ON ORTHOGONAL POLYNOMIALS*

BY

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1. Let $f(z)$ be a function which is analytic inside and on the ellipse C, having the points ± 1 for its foci. We suppose also that, for real x in the interval $(-1, +1)$, $f(x)$ is real. We have

$$f(y) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-y}. \tag{1}$$

Consider now the function $p(x)$, summable and not negative in the interval $(-1, +1)$ and satisfying the condition that

$$\int_{-1}^{1}\frac{\log p(x)dx}{(1-x^2)^{1/2}}$$

exists.

Consider also the normal orthogonal polynomials P_0, $P_1(x)$, $P_2(x)$, $\cdots$,

$$P_k(x) = d_0^{(k)}x^k + d_1^{(k)}x^{k-1} + \cdots + d_k^{(k)} \qquad [d_0^{(k)} > 0],$$

corresponding to the characteristic function $p(x)$, i.e.

$$\int_{-1}^{1} p(x)P_k(x)P_s(x)dx = \begin{cases} 0, & k \neq s, \\ 1, & k = s. \end{cases}$$

Then the series

$$\frac{1}{z-y} = \sum_{k=0}^{\infty} a_k P_k(y) \qquad \left(a_k = \int_{-1}^{1}\frac{p(y)P_k(y)}{z-y}dy\right)$$

converges absolutely and uniformly with respect to y, if y lies in any domain lying wholly inside the ellipse C, which passes through the point z and has the points ± 1 for its foci,† i.e.,

$$\left|\frac{y+(y^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right| < 1-\epsilon \qquad (\epsilon > 0 \text{ arbitrarily small}),$$

$$\left|z+(z^2-1)^{1/2}\right| > 1, \ \left|y+(y^2-1)^{1/2}\right| \geqq 1.$$

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* Presented to the Society, December 30, 1930; received by the editors in June, 1929.

† G. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynómen eines Orthogonalsystems*, Mathematische Annalen, vol. 82 (1920), p. 209.

Hence, introducing the functions $Q_k(z)$ of the second kind*

$$a_k = Q_k(z) \qquad (k = 0, 1, 2, 3, \cdots), \tag{2}$$

$$\frac{1}{z - y} = \sum_{k=0}^{\infty} P_k(y)Q_k(z). \tag{3}$$

Inserting this value of $1/(z-y)$ in (1) we get

$$f(y) = \frac{1}{2\pi i}\int_C \left\{ \sum_{k=0}^{\infty} P_k(y)Q_k(z) \right\} f(z)dz = \sum_{k=0}^{\infty} C_k P_k(y),$$

$$C_k = \frac{1}{2\pi i}\int_C f(z)Q_k(z)dz \qquad (k = 0, 1, 2, 3, \cdots). \tag{4}$$

The coefficients C_k may be found independently:

$$C_k = \int_{-1}^{1} p(x)f(x)P_k(x)dx \qquad (k = 0, 1, 2, 3, \cdots), \tag{5}$$

whence we find the relation

$$\frac{1}{2\pi i}\int_C f(z)Q_k(z)dz = \int_{-1}^{1} p(x)f(x)P_k(x)dx \quad (k = 0, 1, 2, 3, \cdots). \tag{6}$$

Put $k=0$ *and* $f(x) = P_s(x)P_r(x)$. Then

$$\frac{1}{2\pi i P_0}\int_C Q_0(z)P_s(z)P_r(z)dz = \int_{-1}^{1} p(x)P_s(x)P_r(x)dx = \begin{cases} 0, & s \neq r, \\ 1, & s = r. \end{cases} \tag{7}$$

This formula shows that *the polynomials* $P_k(x)$, *which are orthogonal and normal in the interval* $(-1, +1)$ *with the characteristic function* $p(x)$, *have the same property* (x *being replaced by the complex variable* z) *on the contour* C *with the characteristic function*

$$\frac{Q_0(z)}{P_0} = \int_{-1}^{1} \frac{p(x)dx}{z - x}. \dagger \tag{8}$$

In particular, we find that *the normalized trigonometric polynomials*

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* G. Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres*, Journal de Mathématiques Pures et Appliquées, (3), vol. 4 (1878), p. 414.

† Cf. J. Sokhotzki, *The Theory of Integral Residues with Applications* (Thesis in Russian), St. Petersburg, 1868, p. 59, where formula (7) was established in a different way.

$$T_k(z) = \left(\frac{2}{\pi}\right)^{1/2} \cos k \arc\cos z$$

(9) $$= \left(\frac{2}{\pi}\right)^{1/2} \frac{(z + (z^2 - 1)^{1/2})^k + (z - (z^2 - 1)^{1/2})^k}{2}$$

$$\left(k = 1, 2, \cdots; T_0 = \left(\frac{1}{\pi}\right)^{1/2}\right),$$

orthogonal on $(-1, 1)$ *with the characteristic function* $p(z) = 1/(1-z^2)^{1/2}$, *are orthogonal on the contour C with the characteristic function* $\pi/(z^2-1)^{1/2}$.

In fact, according to S. Bernstein,* we have

(10) $$Q_0(z) = \left(\frac{\pi}{z^2 - 1}\right)^{1/2}, \quad Q_k(z) = \frac{(2\pi)^{1/2}}{(z^2 - 1)^{1/2}\{z + (z^2 - 1)^{1/2}\}^k}$$

$$(k = 1, 2, \cdots).$$

2. A well known property of the polynomials (9) is the following. The formal developments

$$f(x) \sim \sum_{k=0}^{\infty} A_k T_k(x), \quad \phi(x) \sim \sum_{k=0}^{\infty} B_k T_k(x)$$

$$\left(A_k = \int_{-1}^{1} \frac{f(x)T_k(x)}{(1 - x^2)^{1/2}} dx, \qquad B_k = \int_{-1}^{1} \frac{\phi(x)T_k(x)}{(1 - x^2)^{1/2}} dx\right)$$

imply, provided the integrals $\int_{-1}^{1}(f^2(x)/(1-x^2)^{1/2})dx$ and $\int_{-1}^{1}(\phi^2(x)/(1-x^2)^{1/2})dx$ exist,

(11) $$\int_{-1}^{1} \frac{f(x)\phi(x)}{(1-x^2)^{1/2}} dx = \sum_{k=0}^{\infty} A_k B_k.$$

Apply (11) to $f(x) = p(x)(1-x^2)^{1/2}$, $\phi(x) = 1/(z-x)$, assuming that

(12) $$\int_{-1}^{1} p^2(x)(1 - x^2)^{1/2} dx = \int_0^{\pi} p^2(\cos\phi) \sin^2 \phi d\phi$$

exists. Thus we get, writing

(13) $$p(x)(1 - x^2)^{1/2} \sim \sum_{k=0}^{\infty} c_k T_k(x) \qquad \left(c_k = \int_{-1}^{1} p(x) T_k(x) dx\right),$$

(14) $$\frac{1}{z - x} = \sum_{k=0}^{\infty} T_k(x) Q_k(z),$$

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* S. Bernstein, *Sur la valeur asymptotique de la meilleure approximation* (in Russian), Proceedings of the Kharkow Mathematical Society, 1913.

$$(15)\qquad \frac{Q_0(x)}{P_0(x)} = \int_{-1}^{1}\frac{p(x)(1-x^2)^{1/2}}{z-x}\,\frac{dx}{(1-x^2)^{1/2}} = \sum_{k=0}^{\infty} c_k Q_k(z)$$

(see (3), $Q_k(z)$ given by (10)).

Hence, the formal trigonometric expansion

$$(16)\qquad p(\cos\phi)\sin\phi \sim \left(\frac{2}{\pi}\right)^{1/2}\left\{\frac{a_0}{2^{1/2}} + \sum_{k=1}^{\infty} a_k \cos k\phi\right\} \qquad \left(a_k = \int_0^{\pi} p(\cos\phi)\sin\phi\cos k\phi d\phi\right)$$

yields at once, under condition (12), the expansion for $Q_0(z)/P_0$ with the same coefficients:

$$(17)\qquad \frac{Q_0(z)}{P_0} = \left(\frac{2\pi}{z^2-1}\right)^{1/2}\left\{\frac{a_0}{2^{1/2}} + \sum_{k=1}^{\infty} a_k[z-(z^2-1)^{1/2}]^k\right\}.$$

If $p(\cos\phi)\sin\phi$ is a finite trigonometric sum, then $Q_0(z)/P_0$ is also a finite sum. For example, taking

$$(18)\qquad p(x) = (1-x^2)^{1/2},\ \ p(\cos\phi)\sin\phi = \sin^2\phi = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\phi,$$

we find

$$(19)\qquad \frac{Q_0(z)}{P_0} = \pi\left\{z-(z^2-1)^{1/2}\right\}.$$

In other words, *the polynomials*

$$(20)\ P_0 = \left(\frac{2}{\pi}\right)^{1/2},\ \ P_k(z) = \frac{\left\{z+(z^2-1)^{1/2}\right\}^{k+1} - \left\{z-(z^2-1)^{1/2}\right\}^{k+1}}{(2\pi)^{1/2}(z^2-1)^{1/2}} \qquad (k = 1, 2, 3, \cdots)$$

are orthogonal and normal on the contour C with the characteristic function $\pi\{z-(z^2-1)^{1/2}\}$.

3. We proceed to derive some interesting properties of the functions $Q_n(z)$. Darboux has shown* that they satisfy the same recurrence relation as the $P_n(x)$:

$$(21)\qquad \begin{aligned} A_{n+1}Q_{n+1}(z) + A_nQ_{n-1}(z) &= (B_n+z)Q_n(z) \quad (n = 1, 2, 3, \cdots),\\ A_1Q_1(z) &= (B_0+z)Q_0(z) - \frac{1}{P_0} \qquad (A_i = \text{const.}).\end{aligned}$$

* Loc. cit., p. 415.

We multiply both members of Darboux's formula*

$$(22)\quad \sum_{k=0}^{n} P_k\ (x)P_k(y) = A_{n+1}\frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x-y} \quad \left(A_{n+1} = \frac{d_0^{(n)}}{d_0^{(n+1)}}\right)$$

by $p(x)$ and $p(x)/(z-x)$, and integrate between -1 and 1. We get

$$Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) = \frac{1}{A_{n+1}}, \dagger$$

$$(23)\quad \sum_{k=0}^{n} P_k(y)Q_k(z) = A_{n+1}\frac{Q_{n+1}(z)P_n(y) - Q_n(z)P_{n+1}(y)}{z-y} + \frac{1}{z-y}\cdot \ddagger$$

Suppose now that $p(-x)\equiv p(x)$ ("symmetric" orthogonal polynomials). Then, as is known,

$$P_k(-x) \equiv (-1)^k P_k(x),$$

and we get from (22), denoting by $[m]$ the greatest integer $\leqq m$,

$$\sum_{s=0}^{[n/2]} P_{2s}(x)P_{2s}(y) = A_{n+1}\frac{xP_{n+1}(x)P_n(y) - yP_n(x)P_{n+1}(y)}{x^2-y^2},$$

which, combined with the recurrence relation for $P_k(x)$, gives

$$(24)\quad \sum_{s=0}^{[n/2]} P_{n-2s}(x)P_{n-2s}(y) = A_{n+1}A_{n+2}\frac{P_{n+2}(x)P_n(y) - P_n(x)P_{n+2}(y)}{x^2-y^2}\cdot \S$$

The same method applied to (23) gives

$$(25)\quad \sum_{s=0}^{n/2} P_{n-2s}(y)Q_{n-2s}(z) = A_{n+1}A_{n+2}\frac{Q_{n+2}(z)P_n(y) - Q_n(z)P_{n+2}(y)}{z^2-y^2} + \frac{z}{z^2-y^2} \quad (n \text{ even}),$$

$$(26)\quad \sum_{s=0}^{(n-1)/2} P_{n-2s}(y)Q_{n-2s}(z) = A_{n+1}A_{n+2}\frac{Q_{n+2}(z)P_n(y) - Q_n(z)P_{n+2}(y)}{z^2-y^2} + \frac{y}{z^2-y^2} \quad (n \text{ odd}).$$

4. Assume now again that the integral

$$\int_{-1}^{1}\frac{\log p(x)}{(1-x^2)^{1/2}}dx$$

* Loc. cit., p. 413.

† F. Neumann, *Beiträge zur Theorie der Kugelfunktionen*, 1878, p. 71 ($p(x)\equiv 1$).

‡ Darboux, loc. cit., p. 415.

§ Cf. ($p(x)\equiv 1$) C. Neumann, *Über einige Reihenentwickelungen die nach Produkten von Kugelfunktionen fortschreiten*, Journal für Mathematik, vol. 135 (1909), p. 165.

exists, and use the results of §1, concerning the expansion

$$\frac{1}{z-y} = \sum_{k=0}^{\infty} P_k(y)Q_k(z). \tag{3}$$

Combining (23), (3), we get the expansion*

$$\sum_{k=1}^{\infty} P_{n+k}(y)Q_{n+k}(z) = A_{n+1}\frac{P_{n+1}(y)Q_n(z) - P_n(y)Q_{n+1}(z)}{z-y}. \tag{27}$$

Multiplying (26), (27) by $p(y)/(x-y)$ and integrating between -1 and 1, we get the expansions

$$\sum_{k=0}^{\infty} Q_k(x)Q_k(z) = \frac{Q_0(x) - Q_0(z)}{P_0(z-x)}, \tag{28}$$

$$\sum_{k=1}^{\infty} Q_{n+k}(x)Q_{n+k}(z) = A_{n+1}\frac{Q_n(z)Q_{n+1}(x) - Q_{n+1}(z)Q_n(x)}{z-x}, \tag{29}$$

which are valid for

$$|x + (x^2-1)^{1/2}| > 1 + \epsilon, \quad |z + (z^2-1)^{1/2}| > 1 + \epsilon_1, \tag{30}$$

where ϵ and ϵ_1 are arbitrarily small but fixed positive constants.

In particular, for $z = x$, we derive from (28), (29),

$$\sum_{k=0}^{\infty} Q_k^2(x) = -\frac{Q_0'(x)}{P_0}, \tag{31}$$

$$\sum_{k=1}^{\infty} Q_{n+k}^2(x) = A_{n+1}\,\{Q_{n+1}(x)Q_n'(x) - Q_n(x)Q_{n+1}'(x\}). \tag{32}$$

In the case of symmetric orthogonal polynomials, we get from (26)

$$\sum_{k=0}^{\infty} P_{2k}(y)Q_{2k}(z) = \frac{z}{z^2-y^2}. \tag{33}$$

Similarly,

$$\sum_{k=0}^{\infty} P_{2k+1}(y)Q_{2k+1}(z) = \frac{y}{z^2-y^2}. \tag{34}$$

From (25) and (33) we find†

$$\sum_{k=1}^{\infty} P_{n+2k}(y)Q_{n+2k}(z) = A_{n+1}A_{n+2}\frac{P_{n+2}(y)Q_n(z) - P_n(y)Q_{n+2}(z)}{z^2-y^2}. \tag{35}$$

* Neumann, Journal für Mathematik, vol. 135, p. 174.

† Neumann, Journal für Mathematik, vol. 135, p. 171.

All these expansions are valid for

$$\left| z + (z^2 - 1)^{1/2} \right| > 1, \quad \left| y + (y^2 - 1)^{1/2} \right| \geqq 1,$$

$$\left| \frac{y + (y^2 - 1)^{1/2}}{z + (z^2 - 1)^{1/2}} \right| < 1 - \epsilon.$$

Multiplying (33), (34), (35) by $p(y)/(x-y)$ and integrating between -1 and 1, we get the expansions

$$\sum_{k=0}^{\infty} Q_{2k}(x) Q_{2k}(z) = \frac{xQ_0(z) - zQ_0(x)}{P_0(x^2 - z^2)}, \tag{36}$$

$$\sum_{k=0}^{\infty} Q_{2k+1}(x) Q_{2k+1}(z) = \frac{zQ_0(z) - xQ_0(x)}{P_0(x^2 - z^2)}, \tag{37}$$

$$\sum_{k=1}^{\infty} Q_{n+2k}(x) Q_{n+2k}(z) = A_{n+1} A_{n+2} \frac{Q_{n+2}(z) Q_n(x) - Q_n(z) Q_{n+2}(x)}{x^2 - z^2}, \tag{38}$$

which are valid under condition (30). Putting $z = x$, we get

$$\sum_{k=0}^{\infty} Q_{2k}^2(x) = \frac{Q_0(x) - xQ_0'(x)}{2P_0(x)}, \tag{39}$$

$$\sum_{k=0}^{\infty} Q_{2k+1}^2(x) = -\frac{Q_0(x) + xQ_0'(x)}{2P_0(x)}, \tag{40}$$

$$\sum_{k=1}^{\infty} Q_{n+2k}^2(x) = A_{n+1} A_{n+2} \frac{Q_{n+2}(x) Q_n'(x) - Q_n(x) Q_{n+2}'(x)}{2x}. \tag{41}$$

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