

ON THE INTERCHANGE OF LIMIT AND LEBESGUE INTEGRAL FOR A SEQUENCE OF FUNCTIONS*

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R. L. Jeffery in a paper on *The integrability of a sequence of functions*† has given a number of necessary and sufficient conditions for $\lim_n \int_E f_n = \int_E F$, where f_n and F are summable on the measurable set E and $\lim_n f_n = F$ on E . The object of this note is to give an additional condition for the validity of this interchange, embodied in the

THEOREM. *If $U(n, \delta)$ is the least upper bound, and $L(n, \delta)$ is the greatest lower bound of $\int_e f_n$ for all measurable subsets e of E for which $m_e \leq \delta$, then a necessary and sufficient condition that $\lim_n \int f_n = \int F$ on E , is that*

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} [U(n, \delta) + L(n, \delta)] = 0.$$

We are using the notation $\lim_\delta \lim_n$ in the standard sense of the common value of $\lim_\delta \underline{\lim}_n$ and $\lim_\delta \overline{\lim}_n$. In (ϵ, δ) form the condition of the theorem is equivalent to the following:‡ For every $\epsilon > 0$ there exists a δ_ϵ such that for every $\delta \leq \delta_\epsilon$, there exists an $n_{\delta\epsilon}$ such that if $n \geq n_{\delta\epsilon}$, then $|U(n, \delta) + L(n, \delta)| \leq \epsilon$.

We establish the equivalence of our condition with the necessary and sufficient condition I of Jeffery, viz.

$$\lim_{l \rightarrow \infty} \int_{C(l, \eta)} f_n = 0 \quad (n \geq l)$$

where $C(l, \eta)$ is the complement relative to E of the set for which

$$|f_n - f| \leq \eta$$

for every $n \geq l$.

By the definition of $U(n, \delta)$, for every ϵ and δ , there exists a subset e of E of measure less than δ such that

$$U(n, \delta) - \epsilon \leq \int_e f_n \leq U(n, \delta).$$

Since $U(n, \delta) \geq 0$, we can obviously assume that $f_n \geq 0$ on e . Now for any l and η

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† Cf. the present number of these Transactions.

‡ Cf., for instance, Hildebrandt, *Note on interchange of limits*, Bulletin of the American Mathematical Society, vol. 34 (1928), p. 80.

$$\left| \int_e f_n - \int_{eC(l,\eta)} f_n \right| \leq \int_{e-eC(l,\eta)} |f_n - F| + \int_{e-eC(l,\eta)} |F|.$$

As a consequence if we take δ so that $\delta\eta \leq \epsilon/2$ and so that, for $m\epsilon \leq \delta$, it is true that $\int_e |F| < \epsilon/2$, then provided $n \geq l$

$$\left| \int_e f_n - \int_{eC(l,\eta)} f_n \right| \leq \epsilon.$$

Let C^+ be the subset of $C(l, \eta)$ for which $f_n \geq 0$, and C^- the set for which $f_n < 0$. If then l be chosen so that $mC(l, \eta) \leq \delta$, and $n \geq l$, then

$$U(n, \delta) - 2\epsilon \leq \int_{eC(l,\eta)} f_n = \int_{eC^+} f_n \leq \int_{C^+} f_n \leq U(n, \delta).$$

In a similar way we show that

$$L(n, \delta) \leq \int_{C^-} f_n \leq L(n, \delta) + 2\epsilon.$$

Hence if $\delta \leq \delta_\epsilon$, and $l \geq l_{\delta_\epsilon}$, with $n \geq l$, we have

$$\left| \int_{C(l,\eta)} f_n - [U(n, \delta) + L(n, \delta)] \right| \leq 2\epsilon.$$

From this statement follows the equivalence of the conditions

$$\lim_{l \rightarrow \infty} \int_{C(l,\eta)} f_n = 0 \quad (n \geq l) \quad \text{and} \quad \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} [U(n, \delta) + L(n, \delta)] = 0.$$

In so far as

$$\lim_{\delta \rightarrow 0} [U(n, \delta) + L(n, \delta)] = 0$$

for every n , our condition is equivalent to the equality of the iterated limits $\lim_\delta \lim_n$ and $\lim_n \lim_\delta$ of the function $U(n, \delta) + L(n, \delta)$. Since the existence of the double limit $\lim_{(n,\delta) \rightarrow (\infty,0)}$ is sufficient for this interchange it follows that

$$\lim_{(n,\delta) \rightarrow (\infty,0)} [U(n, \delta) + L(n, \delta)] = 0$$

is also a sufficient condition for $\lim_n \int_E f_n = \int_E F$. That it is not necessary follows from the example given by Jeffery in §3.

If the integrals $\int f_n$ are equicontinuous on E , i.e., if

$$\lim_{m\epsilon \rightarrow 0} \int_e f_n = 0$$

uniformly in n , then obviously $\lim_{\delta} [|U(n, \delta)| + |L(n, \delta)|] = 0$, uniformly in n , U and L being even taken with respect to any subset e of E . Applying the standard theorem on interchange of iterated limits, we get at once the Vitali theorem for all subsets e of E . On the other hand, if all the functions f_n are positive on E , then $U(n, \delta)$ converges to zero monotonically in δ , while $L(n, \delta) = 0$. By applying the following generalization of Dini's Theorem:*

If $\lim_n \lim_{\delta} U(n, \delta) = \lim_{\delta} \lim_n U(n, \delta)$, and $U(n, \delta)$ is monotone in δ , then $\lim_{\delta} U(n, \delta)$ exists uniformly in n ,

we find the well known result that when $f_n \geq 0$, equicontinuity of $\int f_n$ is a necessary condition for $\lim_n \int f_n = \int F$ on E .

The theorem of this note is still valid if the convergence of f_n to f on E is convergence in a measure,† i.e., if $D(n, \eta)$ is the set of points of E for which $|f_n - f| > \eta$ then $\lim_n mD(n, \eta) = 0$ for each η . Jeffery's Theorem must be altered so that the condition becomes

$$\lim_n \int_{D(n, \eta)} f_n = 0$$

for every η . The proofs for this more general case require only slight changes from those given.

* Cf. Bulletin of the American Mathematical Society, vol. 21 (1914), p. 113.

† The possibility of this extension was suggested by L. M. Graves.

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