NON-SEPARATED CUTTINGS OF
CONNECTED POINT SETS*

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1. We shall consider a connected, metric and separable space which we
denote by \( M \). A subset \( X \) of \( M \) is called a cutting of \( M \) provided that the
complement \( M - X \) of \( X \) is not connected and hence is the sum of two mutually
separated sets \( M_1(X) \) and \( M_2(X) \); \( X \) is said to separate two points or
point sets \( A \) and \( B \) in \( M \) when the sets \( M_1(X) \) and \( M_2(X) \) can be so chosen
that \( M_1(X) \supset A \) and \( M_2(X) \supset B \), and is said to separate a single set \( N \) in \( M \)
when \( M_1(X) \) and \( M_2(X) \) can be chosen so that \( N \cdot M_1(X) \neq 0 \neq N \cdot M_2(X) \).

A collection \( G \) of subsets of \( M \) will be called non-separated provided that
the elements of \( G \) are mutually exclusive and no element of \( G \) separates any
other element of \( G \) in \( M \).

A subset \( P \) of \( M \) is said to have the potential order \( \alpha \) in \( M \) relative to a
given collection \( G \) of subsets of \( M \) provided that \( \alpha \) is the least cardinal num-
ber such that there exists a monotonic decreasing sequence \( \{ U_i \} \) of neighbor-
hoods of \( P \) such that \( P = \bigcap U_i \) and such that for each \( i \), the boundary \( F(U_i) \)
of \( U_i \) is a subset of the sum of \( \alpha \) of the sets of the collection \( G \).

In this paper we shall show, first, that if \( G \) is any uncountable non-
separated collection of cuttings of \( M \) then \textit{all save a countable number of the}
elements of \( G \) have the potential order 2 in \( M \) relative to \( G \). Now obviously if
the elements of any collection \( G \) of mutually exclusive cuttings of \( M \) are con-
nectd or if they reduce to single points, then the collection \( G \) is non-
separated. And since for the case where \( M \) is compact, the potential order of
a point of \( M \) is the same as its order in the Menger-Urysohn sense, our
theorem yields as corollaries many important known results concerning the
cut points and connected cuttings of connected sets and of continua; for
example: (1) the theorem of Wazewski-Menger‡ that the ramification points
of any acyclic continuous curve are countable, (2) the theorem of Kuratowski
and Zarankiewicz§ that the set of all points of any connected set \( M \) whose
complement in \( M \) is neither connected nor the sum of two connected point

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† Fellow, John Simon Guggenheim Memorial Foundation.
‡ See Wazewski, Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 49; and
sets is countable; (3) the theorem of the author* that all save a countable number of the cut points of any continuum are points of order 2 of $M$ in the Menger-Urysohn sense; and (4) other results concerning cuttings due to Zarankiewicz† and to the author.‡

Second, with the aid of this theorem we shall show that if the space $M$ contains an uncountable non-separated collection $G$ of cuttings, then there exists an upper semi-continuous collection $S$ of elements such that all save a countable number of the sets of $G$ are elements of $S$ and such that every two elements of $S$ may be separated in $M$ by some third element. In case $M$ is compact, the decomposition space $S$ is an acyclic continuous curve.

Finally, we shall prove an existence theorem to the effect that every locally connected space $M$ contains an uncountable non-separated collection of cuttings. Therefore, the above mentioned decomposition is always realisable for locally connected sets $M$, and notably for the case where $M$ is a continuous curve, this decomposition gives rise to a decomposition space which is a non-degenerate acyclic continuous curve.

2. Preliminary lemmas. Let $X$ and $Y$ be any two cuttings of $M$ and set

\[(i) \quad M - X = M_1(X) + M_2(X),\]
\[(ii) \quad M - Y = M_1(Y) + M_2(Y),\]

representing decompositions of $M - X$ and $M - Y$ respectively into mutually separated sets. Then if $i, j, r, \text{and } s$ are positive integers such that $i + j = 3 = r + s$, it follows immediately that the following equation is valid:

\[(2.1) \quad M = M_1(X) + M_2(Y) + M_1(X) \cdot M_2(Y) + X + Y.\]

With the aid of this equation, we deduce at once the result

\[(2.2) \quad \text{If neither of the sets } X \text{ and } Y \text{ separates the other, we may choose the indices } i \text{ and } r \text{ such that}\]

(a) $X \subset M_r(Y) \text{ and } Y \subset M_i(X)$;

and these relations imply also the relations

(b) $M_j(X) \cdot M_s(Y) = 0$, $M_j(X) + X \subset M_r(Y)$, and $M_s(Y) + Y \subset M_i(X)$.

Clearly this is the case, because by virtue of the relations (a) we may omit the last two terms in equation (2.1); and since $M$ is connected, the term $M_j(X) \cdot M_s(Y)$ must vanish. This fact gives at once the remaining two relations (b).

Now let $G$ be any non-separated collection of cuttings of $M$ and let $E(a, b)$ be the collection of all those elements of $G$ which separate two given points $a$ and $b$ in $M$. Let $X$ and $Y$ be any two elements of $E(a, b)$ and let the indices in (i) and (ii) be chosen so that

(iii) $M_1(X) \cdot M_1(Y) \supset a$ and $M_2(X) \cdot M_2(Y) \supset b$.

The element $X$ of $E(a, b)$ will be said to precede the element $Y$, and this fact is indicated by the notation $X < Y$, provided that for at least one set of decompositions satisfying (i), (ii) and (iii) it is true that $X \subset M_1(Y)$. We shall now show that this definition gives a natural order to the elements of $E(a, b)$.

First, for any two elements $X$ and $Y$ of $E(a, b)$, at least one of the relations $X < Y$ and $Y < X$ must be valid. For if $X$ does not precede $Y$, then by (2.2), (a), $r = 2$ and hence $s = 1$. By (b) and (iii) it follows that $j = 2$ and hence $i = 1$. Therefore by (a), $Y \subset M_1(X)$, which means $Y < X$.

Second, only one of the relations $X < Y$ and $Y < X$ can be valid. For if $X < Y$, then for any set of decompositions whatever satisfying (i), (ii), (iii), in (2.2), $r = 1$ and hence $s = 2$. By (b) and (iii) it follows that $j = 1$ and hence $i = 2$. Therefore by (a), $Y \subset M_2(X)$, which is incompatible with $Y < X$.

Finally, for any three elements $Z$, $X$ and $Y$ of $E(a, b)$, the relations $Z < X$, $X < Y$ imply that $Z < Y$. For then $Z \subset M_1(X)$ and $X \subset M_1(Y)$. Hence in (2.2), $r = 1$ and $s = 2$. By (b) and (iii) it follows that $j = 1$. Therefore by the second relation in (b), $Z \subset M_1(X) + X \subset M_1(Y)$, which gives $Z < Y$.

Thus we have proved the following result:

(2.3) If each element of the non-separated collection $E(a, b)$ of subsets of $M$ separates the two points $a$ and $b$ in $M$, then the collection $E(a, b)$ possesses a natural order.

For convenience we give here a lemma concerning ordered sets due to Zarankiewicz* which will be used below.

**Lemma (Zarankiewicz).** If $K$ is any ordered subset of $M$, then the set $H$ of all points $p$ of $K$ such that $p$ is not at the same time a limit point of the set $P_p$ of all points of $K$ preceding $p$ and also of the set $F_p$ of all points of $K$ following $p$ is countable.

The space $M$ being separable and metric, it therefore contains a countable sequence $R_1, R_2, R_3, \cdots$ of open sets which is equivalent to the set of all open subsets of $M$. Now let $H_1$ be the set of all points of $K$ which are not limit points of their predecessors, and let $H_2 = H - H_1$. For each point $p$ of $H_1$ let $n(p)$ be the least positive integer such that $R_{n(p)}$ contains $p$ but contains no point of $K$ which precedes $p$. Then if $p$ and $q$ are distinct points of $H_1$ and

p < q, then since $R_n(q)$ does not contain $p$, it follows that $n(p) \neq n(q)$, and therefore $H_1$ is countable. A similar argument proves $H_2$ countable; and hence $H$ is countable.

3. Theorem. If $G$ is any uncountable non-separated collection of cuttings of a connected, metric, and separable space $M$, then all save possibly a countable number of the elements of $G$ have the potential order 2 in $M$ relative to $G$.

Suppose, on the contrary, that $G$ contains an uncountable subcollection $G_1$ no element of which has the potential order 2 in $M$ relative to $G$. Now there exist two points $a$ and $b$ of $M$ such that the collection $E(a, b)$ of all those elements of $G_1$ which separate $a$ and $b$ in $M$ is uncountable; for $M$ being separable, there exists a countable subset $D$ of $M$ such that $D = M$; and since every element of $G_1$ which contains no point of $D$ must separate some pair of points of $D$ in $M$, and since the set of all pairs of points of $D$ is countable, it follows that for at least one pair of points $a$, $b$ of $D$, the set $E(a, b)$ is uncountable.

By §2 the elements of the collection $E(a, b)$ possess a natural order; and if $K$ is a point set which contains exactly one point $x$ of each element $X$ of $E(a, b)$ and contains no other points, then $K$ is an ordered point set. Indeed for each pair $x$, $y$ of points of $K$, set $x < y$ provided that $X < Y$. By the Zarankiewicz lemma, the set $H$ of all points $p$ of $K$ which are not at the same time a limit point both of their predecessors and of their successors is countable. Let $H(a, b)$ be the collection of all those sets $X$ of $E(a, b)$ such that the corresponding point $x$ in $K$ belongs to $K - H$. Then $H(a, b)$ is uncountable and each element $X$ of $H(a, b)$ contains a point $x$ which is a limit point of the sum of the predecessors of $X$ and also of the sum of the successors of $X$.

Now for each element $X$ of $H(a, b)$, there exist mutually separated sets $M_1(X)$ and $M_2(X)$ such that

$$M - X = M_1(X) + M_2(X), \quad M_1(X) \supset a \quad \text{and} \quad M_2(X) \supset b.$$ 

And with the aid of what has just been shown it follows immediately that there exist two infinite sequences of elements $X_1, X_2, X_3, \ldots$ and $Y_1, Y_2, Y_3, \ldots$ of $H(a, b)$ such that, for each $n$,

$$X_n < X_{n+1} < X < Y_{n+1} < Y_n,$$

and such that $X$ contains a point which is a limit point both of $\sum X_n$ and $\sum Y_n$.

Since by supposition no element of $H(a, b)$ can have the potential order 2 in $M$ relative to $G$, it follows that if for each element $X$ of $H(a, b)$, $V_n(X)$ denotes the set of points $M - [M_1(X_n) + M_2(Y_n)]$, then there exists at least one point $p_n$ belonging to the point set
for if this were not the case, then by virtue of (1) and equation (2.1) in which substitute \( X_n \) for \( X \), \( Y_n \) for \( Y \), 1 for \( i \) and 2 for \( r \), it follows that \( V_n(X) \supseteq M_2(X_n) \cdot M_1(Y_n) \supseteq X \); and if for each point \( p \) of \( M_2(X_n) \cdot M_1(Y_n) \) we take a neighborhood \( N_p \) of \( p \) of diameter less than 1/4 the distance from \( p \) to the set of points \( M_1(X_n) + M_2(Y_n) \), and call \( U_n(X) \) the sum of all the neighborhoods \( N_p \), then it follows readily that

\[
X \subseteq M_2(X_n) \cdot M_1(Y_n) \subseteq U_n(X) \subseteq \overline{U_n(X)} \subseteq V_n(X);
\]

and hence \( F[U_n(X)] \subseteq X_n + Y_n \), \( U_n(X) \subseteq U_{n-1}(X) \) and \( X = \prod U_n(X) \); but then \( X \) has the potential order 2 in \( M \) relative to \( G \), contrary to supposition.

Now if \( X \) and \( Y \) are any two elements of \( H(a, b) \), \( X \neq Y \), it follows that \( p_x \neq p_y \). For suppose \( X < Y \). Then since \( X \) contains a limit point of the sum of its successors in \( E(a, b) \) but contains no limit point of \( M_2(Y) \), it follows that there exist two elements \( Y_k \) and \( Y_m \) in the “\( Y \)-sequence” in (1) for the element \( X \) such that

\[
X < Y_k < Y_m < Y;
\]

and since \( Y \) contains a limit point of the sum of its predecessors in \( E(a, b) \) but contains no limit point of \( M_1(Y_m) \), it follows that there exists an element \( X_n \) of the “\( X \)-sequence” for \( Y \) in (1) such that

\[
X < Y_k < Y_m < X_n < Y.
\]

Consequently it follows with the aid of (2.2) that

\[
p_x \subseteq M_1(Y_k) + Y_k \subseteq M_1(Y_m)
\]

and

\[
p_y \subseteq M_2(X_n) + X_n \subseteq M_2(Y_m),
\]

and hence \( p_x \neq p_y \).

Now let \( L \) denote the set of all points \( [p_x] \) for all elements \( X \) of \( H(a, b) \). Then \( L \) is uncountable and is an ordered set; indeed, it is only necessary to set \( p_x < p_y \) when \( X < Y \). Therefore by the Zarankiewicz lemma, there exists a point \( p_x \) of \( L \) which is a limit point both of its predecessors and of its followers, and hence both of \( \sum X_n \) and of \( \sum Y_n \), where the sequences \( [X_n] \) and \( [Y_n] \) satisfy (1). But \( \sum X_n \subseteq M_1(X) \) and \( \sum Y_n \subseteq M_2(X) \); and \( p_x \) must then belong either to \( M_1(X) \) or to \( M_2(X) \) and be a limit point of the other, contrary to the fact that these two sets are mutually separated. Thus the supposition that our theorem is false leads to a contradiction.
4. Consequences of §3. Let $G$ be any uncountable non-separated collection of cuttings of $M$. Then since the product of any family $[\overline{U}_n]$ of closed sets is closed, §3 yields at once the result

(a) All save a countable number of the elements of $G$ are closed point sets.

Now if $X$ is any element of $G$ such that $M - X$ is not the sum of two connected point sets, $X$ cannot have a potential order 2 in $M$ relative to $G$. For $M - X = M_1(X) + M_2(X) + M_3(X)$, where the sets $M_1(X)$, $M_2(X)$, and $M_3(X)$ are mutually separated and contain points $a_1$, $a_2$ and $a_3$ respectively; and if $X$ had the potential order 2 relative to $G$, there would exist two elements $X_1$ and $X_2$ of $G$ and a neighborhood $U$ of $X$ such that $F(U) \subset X_1 + X_2$, $X_1 \subset M_1(X)$, $X_2 \subset M_2(X)$ and $\overline{U} \cdot (a_1 + a_2 + a_3) = 0$; but then it would readily follow that the point set $M_3(X) \cdot (M - \overline{U})$ is non-vacuous and is both open and closed, contrary to the fact that $M$ is connected. Thus in consequence of the theorem in §3 we have

(b) The complement of each element of $G$, with the exception of a countable number of such elements, consists of exactly two components.

Let us denote by $\rho$ the property of any subset $N$ of $M$ not to be separated in $M$ by any single element of $G$. Clearly each element $X$ of $G$ has the property $\rho$. We shall now show that

(γ) All save a countable number of the elements of $G$ are saturated in $M$ relative to the property $\rho$.

If, on the contrary, $G$ contains an uncountable subcollection $G_1$ no element of which is saturated relative to the property $\rho$, then for each element $Z$ of $G_1$ there exists at least one point $p_x$ which is not separated from $Z$ in $M$ by any single element of $G$. Under these conditions it follows by the theorem and proof in §3 that there exist two points $a$ and $b$ of $M$ and three elements $Z, X$ and $Y$ of $E(a, b)$ (the collection of all those elements of $G_1$ which separate $a$ and $b$) such that $X < Z < Y$, and $M_2(X) \cdot M_1(Y)$ contains $Z$ but does not contain the point $p_x$ and also such that $X + Y$ does not contain $p_x$. But then by equation (2.1) we have either $p_x \subset M_1(X)$ or $p_x \subset M_2(Y)$. This is impossible because in the first case $X$ separates $p_x$ and $Z$ in $M$ and in the second case $Y$ separates $p_x$ and $Z$ in $M$.

A cutting $X$ of $M$ is said to be an irreducible cutting of $M$ provided that no proper subset of $X$ is a cutting of $M$.

(δ) All save a countable number of the elements of $G$ are irreducible cuttings of $M$.

If this is not so, there exists an uncountable collection $G^0$ of cuttings of $M$ such that for each element $X^0$ of $G^0$ there exists an element $X$ of $G$ and a point $p_x$ of $X$ such that $X^0 \subset X - p_x$. Since $G$ is non-separated, it follows at once that $G^0$ is non-separated. Therefore by (γ) there exists an element $X^0$
of $G^0$ which is saturated relative to the property $\rho$ defined by the collection $G^0$. Consequently there exists an element $Y^0$ of $G^0$ which separates $X^0$ and $p_x$ in $M$, and one has $M - Y^0 = M_1(Y^0) + M_2(Y^0)$, where $M_1(Y^0) \triangleright X^0$ and $M_2(Y^0) \triangleright p_x$. But then $M - Y = M_1(Y^0) \cdot (M - Y) + M_2(Y^0) \cdot (M - Y)$, and thus $Y$ separates $X$ in $M$ (for $Y \cdot (X^0 + p_x) \subset Y \cdot X = 0$), which contradicts the non-separatedness of $G$.

We prove now the following general theorem:

**Theorem.** Every uncountable non-separated collection $G$ of cuttings of a connected, metric, and separable space $M$ contains a subcollection $Q$ which contains all save possibly a countable number of the elements of $G$ and such that each element $X$ of $Q$ has the following properties: (a) $X$ is closed; (b) $M - X$ is the sum of two mutually separated connected point sets; (c) $X$ is saturated in $M$ relative to the property $\rho$ defined by the collection $Q$, i.e., for every point $p$ of $M - X$, there exists an element $Y$ of $Q$ which separates $X$ and $p$ in $M$; (d) $X$ is an irreducible cutting of $M$; and (e) $X$ has the potential order 2 in $M$ relative to $Q$.

To obtain the collection $Q$, let $D$ be a countable subset of $M$ which is dense in $M$ and let us omit from $G$: (1) every element which does not possess each of the properties (a), (b), and (d); (2) every element which separates in $M$ some pair of points $a, b$ of $D$ which are separated by only a countable number of elements of $G$; (3) every element which separates some pair $a, b$ of points of $D$ and contains no point $p$ having the property that every neighborhood of $p$ contains points of uncountably many distinct elements of $G$ which separate $a$ and $b$. Let $G_1$ denote the collection of the elements of $G$ remaining after these omissions. Then by virtue of (α), (β) and (δ), together with the facts that there are only a countable number of pairs of points of $D$ and that in the space $M$ every uncountable set of points contains a point of condensation of itself, it follows that $G_1$ contains all save possibly a countable number of the elements of $G$.

Now let us omit from $G_1$ every element which is not saturated in $M$ relative to the property $\rho$ defined by the collection $G_1$ and also every element which does not have the potential order 2 in $M$ relative to $G_1$. Let $Q$ be the collection of elements of $G_1$ remaining after these omissions. Then $Q$ contains all save a countable number of the elements of $G_1$ and hence also of $G$, and every element $X$ of $Q$ has the desired properties (a)-(e). Clearly $X$ has properties (a), (b) and (d), for every element of $G_1$ has these properties. It remains to show that $X$ has properties (c) and (e).

To show that $X$ has property (c), let $p$ be any point of $M - X$. There exists an element $Y$ of $G_1$ which separates $X$ and $p$, because every element of $Q$ is saturated in $M$ relative to the property $\rho$ defined by $G_1$. Hence $M - Y$
= M_1(Y) + M_2(Y), where M_1(Y) \supset X and M_2(Y) \supset p. Also M - X = M_1(X) + M_2(X), where M_2(X) \supset Y. Thus if a and b are points of M_1(X) and M_2(Y) respectively belonging to D, both X and Y separate a and b in M, and we have X < Y in the order from a to b. Now there exists also an element Z of G_1 which separates X and Y in M, and it follows from \S 2 that Z also separates a and b in M, and we have the order X < Z < Y. Thus Z \in M_2(X) \cdot M_1(Y).

Since X and Y are closed, M_2(X) \cdot M_1(Y) is a neighborhood of Z, and hence it contains points of (and therefore contains all of) uncountably many elements of G which separate a and b in M. Therefore there exists at least one of these elements, say W, which belongs to Q, for all but a countable number of the elements of G belong to Q. Thus we have the order X < W < Y; and since p \in M_2(Y), it follows that W separates X and p in M. Consequently X has property (c).

Since X has the potential order 2 in M relative to G_1, there exist, as shown in \S 3, two points a and b of M such that X belongs to the collection E(a, b) of all those elements of G_1 which separate a and b in M and such that there exist two sequences X_1, X_2, \ldots and Y_1, Y_2, \ldots of elements of E(a, b) so that

\[ X_n < X_{n+1} < X < Y_{n+1} < Y_n \]

and such that if U_n = M_2(X_n) \cdot M_1(Y_n), then X = \prod U_n. Now for each n there exist, by virtue of property (c), two elements X'_n and Y'_n of Q belonging to E(a, b) such that X_n < X'_n < X < Y'_n < Y_n. Hence if U'_n denotes the point set M_2(X'_n) \cdot M_1(Y'_n), one has U'_n \supset U_n. Hence X = \prod U'_n, and since F(U'_n) \subset X'_n + Y'_n and since clearly the sequence \{U_n\} contains an infinite subsequence \{U_{n_k}\} such that U_{n_{k+1}} \subset U_{n_k}, it follows that X has the potential order 2 in M relative to Q. This completes the proof.

5. Decomposition of M by means of a non-separated collection G every element of which is saturated relative to property p. Let G be any non-separated collection of subsets of M each of which is saturated in M relative to the property p defined by G. For each point e of M which belongs to no element of G, let E denote the point set consisting of e together with all points p of M which are not separated in M from e by any single element of G. Let S denote the collection whose elements are the elements of G together with all such point sets E thus defined. Clearly each element of S is closed and every point of M belongs to exactly one element of S. We shall show next that the collection S is non-separated.

Suppose, on the contrary, that some element X of S separates some pair of points p and q belonging to an element Y of S. Then M - X = M_1(X) + M_2(X), where M_1(X) \supset p and M_2(X) \supset q. Now by virtue of the definition of the collections G and S, it follows that there exists an element Z of G which
separates $X$ and $p$ in $M$. Hence $M-Z = M_1(Z)+M_2(Z)$, where $M_1(Z) \supset X$ and $M_2(Z) \supset p$. Since $Z$ belongs to $G$, it cannot separate $Y$ in $M$; and therefore $p+q \in Y \subset M_2(Z)$. But then

$$M-Z = [M_1(Z) + M_1(X) \cdot M_2(Z)] + M_2(X) \cdot M_2(Z),$$

and we have a separation of $M-Z$ into two mutually separated sets containing the points $p$ and $q$ respectively of $Y$, contrary to the fact that since $Z$ belongs to $G$ it cannot separate $Y$ in $M$. Therefore $S$ is non-separated.

Now clearly every element of $S$ is saturated in $M$ relative to the property $\rho$ defined by the collection $S$. Consequently every two elements $X$ and $Y$ of $S$ are separated in $M$ by some third element of $S$. With the aid of this property it follows immediately that the collection $S$ is upper semi-continuous, i.e., there does not exist a sequence $X_1, X_2, X_3, \ldots$ of elements of $S$ and two sequences $[p_i]$ and $[q_i]$ of points such that $p_i+q_i \in X_i$ and which have sequential limit points $p$ and $q$ respectively belonging to two different elements $P$ and $Q$ respectively of $S$. For there exists an element $X$ of $S$ such that $M-X = M_1(X)+M_2(X)$ where $M_1(X) \supset P$ and $M_2(X) \supset Q$; and since for each $i$, $X_i$ is a subset either of $M_1(X)$ or of $M_2(X)$, either $M_1(X)$ or $M_2(X)$ contains $X_i$ for infinitely many $i$'s; but this is impossible, for both $p$ and $q$ are limit points of every infinite subsequence of $X_1, X_2, X_3, \ldots$.

Now in case the space $M$ is compact, the elements of $S$ are closed and compact, and if for each pair of elements $X$ and $Y$ of $S$ we define the distance $\rho(X, Y)$ between $X$ and $Y$ as the upper limit of the distances $\rho(x, y)$, where $x$ and $y$ are points of $X$ and $Y$ respectively, it readily follows that the space $S'$ so obtained is compact, separable, metric and connected; and since it readily follows that every two "points" of $S'$ are separated in $S'$ by some third "point" of $S'$, therefore $S'$ is an acyclic continuous curve.

6. Existence Theorem. If the space $M$ is connected im kleinen, there exists an uncountable non-separated collection of cuttings of $M$.

Let $a$ and $b$ be any two points of $M$, and for each positive number $r$ which is less than the distance from $a$ to $b$, let $S(a, r)$ denote the set of all points of $M$ whose distance from $a$ is equal to $r$ and let $I(a, r)$ denote the set of all points at a distance $< r$ from $a$. Then for each $r$, $S(a, r)$ separates $a$ and $b$ in $M$. Let $R(a, r)$ denote the component of $M-S(a, r)$ containing $a$, let $R(b, r)$ denote the component of $M-R(a, r)$ containing $b$, and let $X_r$ denote the point set $R(a, r) \cap R(b, r)$. Then clearly $X_r$ separates $a$ and $b$ in $M$ and we have

\begin{align*}
(i) & \quad X_r \subset F[R(a, r)] \subset S(a, r), \quad \text{and} \quad X_r = F[R(b, r)]; \\
(ii) & \quad R(a, r) \subset I(a, r).
\end{align*}

* See R. L. Moore, these Transactions, vol. 27 (1925), pp. 416-428.
Obviously the collection of cuttings \([X_r]\) is uncountable. It remains to show that it is non-separated. Let \(X_r\) and \(X_s\) be any two elements of this collection and suppose \(r_1 < r_2\). By (ii) it follows that \(R(a, r_1) \subset R(a, r_2)\). Thus \(X_r \subset R(a, r_2)\), and therefore \(X_r\) does not separate \(X_s\) in \(M\). From the inclusion \(R(a, r_1) \subset R(a, r_2)\) and (i) it follows that \(R(b, r_2) = R(b, r_2) + X_{r_2} \subset R(b, r_1)\), and consequently \(X_r\) does not separate \(X_{r_2}\) in \(M\). Thus the collection \([X_r]\) is non-separated, and the theorem is proved.

Now since \(a\) may be any point whatever of \(M\) and since every neighborhood of \(a\) contains uncountably many of the sets \([X_r]\), it follows by §4, (δ), that every such neighborhood contains at least one set \(X_r\) which is an irreducible cutting of \(M\). Thus we have the following

**Corollary.** Every open subset of a connected and connected im kleinen point set \(M\) lying in a separable metric space contains an irreducible cutting \(I\) of \(M\).

This corollary answers a question raised by the author.*

As a result of this existence theorem it follows that the decomposition treated in §5 is always realisable in case \(M\) is locally connected; and in case \(M\) is a continuous curve, \(M\) may be decomposed upper semi-continuously into a collection \(S\) of the type attained in §5, and the decomposition space \(S'\) is a non-degenerate acyclic continuous curve.

7. **Concluding remarks.** Although it is easily seen with the aid of a very simple example that two cuttings \(X\) and \(Y\) of \(M\) may have the property that neither of them separates the other in \(M\) and yet the set \(M_2(X) + X\) not be connected, where \(M_2(X) \ni Y\), nevertheless the following lemma is true.

**Lemma.** If \(a\) and \(b\) are two points of \(M\) and \(X_1, X_2, X_3, \ldots\) is any infinite sequence of distinct mutually exclusive sets each of which separates \(a\) and \(b\) in \(M\) and no one of which separates any other one, and we have

\[ X_1 < X_2 < X_3 < \cdots, \]

then the set of points \(\sum_{i=1}^\infty M_1(X_i)\) is connected.

For if on the contrary this set of points is the sum of two mutually separated sets \(N_1\) and \(N_2\), then since \(a\) belongs to all of the sets \(M_1(X_i)\), there exists an integer \(n\) such that \(N_1 \cdot M_1(X_n) \neq 0 \neq N_2 \cdot M_1(X_n)\). Since by (2.2), (b), it follows that \(M_1(X_n) \subset M_1(X_{n+1})\), therefore \(N_1 \cdot M_1(X_{n+1}) \neq 0 \neq N_2 \cdot M_1(X_{n+1})\). Since these two sets are mutually separated, one of them, say \(N_1 \cdot M_1(X_{n+1})\),

* See Fundamenta Mathematicae, vol. 13 (1929), p. 50, where the question is raised for continuous curves \(M\). A solution of this problem for the case where \(M\) is a plane continuous curve has been given by J. H. Roberts; see these Transactions, vol. 32 (1930), p. 19.
contains $X_n$. But then it is easily seen that the sets $N_2 \cdot M(X_n)$ and $M - N_2 \cdot M(X_n)$ are mutually separated, contrary to the fact that $M$ is connected.

With the aid of this lemma it can be shown without great difficulty that if $X$ is any element of a non-separated collection $G$ of subsets of $M$ which is saturated in $M$ relative to the property $\rho$ defined by the collection $G$, then

1. each component of $M - X$ is open in $M$;
2. the components of $M - X$ are countable;
3. $X$ is a potentially regular element of $G$ in $M$ relative to $G$, i.e., a monotone decreasing sequence of neighborhoods $[U_i]$ of $X$ exists such that $F(U_i)$ is a subset of a finite number of the elements of $G$ and $X = \prod_i^\omega \overline{U_i}$;
4. the potential order of $X$ in $M$ relative to $G$ is equal to the number of components of $M - X$ when this number is finite, and is equal to $\omega$ (i.e., $X$ is of increasing order) when and only when this number is infinite.

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