ON A THEOREM OF S. BERNSTEIN-WIDDER*

BY

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The present note is merely a comment to the preceding paper by D. V. Widder, and was suggested by the reading of its manuscript. It gives a simplified proof of the following important theorem discovered recently by S. Bernstein, and subsequently, but independently, by Widder, whose proof is based upon entirely different principles.

**Theorem.** A necessary and sufficient condition that the function \( f(x) \) should be completely monotone in the interval \( c < x < \infty \) is that

\[
f(x) = \int_0^\infty e^{-xt}d\alpha(t),
\]

where \( \alpha(t) \) is a non-decreasing function of such a nature that the integral converges for \( x > c \).

The sufficiency of the condition is obvious since

\[
f^{(n)}(x) = (-1)^n \int_0^\infty e^{-xt}t^n d\alpha(t), \quad x > c \quad (n = 0, 1, 2, \ldots).
\]

Conversely let \( f(x) \) be completely monotone in the interval \( c < x < \infty \). Let \( a \) be an arbitrary constant greater than \( c \) and set \( c_i = f^{(i)}(a) \). It follows from the monotonic character of \( f(x) \) that the quadratic form

\[
Q_n(x) = \sum_{i=0}^n \sum_{j=0}^n c_{i+j}x_i x_j \quad (n = 0, 1, 2, \ldots)
\]

is non-negative. This fact is sufficient to ensure the existence of at least one non-decreasing function \( \rho(t) \) such that†

\[
c_i = \int_{-\infty}^\infty t^i d\rho(t) \quad (i = 0, 1, 2, \ldots).
\]

We now distinguish two cases:

**Case I.** The function \( \rho(t) \) is a step-function with a finite number of jumps.

**Case II.** The function \( \rho(t) \) is any other non-decreasing function.

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* Presented to the Society, September 9, 1931; received by the editors June 13, 1931.
† See, for example, Marcel Riesz, *Sur le problème des moments*, Arkiv för Matematik, Astronomi och Fysik, vol. 17, no. 16 (1923).
Case I. If \( \rho(t) \) is a step-function with \( p \) positive jumps at the points \(-\lambda_1, -\lambda_2, \ldots, -\lambda_p\) we have

\[
c_m = \sum_{k=1}^{p} \sigma_k (\lambda_k)^m, \quad \sigma_k > 0.
\]

From the Taylor development of \( f(x) \) we obtain

\[
f(x) = \sum_{i=0}^{\infty} \frac{c_i (x - a)^i}{i!} = \sum_{i=0}^{\infty} \sum_{k=1}^{p} \sigma_k (\lambda_k)^i (x - a)^i/i!
\]

\[= \sum_{k=1}^{p} \sigma_k e^{\lambda_k x} e^{-\lambda_k^2 x}.\]

We can now show that all the \( \lambda_k \) are positive or zero. It is only a matter of notation to suppose that \( \lambda_1 < \lambda_2 < \cdots < \lambda_p \). Suppose that \( \lambda_1 \) were negative. We should have

\[
f(x) = \sum_{i=0}^{\infty} \frac{c_i (x - a)^i}{i!} = \sum_{i=0}^{\infty} \sum_{k=1}^{p} \sigma_k (-\lambda_k)^i (x - a)^i/i!
\]

\[= \sum_{k=1}^{p} \sigma_k e^{\lambda_k x} e^{-\lambda_k^2 x} = -\lambda_1 \sigma_1 e^{\lambda_1 x} + \sum_{k=2}^{p} \sigma_k e^{\lambda_k x} (-\lambda_k) e^{-x(\lambda_k - \lambda_1)}.\]

From the latter equation it is clear that \( f'(x) e^{\lambda_1 x} \) tends to a limit as \( x \) becomes infinite, in fact to the positive limit \(-\lambda_1 \sigma_1 e^{\lambda_1 x}\). But since \( f(x) \) is completely monotonic for \( x > c \) we have \( f'(x) \leq 0 \) and

\[
\lim_{x \to \infty} f'(x) e^{\lambda_1 x} \leq 0.
\]

The contradiction shows that \( \lambda_1 \) must be positive or zero.\(^*\) But equation (1) may be written in the form

\[
f(x) = \int_{0}^{\infty} e^{-x t} d\alpha(t),
\]

where \( \alpha(t) \) is a non-decreasing function. Hence the theorem is established in Case I.

Case II. If \( \rho(t) \) is not a step-function then the quadratic form is not only non-negative but is a positive definite form. For,

\[
Q_n(x) = \int_{-\infty}^{\infty} \left( \sum_{i=0}^{n} t^i x_i \right)^2 d\rho(t) \quad (n = 0, 1, 2, \ldots),
\]

\(^*\) That it may be zero is seen by the example \( f(x) = 1 + e^{-x} \), which is certainly completely monotonic for all \( x \).
and this is clearly positive unless \( x_0 = x_1 = \cdots = x_n = 0 \). It follows that

\[
\begin{vmatrix}
  c_0 & c_1 & \cdots & c_n \\
  c_1 & c_2 & \cdots & c_{n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_n & c_{n+1} & \cdots & c_{2n}
\end{vmatrix} > 0 \quad (n = 0, 1, 2, \cdots).
\]

We may also show in this case that

\[
\begin{vmatrix}
  c_1 & c_2 & \cdots & c_{n+1} \\
  c_2 & c_3 & \cdots & c_{n+2} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_n & c_{n+1} & \cdots & c_{2n+1}
\end{vmatrix} > 0 \quad (n = 1, 2, 3, \cdots).
\]

For, since \(-f'(x)\) is itself a completely monotonic function, the two cases applicable to \( f(x) \) are also applicable to \(-f'(x)\). In the second of these cases we have (3) (which is merely (2) with all subscripts increased by unity). In the first of these cases we are led to a contradiction. For we should have

\[
-f'(x) = \sigma_0' + \sum_{k=1}^{p} \sigma_k' e^{\lambda_k' x} e^{-\lambda_k' x},
\]

\[
0 < \lambda_1' < \lambda_2' < \cdots < \lambda_p'; \quad \sigma_0' \geq 0, \quad \sigma_k' > 0 \quad (k = 1, 2, \cdots, p).
\]

Integrating equation (4) we should obtain

\[
f(x) = -\sigma_0' x + \sum_{k=1}^{p} \sigma_k' e^{\lambda_k' x} e^{-\lambda_k' x} / \lambda_k' + C,
\]

where \( C \) is a constant of integration. But \( \sigma_0' \) must be zero, for otherwise

\[
\lim_{x \to \infty} f(x) = -\infty.
\]

This is impossible since \( f(x) \geq 0 \). But if \( f(x) \) has the form (5) it is clear that the functions \( f(x), f'(x), f''(x), \cdots, f^{(p+1)}(x) \) are linearly dependent. Hence the Wronskian determinant of these functions must vanish identically. But this determinant reduces to (2) for \( x = a, n = p + 1 \). We thus reach a contradiction. It follows that both (2) and (3) must hold in Case II. Hence we are in a position to apply a theorem of Hamburger* and obtain

f(x) = \int_0^\infty e^{-xt}d\alpha(t),

where \alpha(t) is a non-decreasing function. The theorem is thus established in all cases.

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