A CERTAIN TYPE OF CONTINUOUS CURVE AND RELATED POINT SETS*

BY

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DEFINITION. A continuous curve every subcontinuum of which is a continuous curve will be called a *perfect continuous curve*.† In this paper, among other things, a study is made of this type of continuous curve.

An important problem in analysis situs is to determine whether a given point set is arc-wise connected. It is known that a connected subset of a perfect continuous curve is not necessarily arc-wise connected.‡ Here, however, it is shown in Theorem 5 that, if a and b are two points of a connected subset N of a perfect continuous curve, then the set common to N and the arcs ab of N' is connected. And in Theorem 7 it is shown that if N' contains but a countable number of arcs ab then N contains an arc ab.§

A generalization of the problem as to whether there exists an arc joining two points in a given point set is the following: when do there exist in a given point set where q is a given positive integer, q arcs, distinct except for end points, joining two given points; or still a further generalization would be to ask when do there exist q such arcs joining two distinct closed point sets? This problem is considered in §II of this paper. It might also be asked when do there exist q distinct arcs joining two distinct closed point sets? This problem is considered in §I.

I wish to express my thanks and to acknowledge my great indebtedness to Professor R. L. Wilder for his suggestions, criticisms, and constant encouragement.*

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* Presented to the Society, April 6, 1928; received by the editors in June, 1929, and June, 1930. Because of the delay in publishing this paper the privilege has been taken of using some recent results of other authors to abbreviate proofs and to obtain more general results than were originally obtained with respect to space and boundedness. These changes have been made in footnotes and in Section V with the exception that recent results have been used in proving Theorem 7. In general proofs which are well known in recent papers have been omitted here.

† In this paper a connected im kleinen continuum will be called a *continuous curve*. For definitions see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 289–302.

‡ B. Knaster and C. Kuratowski, *A connected and connected im kleinen point set which contains no perfect subset*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 106–109.

§ For theorems on arc-wise connected subsets of a perfect continuous curve see R. L. Wilder: *Characterizations of continuous curves that are perfectly continuous*, Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 614–621, Theorem 3; *Concerning perfect continuous curves*, Proceedings of the National Academy of Sciences, vol. 16 (1930), pp. 233–240, Theorems 1 and 2.

I. Distinct arcs in a point set

In this section the following problem will be considered: when can it be said that in a set M, for any given positive integer q, either there exist q distinct arcs† of M joining two distinct closed point sets‡ A and B or there exist $q-1$ points such that every arc of M joining A and B contains at least one of these points? If M contains no arc joining A and B it will be understood that there exist $1-1=0$ points contained in every arc of M joining A and B. With this understanding it will be shown that one of the two above possibilities holds for every set M.

Lemma 1. *If (t) is a set of arcs contained in an arc ab such that every point of ab except a and b is an interior point of an arc of (t) and both a and b are end points of at least one arc of (t), then there exists a simple chain of arcs§ of (t) joining a and b.*

This follows in a manner very similar to that used by R. L. Moore in proving Theorem 10 of his paper *On the foundations of plane analysis situs*.‖

Lemma 2. *In E_n¶ let C and K be distinct, closed, and bounded point sets, where K is connected and does not separate E_n, and let a_iw_i $(i=1, 2, \cdots, m)$ be m distinct arcs such that $a_iw_i \times K = w_i$. Then there exists a positive number d such that for any positive number r less than d there exist m distinct arcs a_iu_i of a_iw_i and m arcs u_ih distinct except for h such that (1) every point of u_ih is at a distance less than r from K, (2) the arc w_iu_i of a_iw_i has nothing in common with C, and (3) for any j, $u_jh \times ((a_iu_i)^{**}+C) = u_jh \times u_ja_j = u_j$.*

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* Since the theorems of this paper were proved a paper has appeared giving an interesting result in this connection: see N. E. Rutt, *Concerning the cut points of a continuous curve where the arc curve, AB, contains exactly N independant arcs*, American Journal of Mathematics, vol. 51 (1929), p. 218. For another interesting result see W. L. Ayres, *Concerning continuous curves in metric space*, ibid., pp. 577–594, Theorem 6. See also K. Menger, *Zur allgemeinen Kurventheorie*, Fundamenta Mathematicae, vol. 10, p. 100, Theorem β. In each of these papers the q arcs considered lie in a continuous curve in contrast to the results of this paper where the containing set may not even be connected in Theorem 1 and in Theorem 3 is connected but not necessarily closed.

† Two point sets are *distinct* when they have no common point.

‡ An arc ab will be said to join the point sets A and B if $A\times ab=a$ and $B\times ab=b$. Under these conditions the arc ba will be said to join B and A.

§ If a and b are distinct points, then a *simple chain* from a to b is a finite sequence of arcs $t_i(i=1, 2, \cdots, m)$ such that (1) t_i contains a if and only if $i=1$, (2) t_i contains b if and only if $i=m$, and (3) if $1\leqq i\leqq m$ and $1\leqq j\leqq m$, $i<j$, then t_i has a point in common with t_j if and only if $j=i+1$.

‖ These Transactions, vol. 17 (1916), pp. 131–164.

¶ The notation E_n will be used to denote a euclidean space of n dimensions.

** Throughout this paper the set $a_1+a_2+\cdots+a_m$ will be designated by (a_i) $(i=1, 2, \cdots, m)$. Thus $(a_i)+(b_i)=a_1+a_2+\cdots+a_m+b_1+b_2+\cdots+b_m$. The range of values for i, if it is not given, will refer to the last mentioned range of values for i.

The proof will first be given for E_2 as this is the most difficult case. Let z_i be the first point of $C+a_i$ on w_ia_i and let $a_iz_i=a_i$ if $z_i=a_i$. Let d be a positive number such that every point of $C+(a_iz_i)$ is at a distance greater than d from K. Let $M=K+C+(a_iz_i)$. Then for any positive number r less than d there exists* a simple closed curve J bounding a region R such that $J\times M=0$, $R\times M=K$, and every point of R is at a distance less than r from K. Let the first point of J on a_iw_i be u_i, and let a_iu_i be an arc of a_iw_i. Let h be any point of R. Then there exist in R the m arcs hu_i, distinct except for h, which together with the arcs a_iu_i have the properties described in this theorem.

For E_n, $n>2$, a connected domain R can be obtained by means of a finite set of spheres of radius r and covering K. The proof is then similar to the above.

It will simplify the proof of the main theorem of this section to prove first the following lemma.

LEMMA 3. *Suppose for the integer k it is true that for every point set G, either there exist $k-1$ points such that any arc of G, joining any two distinct closed point sets, contains at least one of these points or there exist k distinct arcs of G joining these two distinct closed point sets. Then in E_n let a_ib_i $(i=1, 2, \cdots, k)$ be k distinct arcs joining two distinct closed point sets A and B and A_iB_i be k distinct arcs such that* (1) *A contains $(A_i)\times(A+B)$ and B contains $(B_i)\times(A+B)$,* (2) *$A_iB_i\times a_1b_1=A_i+B_i-(A_i+B_i)\times(A+B)$,* (3) *$A_iB_i\times(A+B)$ contains at most A_i+B_i, and* (4) *a_1A_i of a_1b_1 does not contain a point of (B_i). Then there exist k distinct arcs w_i of $M=(a_ib_i)+(A_iB_i)$ joining A and B and there exists an arc $w_{k+1}=W_1W_2$ of a_1b_1, where W_1 is contained in $(A_i)\times a_1b_1+a_1$ and W_2 in $(B_i)\times a_1b_1+b_1$, such that $(w_i)\times w_{k+1}$ contains at most W_1+W_2, a_1W_1 of a_1b_1 does not contain W_2, and there exists an arc of $(w_i)+w_{k+1}$ which contains b_1W_2 of a_1b_1.*

Consider for example the case where A_3 is the first point of $(A_i)+a_1$ on b_1a_1† and B_4 is the first point of $(B_i)+b_1$ on a_1b_1. Let p_iq_i be an arc of A_iB_i where p_i precedes q_i on A_iB_i, $p_i\neq A_i$, and $q_i\neq B_i$. Let p_iB_i and q_iA_i be arcs of A_iB_i and let a_1A_3 and b_1B_4 be arcs of a_1b_1. Let T_a be composed of $(p_iB_i)+(a_jb_j)+b_1B_4$ $(j=2, 3, \cdots, k)$ and also of q_iA_i if and only if A contains A_i; let T_b be composed of $(q_iA_i)+(a_jb_j)+a_1A_3$ and also of p_iB_i if B contains B_i. Let d be a positive number such that no point of T_a is at a distance less than d from a_1A_3 and no point of T_b is at a distance less than d from b_1B_4.

* R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11, No. 8, pp. 469–476, Theorem 1.

† The first point of a set C on an arc ab is the first point going from a to b; the first point of C on ba is the first point going from b to a.

The sets T_a and a_1A_3 are distinct, closed, and bounded point sets where a_1A_3 is connected and does not separate E_n. Take r less than $d/3$. The arcs q_iA_i, which have a point common with a_1b_1, have the same properties with reference to a_1A_3 as the arcs a_iw_i in Lemma 2 have to K. Say that there are m of these arcs q_iA_i which have a point common with a_1b_1. Then by Lemma 2 there exist m arcs and also there exist $k-m$ vacuous point sets, giving a total of k sets h_1u_{1i}, where h_1u_{1i} is vacuous if A contains A_i and consists of an arc if A does not contain A_i, and similarly there exist k arcs or vacuous sets B_iu_{1i}, where B_iu_{1i} is contained in A_iB_i. It is seen that the sets $B_iu_{1i}+u_{1i}h_1$ are distinct except for h_1, the set $(A_iu_{1i}+u_{1i}h_1)\times T_a=0$, where $A_iu_{1i}=0$ or is an arc according as A does or does not contain A_i, and every point of h_1u_{1i} is at a distance less than r from a_1A_3. From the last mentioned property it follows that the new sets h_1u_{1i} are at a distance greater than $d/3$ from b_1B_4. Then Lemma 2 can be again applied where $T_b+(h_1u_{1i})=C$ and $b_1B_4=K$ are the distinct, closed, and bounded point sets and the arcs B_ip_i, which have a point common with a_1b_1, are the m distinct arcs of that hypothesis. Thus a new set of k arcs or vacuous point sets h_2u_{2i}, distinct except for h_2, and a set of k distinct arcs or vacuous point sets p_iu_{2i}, where p_iu_{2i} is contained in p_iB_i and so in $u_{1i}B_i$, are obtained. There exists then a set of k distinct arcs $u_{1i}u_{2i}$ in (A_iB_i) where $u_{1i}=A_i$ if A contains A_i and $u_{2i}=B_i$ if B contains B_i. Thus a set of k arcs $h_1u_{1i}+u_{1i}u_{2i}+u_{2i}h_2=t_i$, distinct except for h_1 and h_2, are obtained where $(h_1u_{1i}+u_{2i}h_2)$ is at a distance less than r from $a_1A_3+B_4b_1$ and so has no points in common with (a_jb_j). Let $G=(a_jb_j)+t_i$.

Suppose that there exists a set H consisting of $k-1$ points such that every arc of G joining $A+h_1$ and $B+h_2$ contains at least one point of H. Therefore there exists one of these points on each of the $k-1$ arcs a_jb_j. Since then the points of H are on a_jb_j and since $(h_1u_{1i}+u_{2i}h_2)\times(a_jb_j)=0$, the set $H\times(h_1u_{1i}+u_{2i}h_2)=0$. Thus if there exists a point of H on each of the arcs t_i joining $A+h_1$ and $B+h_2$ there must exist a point of H on each of the k distinct arcs $u_{1i}u_{2i}$ of t_i. But as H contains only $k-1$ points this is impossible. Therefore it is necessary under our hypothesis that G contain k distinct arcs, v_i say, joining $A+h_1$ and $B+h_2$.

There exist three cases to consider having obtained the k arcs v_i: (I) the set (v_i) contains h_1+h_2; (II) it contains one and only one of these points; and (III) it contains neither of them.

(I) Under this case there exist two possibilities: one arc of the set (v_i) may contain both h_1 and h_2 or one arc may contain one of these points and another arc contain the other.

(a) Consider for example the case where v_1 contains h_1+h_2. Then v_1 contains a point of (u_{1i}), u_{12} say, and a point of (u_{2i}), u_{23} say. It will now be shown that $w_1=a_1A_2+A_2u_{12}+u_{12}u_{23}+u_{23}B_3+B_3b_1$ is such that $w_1\times(v_j)=0$, where $a_1A_2+B_3b_1$ is contained in a_1b_1, A_2u_{12} in A_2B_2, $u_{23}B_3$ in A_3B_3, and $u_{12}u_{23}$ in v_1. Since $v_1\times(v_j)=0$, $u_{12}u_{23}\times(v_j)=0$. Since v_1 contains h_1+h_2, it is the only one of the arcs of (v_i) which contains a point of $h_1u_{1i}+h_2u_{2i}$. Hence the set $(a_jb_j)+(u_{2i}u_{1i})-(u_{1i})-(u_{2i})$ contains (v_j). But the set $a_1b_1\times((a_jb_j)+(u_{1i}u_{2i}))=0$ for $a_1b_1\times(a_jb_j)=0$ and as a_1b_1 contains at most A_i+B_i of A_iB_i, $a_1b_1\times(u_{1i}u_{2i})=0$. Therefore $(a_1A_2+B_3b_1)\times(v_j)=0$. It remains to prove that $(A_2u_{12}+B_3u_{23})\times(v_j)=0$. This follows from the fact that $(a_jb_j)+(u_{1i}u_{2i})-(u_{1i})-(u_{2i})$ contains (v_j). Therefore $w_1\times(v_j)=0$. Let now $w_j=v_j$ and let $w_{k+1}=A_2B_3$ of a_1b_1. As shown above, $a_1b_1\times(w_j)=0$ and so $w_{k+1}\times(w_j)=0$. And $w_{k+1}\times w_1=A_2+B_3$ while w_1 contains b_1B_3.

(b) Consider now for example the case where $v_1=h_1b$ and $v_2=h_2a$, where A contains a and B contains b. Then v_1 contains a point, u_{11} say, of (u_{1i}), and v_2 one, u_{21} say, of (u_{2i}). Here $(a_jb_j)+(u_{1i}u_{2i})-(u_{1i})-(u_{2i})$ contains $(v_f)+bu_{11}+au_{21}$ $(f=3, 4, \cdots, k)$, where v_1 contains bu_{11} and v_2 contains au_{21}. Thus $u_{11}A_1+A_1a_1+u_{21}B_1+B_1b_1$ does not contain a point of $(v_f)+bu_{11}+au_{21}$ except u_{11} and u_{21}, where A_1B_1 contains $u_{11}A_1+u_{21}B_1$, and a_1b_1 contains $a_1A_1+b_1B_1$. Let $w_f=v_f$, let $w_1=a_1A_1+A_1u_{11}+u_{11}b$, and let $w_2=b_1B_1+B_1u_{21}+u_{21}a$. Let $w_{k+1}=A_1B_1$ of a_1b_1. Now $w_1\times(w_f)=w_2\times(w_f)=0$ and $w_1\times w_2=0$. Also $w_1\times w_{k+1}=A_1$ and $w_2\times w_{k+1}=B_1$, $w_{k+1}\times(w_f)=0$ and w_2 contains b_1B_1 of a_1b_1.

(II) In considering the case where $(v_i)\times(h_1+h_2)$ contains one and only one point, there exist two cases according as h_1 or h_2 is the point contained.

(a) Consider for example the case where $v_1=h_1b$, where B contains b, and say v_1 contains u_{11}. Let $w_j=v_j$ and let $w_1=bu_{11}+u_{11}A_1+A_1a_1$, where bu_{11} is contained in v_1, $u_{11}A_1$ in A_1B_1, and A_1a_1 in a_1b_1. Let $w_{k+1}=A_1b_1$. Then the k arcs w_i are distinct, $w_{k+1}\times(w_j)=0$, and $w_{k+1}\times w_1=A_1$.

(b) Consider now for example the case where $v_1=ah_2$, which contains u_{23}, say, where A contains a. Let $w_j=v_j$ and let $w_1=au_{23}+u_{23}B_3+B_3b_1$, where au_{23} is contained in v_1, $u_{23}B_3$ in A_3B_3, and B_3b_1 in a_1b_1. Let $w_{k+1}=a_1B_3$ of a_1b_1. Here $(w_1+w_{k+1})\times(w_j)=0$ and $w_1\times w_{k+1}=B_3$ while the arc w_1 contains b_1B_3.

(III) For the case where $(v_i)\times(h_1+h_2)=0$ let $w_i=v_i$ and $w_{k+1}=a_1b_1$.

In this manner the lemma is proved for every possible case.

LEMMA 4. *Suppose for the integer k it is true that for every point set G, either there exist $k-1$ points such that any arc of G, joining any two distinct closed point sets, contains at least one of these points, or there exist k distinct arcs of G joining these two distinct closed point sets. Then in E_n let a_ib_i $(i=1, 2, \cdots, k)$ be k distinct arcs joining two distinct closed point sets A and B, and ay an arc such that $A\times ay=a$ and $a_1b_1\times ay=y$ and $(a_jb_j)\times ay=0$ $(j=2, 3, \cdots, k)$; also* *let* A_iB_i *be* k *distinct arcs such that* (1) A *contains* $(A_i)\times(A+B)$ *and* B *contains* $(B_i)\times(A+B)$, (2) $A_iB_i\times(a_1b_1+ay)=A_i+B_i-(A_i+B_i)\times(A+B)$, (3) $A_iB_i\times(A+B)$ *contains at most* A_i+B_i, (4) a_1y *of* a_1b_1 *and* ay *do not contain a point of* (B_i), *and* (5) b_1y *of* a_1b_1 *does not contain a point of* (A_i). *Then there exist* k *distinct arcs* w_i *of* $M=(a_ib_i)+(A_iB_i)+ay$ *joining* A *and* B *and an arc* $w_{k+1}=a_0x$, *where* a_0 *is contained in* A *and* x *in* $(B_i)\times a_1b_1+b_1$, *and such that if there does not exist an arc* w_i *containing* b_1 *then* $w_{k+1}=a_1b_1$ *or* $w_{k+1}=ay+yb_1$ *and* $w_{k+1}\times(w_j)=0$, *but if* w_1, *say, contains* b_1 *then* $w_1\times w_{k+1}=x$ *and* $w_{k+1}\times(w_j)=0$, *and in every case* $(w_i)+w_{k+1}$ *contains* b_1y *and there exists a* g *such that* w_g *contains* b_1x *of* a_1b_1.

The proof is similar to that of Lemma 3.

THEOREM 1. *If* M *is any point set in* E_n, *and* A *and* B *are any two distinct closed point sets, then, for any positive integer* q, *either there exists a point set* N *containing* $q-1$ *points such that every arc of* M *joining* A *and* B *contains at least one point of* N *or there exist at least* q *distinct arcs of* M *joining* A *and* B.

If $q=1$, it is evident that the theorem is true. Assume that it is true for $q=k$. It will now be proved to be true for $q=k+1$.

There are two cases to consider according as either (I) there are $k-1$ points such that every arc of M joining A and B contains at least one of these points, or (II) there are at least k distinct arcs of M joining A and B.

(I) Consider the first case where the $k-1$ points exist. Then if any point is added to these, any arc of M joining A and B contains at least one of these k points. Hence in this case the theorem must be true for $q=k+1$ if it is for $q=k$.

(II) Consider now the case where there are at least k distinct arcs of M joining A and B. Let $a_ib_i(i=1,2,\cdots,k)$ be k such arcs. Consider any point p of a_1b_1. Either (1) there is a set N of $k-1$ points such that every arc of $M-p$ joining A and B contains a point of N or (2) there are k arcs of $M-p$ which are distinct and join A and B.

(1) For the case where the set N exists, $N+p$ is a set of k points such that every arc of M joining A and B contains at least one of these points. Thus in this case also if the theorem is true for $q=k$ it is for $q=k+1$.

(2) Consider now the remaining case where, for every point p of a_1b_1, $M-p$ contains at least k distinct arcs joining A and B. For a certain point p let $e_i(i=1,2,\cdots,k)$ be k such distinct arcs. Take for example the case where $a_1\neq p\neq b_1$. For any i there exists a region R containing p and so containing an arc t of a_1b_1 having p as an interior point, such that $R'\times e_i=0$. Let*

* If M is a point set, M' will denote M together with the limit points of M.

$(a_1b_1-t)'=t_1+t_2$. Then, if t_1 contains a_1, there exists an arc of e_i, s_i say, joining t_1+A and t_2+B. If a_1b_1 contains both end points of s_i, let r_i be the arc of a_1b_1 joining these two points; if one of these end points is in A and one in a_1b_1, let r_i be the arc of a_1b_1 joining a_1 and the end point of s_i in a_1b_1; if B contains one of these end points and a_1b_1 contains the other, let r_i be the arc of a_1b_1 joining b_1 and this end point in a_1b_1; and if A contains one end point and B contains the other, let $r_i=a_1b_1$. Thus to each arc e_i there corresponds an arc r_i of a_1b_1 having p as an interior point. Let h be the arc $r_1\times r_2\times \cdots \times r_k$ of a_1b_1. Thus for each point p of a_1b_1 there exists an arc h, having p as an interior point unless $p=a_1$ or $p=b_1$ in which latter case p is an end point, and a corresponding set, f say, consisting of k distinct arcs e_i joining A and B and having only end points of h common with h. Let (h) be the set of arcs such as h and (f) the set of sets such as f. Then by Lemma 1 there exists a simple chain, $h_1, h_2, \cdots, h_m$ of arcs of (h) joining a_1 and b_1, where h_1 contains a_1 and h_m contains b_1. Also there exists a set f_j of (f) corresponding to each h_j.

The set f_1 contains k distinct arcs $A_{1i}B_{1i}$ joining A and a closed subset of $a_1b_1-a_1+B$ where each a_1B_{1i} of a_1b_1 contains h_1, if a_1b_1 contains B_{1i}. We have here, just as we had in the hypothesis of Lemma 3, k distinct arcs a_ib_i and a set of k distinct arcs $A_{1i}B_{1i}$. Applying this lemma we obtain a set of k distinct arcs $a_{1i}b_{1i}$ joining A and B, which are the arcs w_i of that lemma, and an arc a_1y_1 of a_1b_1 joining a_1 and $(a_{1i}b_{1i})\times(B_{1i})+B$, which is the arc w_{k+1}. The new set $(a_{1i}b_{1i})+a_1y_1$ is contained in $(a_ib_i)+(A_{1i}B_{1i})$. If B contains y_1 then there exist $k+1$ distinct arcs of M joining A and B provided $a_1y_1 \times(a_{1i}b_{1i})\neq b_1$; and if B does not contain y_1, as was shown in Lemma 3, there exists an arc, $a_{11}b_{11}$ say, where $b_{11}=b_1$, containing y_1b_1 of a_1b_1. Thus either there exist $k+1$ distinct arcs of M joining A and B or there exist $k+1$ arcs of M joining A and either $B+b_1$ or $B+b_1y_1$, which are distinct, with the exception that two of them contain a common end point in b_1y_1, and so in some h_j, $j>1$. If this end point is not in h_m, say for example that it is an interior point in h_2. We now have k distinct arcs $a_{1i}b_{1i}$, similar to the arcs a_ib_i of Lemma 4, and an arc a_1y_1 similar to ay. And there exist in f_2 k distinct arcs $A_{2i}B_{2i}$, similar to A_iB_i of this lemma, joining $A+a_1y_1+x_a$ and $B+x_b$, where $(a_{11}b_{11}-h_2)'=x_a+x_b$, and x_b contains y_1b_1 of a_1b_1. Thus by this lemma it follows that there exist k distinct arcs $a_{2i}b_{2i}$ joining A and B and an arc $a_{02}y_2$ joining A and $(B_{2i})\times(a_{2i}b_{2i})+B$. These $k+1$ arcs are distinct or else a new set is obtained, proceeding in the same manner as above, by the use of Lemma 4.

It is necessary then that either $k+1$ distinct arcs joining A and B be obtained or there exist an arc $a_{0g}y_g$ such that h_m of a_1b_1 contains y_g. Hence, by one further application of Lemma 4, $k+1$ distinct arcs joining A and B

must be obtained. Thus if the theorem is true for $q=k$ it is true for $q=k+1$ in this case.

In every case the theorem is true for $q=k+1$ if it is true for $q=k$. The theorem is true for $q=1$. Hence it is true for any value of q.

COROLLARY 1. *If M is a continuous curve in E_n and A and B are any two distinct closed point sets, then either there exist, for any q, at least q distinct arcs of M joining A and B, or there exists a point set N of $q-1$ points such that $M-N$ does not contain a connected subset which contains points of both A and B.*

II. ARCS, DISTINCT EXCEPT FOR END POINTS

Here the following problem will be considered: when can it be said that in a set M either there exists a set of at least q arcs, where q is any positive integer, distinct except for possibly their end points, joining two distinct closed point sets A and B, or there exists a set N of $q-1$ points, contained in $M-A-B$, such that every arc of M joining A and B contains at least one point of N? Here also it will be understood that if M contains no arcs joining A and B, then the set N is vacuous. A complete solution of the above problem is not obtained in this paper.

In proving the next theorem the following lemma is useful and is stated here without proof.*

LEMMA 5. *If, in E_2, M is a continuous curve which contains a subcontinuum which is not a continuous curve and if k is any positive integer, then M contains k distinct arcs a_ib_i $(i=1, 2, \cdots, k)$, and a sequence of distinct arcs x_jy_j $(j=1, 2, \cdots)$ having a sequential limiting set Z, such that $x_jy_j\times a_ib_i\neq 0$, $x_jy_j\times Z=0$, $x_jy_j\times a_1b_1=x_j$, and $x_jy_j\times a_kb_k=y_j$. Furthermore x_j precedes x_{j+1} on a_1b_1, y_j precedes y_{j+1} on a_kb_k, every point of a_ib_i precedes every point of $a_{i+1}b_{i+1}$ on x_jy_j, $a_ib_i\times Z=b_i$, and $a_ib_i\times x_1y_1=a_i$.*

THEOREM 2. *Let q be a given positive integer greater than one. Then in order that a continuous curve M, in E_2, be perfect, it is sufficient, if L is the point set consisting of the points of any set of arcs of M joining any two distinct closed point sets A and B, that either there exist a set N of $q-1$ points, of $L-A-B$, such that every arc of L joining A and B contains at least one point of N, or there exist at least q arcs of L, distinct except for possibly their end points, joining A and B.*

Assume that M contains a subcontinuum which is not a continuous curve. Then by applying Lemma 5 the arcs of the conclusion there are obtained,

* The proof follows from the work of R. L. Wilder, Fundamenta Mathematicae, vol. 7, pp. 362–363, and from Theorem XXI by H. Hahn, Wiener Sitzungsberichte, vol. 123 (Part IIa), p. 2475.

taking $k=q+1$. On the arc x_jy_j there obtained, taking $j=2t-1$ $(t=1, 2, \cdots)$, let u_t be the subarc joining a_1b_1 and a_3b_3; and taking $j=2t$ let v_t be the subarc of x_jy_j joining a_1b_1 and a_2b_2. Let w_t be the subarc of a_2b_2 joining v_t and u_{t+1}; and let z_t be the subarc of a_1b_1 joining u_t and v_t. Let $A=(a_i)$ and $B=Z$. Then every point of $L=(u_t+v_t+w_t+z_t)+a_3b_3+a_4b_4+\cdots+a_{q+1}b_{q+1}$ is contained in an arc of L joining A and B $(t=1, 2, \cdots)$. But as $(u_t+v_t+w_t+z_t)$ does not contain an arc of L joining A and B, L contains at most $q-1$ arcs, distinct except for possibly their end points, joining A and B. However it is evident that $L-A-B$ does not contain $q-1$ points such that every arc of L joining A and B contains at least one of these points. As this is a contradiction with our hypothesis, M must be perfect.

Whether this condition is also necessary is not determined in this paper, except for $q=2$.

The proof of the following lemma can be obtained by means of a theorem by H. M. Gehman.*

LEMMA 6. *If M is a bounded continuous curve in E_2,† then a necessary (and sufficient) condition that M be perfect is the following: let Z be any closed subset of M, W any subset of M such that $Z\times W=0$ and every point of W can be joined to Z by an arc contained in W except for an end point in Z, and x a limit point of W such that $(W+Z)\times x=0$. Then there exists an arc joining x and Z which is contained in W except for its end points.*

THEOREM 3. *If $q=2$ then the condition in Theorem 2 is also necessary, if M is bounded.*‡

Suppose that there exist sets L, A, and B such that there does not exist a point which is contained in $L-A-B$ and in every arc of L joining A and B. Then there exists an arc ab joining A and B. It then follows from the above theorem by H. M. Gehman and from Lemma 6 that for any point p of ab, except possibly a and b, there exists an arc uv of L having the following properties: (1) the set $A+ap-p$ contains u and $B+bp-p$ contains v of uv, where $ap+pb=ab$, and $A+B+ab$ contains only these points of uv, and (2), unless uv joins A and B, in which case the theorem is proved, there does not exist another arc in L, having only its end points in $A+B+ab$, which joins $A+(au-u)$ and $B+(bv-v)$, where au is either an arc of ab or a, and bv is either an arc of ab or b; furthermore if $au\neq a$ there does not exist an arc of L,

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* *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27 (1925), pp. 29–46, Theorem V.

† This lemma is true if "bounded" is omitted. The sufficiency is true in E_n but the necessity is not. See §V.

‡ As shown in §V, "bounded" may be omitted.

having at most one end point in ab, joining $uv-u$ and $A+au-u$ nor if $bv \neq b$ does there exist such an arc joining $uv-v$ and $B+bv-v$. Let w be the arc of ab joining au and bv. Let (uv) be the set of arcs of L such as uv and (w) the corresponding arcs of ab. Every point of $ab-a-b$ is an interior point of one and at most of two of the arcs of (w). And $a+b$ contains the limit points of the end point of the arcs of (w). Since there exist but a countable number of arcs in (w), it is readily shown that $(uv)+(w)$ contains two arcs joining A and B, which are distinct except for possibly their end points. Hence the truth of the theorem is seen.

III. Connected subsets of perfect continuous curves

It is known that a connected subset N of a perfect continuous curve M is not necessarily arc-wise connected. But for any two points a and b of N the set N' contains an arc ab. Here the nature of the point set composed of the points of N, which are also on arcs ab of N', is considered. It is shown that this set must be connected. Some preliminary lemmas will be proved.

Theorem 4. *In order that a bounded continuous curve M in E_2* be perfect it is necessary and sufficient, if p, a, and b be any three points of any subcontinuum N of M such that there does not exist a point distinct from p which is contained in every arc of N joining p and $a+b$, that there exist an arc apb of N.*

By means of Lemma 5, using a device similar to that used in the proof of Theorem 2, it is seen that the condition is sufficient. It is also necessary. For by Theorem 3 there exist two arcs of N joining p and $a+b$ which are either distinct, except for p, or form a simple closed curve, J say, containing p and a point of $a+b$. If they are distinct, they form an arc apb of N; and if they are not distinct, since N contains an arc t joining p and $a+b-J\times(a+b)$ such that $t\times(a+b)\neq a+b$, it is seen that $J+t$ contains an arc apb of N.

Lemma 7. *If p, a, and b are any three distinct points of a bounded,† perfect, continuous curve M in E_2, and if (p) is the set of points, each of which is on every arc of M joining p and $a+b$, then there exists an arc of M which joins a and b and contains a certain point p_1 of (p) and every arc of M joining a and b contains at most p_1 of (p).*

As (p) is closed it is seen that there exists a first point of (p) on an arc of M, and so on every arc of M, joining p and $a+b$. And this point, p_1 say, has the property that there exists no other point common to every arc of M

* In §V it will be shown that "bounded" may be omitted here and that the necessity is true for E_n.

† As the proof depends entirely upon Theorem 4, "bounded" can be omitted and the theorem stated for E_n.

joining it and $a+b$. Hence by Theorem 4 there exists in M an arc ap_1b. And it is seen that every arc of M joining a and b contains at most this point of (p).

LEMMA 8. *In order that a bounded* continuous curve M in E_2 be perfect it is necessary and sufficient that, if N is any connected subset of M, a and b are any two points of N, L is a subset of N' consisting of all points contained in arcs ab of N', $Q=L\times N$, p is a point of $N-Q$, and (p) is the set of all points common to every arc of N' joining p and $a+b$, then there exists a point p_1 of $(p)\times Q$ and a subset W of $N'-p_1$, which contains p and is such that $W\times(a+b)=0$, $N'-p_1=W+(N'-p_1-W)$ separate, and every point of W can be joined to p_1 by an arc of $W+p_1$.*

In showing that the condition is necessary it is seen at once, by means of Lemma 7, that the point p_1 and an arc ap_1b exist. Also there exists a set W composed of the points of N' which can be joined to p_1 by an arc but cannot be joined to $a+b$ by an arc of N' which does not contain p_1. As $N'-p_1$ is an open subset of N', since $N'-p_1$ does not contain an arc joining p and $a+b$, it does not contain a connected subset which joins p and $a+b$.† Hence p and $a+b$ are separated in the weak sense and so in the strong sense‡ in N'. Thus the condition is necessary. That it is sufficient follows from Lemma 5.

THEOREM 5. *In order that a bounded continuous curve M in E_2§ be perfect it is necessary and sufficient that, if N is any connected subset of M of which a and b are any two points, L is the point set consisting of the arcs ab of N', and $Q=L\times N$, then Q is a non-vacuous connected point set.*

To show that the condition is necessary it is seen that, since N' is a continuous curve, L and so Q is a non-vacuous set. Assume that $Q=Y+Z$ separate. By Lemma 8, for each point p of $N-Q$, there exists a point p_1 and a subset W of $N'-p_1$ such that $N'-p_1=W+(N'-p_1-W)$ separate, where $W+p_1$ contains an arc pp_1. Thus sets (W) and (p_1) are obtained. And $(W)=(W)_1+(W)_2$, where a set W of $(W)_1$ is such that the corresponding p_1 of (p_1) is in Y and otherwise W is in $(W)_2$. Let $E=Y+(W)_1$ and $F=Z+(W)_2$. Then $N=Q+(W)\times N=F\times N+F\times N$. It is possible to show then that $N=E\times N+F\times N$ separate, which is a contradiction. And that the condition is sufficient is seen by assuming that it is not, and so obtaining by Lemma 5

* "Bounded" can be omitted and the necessity proved for E_n.

† R. L. Wilder, *Characterizations of continuous curves that are perfectly continuous*, Theorem 3, loc. cit.

‡ R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11, pp. 127–131, Lemma 2.

§ "Bounded" may be omitted, as seen in §V, the necessity holds in E_n, as Lemma 8 does, and the sufficiency holds in E_n, as may be seen from the Moore-Wilder lemma.

$N = a_1b_1 + x_1y_1 + x_2y_2 + \cdots + b_2$, letting $a = a_1$, and $b = b_2$, and thus obtaining a contradiction

IV. Additional theorems

The main problem considered here is the following: if N is a connected subset of a perfect continuous curve M, when does there exist an arc of N joining two points of N? Also a property is obtained of a set of arcs of M which join two distinct closed point sets.

Theorem 6. *In order that a bounded* continuous curve M in E_2 be perfect it is necessary and sufficient, if L is the point set consisting of the points of any set† of arcs of M joining any two distinct closed point sets A and B, that every point p of L' be contained in an arc of $L+p$ joining A and B.*

That the condition is necessary follows from a theorem by H. M. Gehman‡ and that it is sufficient is seen by means of the Moore-Wilder lemma.§

Theorem 7. *In order that a bounded|| continuous curve M in E_2 be perfect it is necessary and sufficient, if a and b are any two points of any connected subset N of M such that N' contains but a countable number of possible arcs ab, that N contain an arc ab.*

The condition is necessary. For assume that N does not contain an arc ab. Then every arc ab of N' contains a point of $N'-N$. Since these arcs are countable in number, let (z) be a set of points obtained by taking a point of $N'-N$ from each arc ab of N'. For each point z of (z), $N'-z$ is an open subset of N', and there exist a countable infinity of such sets. As N is common to all these open sets, it is contained in the quasi-open subset, Q, of N' determined by this countable infinity of open subsets of N'. But as Q is arc-wise connected,¶ it contains an arc ab. But as this arc does not contain a point of (z) a contradiction has been obtained.

The sufficiency follows from Lemma 5.

V. Note

Many of the theorems of this paper have been stated as holding for a bounded continuous curve. This is due to the fact that a theorem** by H. M.

* As shown in §V "bounded" may be omitted here; the sufficiency holds in E_n but the necessity does not.

† It is to be noted that this set is not necessarily the set of all arcs of M joining A and B.

‡ Loc. cit., Theorem 5.

§ R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1927), p. 371, Lemma 1.

|| Bounded may be omitted here. The necessity holds for any locally compact metric space.

¶ R. L. Wilder, *Characterizations of continuous curves that are perfectly continuous*, loc. cit., Theorem 3.

** Loc. cit., Theorem 5.

Gehman is used in the proofs, which is known to be untrue for unbounded continuous curves. However the following lemma could have been used everywhere instead, and so the theorems of this paper hold whether the continuous curve is bounded or not.

LEMMA. *In order that a continuous curve M in E_2 be perfect it is necessary and sufficient that M does not contain an infinite sequence of distinct subcontinua with two sequential limit points.**

The sufficiency is evident from the Moore-Wilder lemma which is known to hold for the unbounded case in E_n. And the necessity is proved as follows: let a and b be the sequential limit points referred to in the theorem. By a well known theorem† there exists in M a bounded continuous curve K containing a and all points of M in a certain neighborhood of a, but not containing b. It is clear that there will exist then points of infinitely many of the given subcontinua that are not in K. Consequently, since K is perfect,the theorem of Gehman is violated by portions of the given continua that lie in K.

It is known from an example by G. T. Whyburn‡ that neither the theorem by H. M. Gehman nor the above lemma is true for E_3. This example further shows that neither the necessity in Theorem 6 nor Lemma 6 is true for E_3.

The necessity of the following theorem is seen by means of a theorem by W. L. Ayres§ and the sufficiency is true because of the proof of Theorem 2. The necessity holds for E_n and the sufficiency for $q>2$.

THEOREM 3′. *Let $q=2$. Then in order that a continuous curve M in E_2 be perfect it is necessary and sufficient, if L is the point set consisting of the points of any set of arcs of M joining any two distinct closed point sets A and B, that either there exist a set N of $q-1$ points of $L'-A-B$ such that every arc of L' joining A and B contains at least one point of N or there exist at least q arcs of L', distinct except for possibly their end points, joining A and B.*

It is seen that the proof of Theorem 4 will go through for E_n, if the necessity of the above theorem, which is true for E_n, is used in place of Theorem 3 there.

* A point q is said to be a *sequential limit point* of a set of continua $C_i (i=1, 2, \cdots)$ if every region containing q contains points of all except a finite number of the sets C_i.

† H. Hahn, *Mengentheoretische Characterisierung der stetigen Kurve,* Wiener Sitzungsberichte, vol. 123 (Part IIa), pp. 2433–2489; see Theorem XXI, p. 2475.

‡ Bulletin of the American Mathematical Society, vol. 34 (1928), p. 551. This paper is to appear in Mathematische Annalen. The example has been given in a paper by R. L. Wilder, Proceedings of the National Academy of Sciences, vol. 16 (1930), p. 234. Using the notation of Professor Wilder to show that Theorem 6 does not hold in E_3, let $L=N$, $a=A$, $b=B$, and $p=(0, 0, 2^{1/2}/2)$. Then there does not exist an arc apb of $L+p$. Similarly Lemma 6 is shown to be untrue in E_3.

§ Loc. cit., Theorem 6.

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