phi \right)
\]
yields at once, under condition (12), the expansion for \(Q_0(z)/P_0\) with the same coefficients:

(17) \[
\frac{Q_0(z)}{P_0} = \left(\frac{2\pi}{z^2 - 1}\right)^{1/2} \left\{ \frac{a_0}{2^{1/2}} + \sum_{k=1}^{\infty} a_k [z - (z^2 - 1)^{1/2}]^k \right\}.
\]

If \(\phi(x)\) is a finite trigonometric sum, then \(Q_0(z)/P_0\) is also a finite sum. For example, taking

(18) \[
\phi(x) = (1 - x^2)^{1/2}, \quad \phi(x) \sin \phi = \sin^2 \phi = \frac{1}{2} - \frac{1}{2} \cos 2\phi,
\]

we find

(19) \[
\frac{Q_0(z)}{P_0} = \pi \left\{ z - (z^2 - 1)^{1/2} \right\}.
\]

In other words, the polynomials

(20) \[
P_0 = \left(\frac{2}{\pi}\right)^{1/2}, \quad P_k(z) = \frac{\{z + (z^2 - 1)^{1/2}\}^{k+1} - \{z - (z^2 - 1)^{1/2}\}^{k+1}}{(2\pi)^{1/2}(z^2 - 1)^{1/2}} \quad (k = 1, 2, 3, \ldots)
\]

are orthogonal and normal on the contour \(C\) with the characteristic function \(\pi \{z - (z^2 - 1)^{1/2}\}\).

3. We proceed to derive some interesting properties of the functions \(Q_n(z)\). Darboux has shown\* that they satisfy the same recurrence relation as the \(P_n(x)\):

(21) \[
A_{n+1}Q_{n+1}(z) + A_nQ_{n-1}(z) = (B_n + z)Q_n(z) \quad (n = 1, 2, 3, \ldots),
\]

\[
A_1Q_1(z) = (B_0 + z)Q_0(z) - \frac{1}{P_0} \quad (A_i = \text{const.}).
\]

\* Loc. cit., p. 415.
We multiply both members of Darboux's formula*

\[ \sum_{k=0}^{n} P_k(x)P_k(y) = A_{n+1} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y} \quad (A_{n+1} = \frac{d_0^{(n)}}{d_0^{(n+1)}}) \]

by \( p(x) \) and \( p(x)/(z - x) \), and integrate between \(-1\) and \(1\). We get

\[ Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) = \frac{1}{A_{n+1}}, \quad \uparrow \]

\[ \sum_{k=0}^{n} P_k(y)Q_k(z) = A_{n+1} \frac{Q_{n+1}(z)P_n(y) - Q_n(z)P_{n+1}(y)}{z - y} + \frac{1}{z - y}, \quad \downarrow \]

Suppose now that \( p(-x) = p(x) \) ("symmetric" orthogonal polynomials). Then, as is known,

\[ P_k(-x) = (-1)^k P_k(x), \]

and we get from (22), denoting by \([m]\) the greatest integer \(\leq m\),

\[ \sum_{s=0}^{[n/2]} P_{2s}(x)P_{2s}(y) = A_{n+1} \frac{xP_{n+1}(x)P_n(y) - yP_n(x)P_{n+1}(y)}{x^2 - y^2}, \]

which, combined with the recurrence relation for \( P_k(x) \), gives

\[ \sum_{s=0}^{[n/2]} P_{n-2s}(x)P_{n-2s}(y) = A_{n+1}A_{n+2} \frac{P_{n+2}(x)P_n(y) - P_n(x)P_{n+2}(y)}{x^2 - y^2}. \quad \uparrow \]

The same method applied to (23) gives

\[ \sum_{s=0}^{[n/2]} P_{n-2s}(y)Q_{n-2s}(z) = A_{n+1}A_{n+2} \frac{Q_{n+2}(z)P_n(y) - Q_n(z)P_{n+2}(y)}{z^2 - y^2} + \frac{z}{z^2 - y^2}, \quad (n \text{ even}), \]

\[ \sum_{s=0}^{(n-1)/2} P_{n-2s}(y)Q_{n-2s}(z) = A_{n+1}A_{n+2} \frac{Q_{n+2}(z)P_n(y) - Q_n(z)P_{n+2}(y)}{z^2 - y^2} + \frac{\nu}{z^2 - y^2}, \quad (n \text{ odd}). \]

4. Assume now again that the integral

\[ \int_{-1}^{1} \frac{\log p(x)}{(1 - x^2)^{1/2}} \, dx \]

* Loc. cit., p. 413.
† F. Neumann, Beiträge zur Theorie der Kugelfunktionen, 1878, p. 71 \((p(x) = 1)\).
‡ Darboux, loc. cit., p. 415.
exists, and use the results of §1, concerning the expansion

\[ \frac{1}{z - y} = \sum_{k=0}^{\infty} P_k(y)Q_k(z). \]

Combining (23), (3), we get the expansion*

\[ \sum_{k=1}^{\infty} P_{n+k}(y)Q_{n+k}(z) = A_{n+1} \frac{P_{n+1}(y)Q_n(z) - P_n(y)Q_{n+1}(z)}{z - y}. \]

Multiplying (26), (27) by \( p(y)/(x-y) \) and integrating between -1 and 1, we get the expansions

\[ \sum_{k=0}^{\infty} Q_k(x)Q_k(z) = \frac{Q_0(x) - Q_0(z)}{P_0(z - x)}, \]

\[ \sum_{k=1}^{\infty} Q_{n+k}(x)Q_{n+k}(z) = A_{n+1} \frac{Q_n(z)Q_{n+1}(x) - Q_{n+1}(z)Q_n(x)}{z - x}, \]

which are valid for

\[ |x + (x^2 - 1)^{1/2}| > 1 + \epsilon, |z + (z^2 - 1)^{1/2}| > 1 + \epsilon_1, \]

where \( \epsilon \) and \( \epsilon_1 \) are arbitrarily small but fixed positive constants.

In particular, for \( z = x \), we derive from (28), (29),

\[ \sum_{k=0}^{\infty} Q_k^2(x) = - \frac{Q_0'(x)}{P_0}, \]

\[ \sum_{k=1}^{\infty} Q_{n+k}^2(x) = A_{n+1} \{Q_{n+1}(x)Q_n'(x) - Q_n(x)Q_{n+1}'(x)\}. \]

In the case of symmetric orthogonal polynomials, we get from (26)

\[ \sum_{k=0}^{\infty} P_{2k}(y)Q_{2k}(z) = \frac{z}{z^2 - y^2}. \]

Similarly,

\[ \sum_{k=0}^{\infty} P_{2k+1}(y)Q_{2k+1}(z) = \frac{y}{z^2 - y^2}. \]

From (25) and (33) we find†

\[ \sum_{k=1}^{\infty} P_{n+2k}(y)Q_{n+2k}(z) = A_{n+1}A_{n+2} \frac{P_{n+2}(y)Q_n(z) - P_n(y)Q_{n+2}(z)}{z^2 - y^2}. \]

All these expansions are valid for
\[
| z + (z^2 - 1)^{1/2} | > 1, \quad | y + (y^2 - 1)^{1/2} | \geq 1,
\]
\[
\frac{| y + (y^2 - 1)^{1/2} |}{z + (z^2 - 1)^{1/2}} < 1 - \epsilon.
\]

Multiplying (33), (34), (35) by \( p(y)/(x-y) \) and integrating between \(-1\) and \(1\), we get the expansions

\[
\sum_{k=0}^{\infty} Q_{2k}(x)Q_{2k}(z) = \frac{xQ_0(z) - zQ_0(x)}{P_0(x^2 - z^2)},
\]
\[
\sum_{k=0}^{\infty} Q_{2k+1}(x)Q_{2k+1}(z) = \frac{zQ_0(z) - xQ_0(x)}{P_0(x^2 - z^2)},
\]
\[
\sum_{k=1}^{\infty} Q_{n+2k}(x)Q_{n+2k}(z) = A_{n+1}A_{n+2} \frac{Q_{n+2}(x)Q_n(x) - Q_n(x)Q_{n+2}(x)}{x^2 - z^2},
\]

which are valid under condition (30). Putting \( z = x \), we get

\[
\sum_{k=0}^{\infty} \frac{Q_{2k}(x)}{x} = \frac{Q_0(x) - xQ_0'(x)}{2P_0(x)},
\]
\[
\sum_{k=0}^{\infty} \frac{Q_{2k+1}(x)}{x} = - \frac{Q_0(x) + xQ_0'(x)}{2P_0(x)},
\]
\[
\sum_{k=1}^{\infty} \frac{Q_{n+2k}(x)}{x} = A_{n+1}A_{n+2} \frac{Q_{n+2}(x)Q_n(x) - Q_n(x)Q_{n+2}(x)}{2x}.
\]