ON STIELTJES POLYNOMIALS*

BY
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Introduction

1. According to a theorem of Heine,† there exist at most

$$\frac{(n+1)(n+2)\cdots(n+p-2)}{1\cdot 2\cdots(p-2)}$$

polynomials $\Phi(z)$ of degree $p-2$ such that the differential equation‡

$$\text{(E)} \qquad \frac{d^2w}{dz^2}+\left(\sum_1^p \frac{\alpha_j}{z-a_j}\right)\frac{dw}{dz}+\frac{\Phi(z)}{\prod_1^p (z-a_j)}\,w=0,$$

where

$$-\gamma \leqq \arg \alpha_j \leqq \gamma < \frac{\pi}{2}, \qquad \text{all } j,$$

has as solution a polynomial of the nth degree. Let us call each of these determinations of $\Phi(z)$ a *characteristic polynomial* and the corresponding solution a *Stieltjes polynomial.*

If

$$S_n(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$$

is a Stieltjes polynomial, it follows from (E) that

$$S_n''(z_k)+\left(\sum_{j=1}^p \frac{\alpha_j}{z_k-a_j}\right)S_n'(z_k)=0 \qquad (k=1,2,\cdots,n).$$

Should $S_n'(z_k)=0$, the z_k would coincide with an a_j, because, otherwise, $S_n''(z_k)=0$ and therefore $S_n^{(i)}(z_k)=0$ all i, and $S_n(z)\equiv 0$. Should $S_n'(z_k)\neq 0$,

$$S_n(z)=(z-z_k)T_{n-1}(z), \qquad T_{n-1}(z_k)\neq 0,$$

and, hence,

$$\frac{S_n''(z_k)}{S_n'(z_k)}=\frac{2T_{n-1}'(z_k)}{T_{n-1}(z_k)}=\sum_{j=1,\,j\neq k}^{n}\frac{2}{z_k-z_j}.$$

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* Presented to the Society, September 12, 1930, and September 9, 1931; received by the editors March 27 and June 3, 1931.

† Heine, *Kugelfunktionen*, Berlin, 1878, pp. 472–479.

‡ A generalization of the Lamé equation ($\alpha_j=1/2$, all j) and of the hypergeometric equation ($p=3$).

Hence, the zeros of $S_n(z)$ are either points a_j or solutions of the system of equations whose left members are the linear partial fractions

$$\text{(S)}\qquad \sum_{j=1}^{p} \frac{\alpha_j}{z_k - a_j} + \sum_{j=1, j\neq k}^{n} \frac{2}{z_k - z_j} = 0 \qquad (k = 1, 2, \cdots, n).$$

When $n=1$, system (S) reduces to a single equation whose left member is a linear partial fraction of the kind investigated in our previous papers.*

Likewise, if t_k is a zero of the characteristic polynomial for which $S_n(z)$ is a solution of (E),

$$S_n''(t_k) + \left(\sum_{j=1}^{p} \frac{\alpha_j}{t_k - a_j} \right) S_n'(t_k) = 0.$$

Should $S_n'(t_k)=0$, the t_k would coincide with an a_j, for, otherwise, $S_n''(t_k)=0$ and therefore $S_n^{(i)}(t_k)=0$, all i, and $S_n(z) \equiv$ constant. Should $S'(t_k) \neq 0$, let us write

$$S_n'(z) \equiv n(z - z_1')(z - z_2') \cdots (z - z_{n-1}').$$

As

$$\frac{S_n''(z)}{S_n'(z)} = \sum_{1}^{n-1} \frac{1}{z - z_j'},$$

the zeros of the characteristic polynomials are either points a_j or zeros of the linear partial fraction

$$\text{(F)}\qquad \sum_{j=1}^{p} \frac{\alpha_j}{t_k - a_j} + \sum_{j=1}^{n-1} \frac{1}{t_k - z_j'} = 0.$$

Thus, if we knew the exact positions of the points a_i, we could, by solving equations (S) and (F), locate exactly the points z_j and t_k. *Given, however, only that the points a_i lie in a given convex region K, can we find a second convex region K' in which will lie the points z_j and t_k for all values of p and n?* This is the question which in the present paper we propose to discuss.

Case $\gamma=0$

2. For the case that $\gamma=0$, this question has already been partially answered, as follows.

Theorem 1a. *If all the points a_j lie on the segment σ of the real axis, the zeros of every Stieltjes polynomial will also lie on σ.*†

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* M. Marden, these Transactions, vol. 32 (1930), pp. 658–668.

† Proved first by Stieltjes (Acta Mathematica, vols. 6–7 (1885–86), pp. 321–326) as a problem in the equilibrium of particles; later by Klein (*Ueber lineare Differentialgleichungen der zweiten Ordnung*, Göttingen, 1894, pp. 211–218) by a method of conformal mapping; still later by Bôcher (Bulletin of the American Mathematical Society, vol. 4 (1897), pp. 256–258) by means of simple function-theoretic considerations.

THEOREM 1b. *Under the hypothesis of Theorem* 1a, *the zeros of every characteristic polynomial will also lie on* σ.*

THEOREM 2a. *Any convex polygon which contains all the points* a_i *will also contain the zeros of every Stieltjes polynomial.*†

To the last theorem we add

THEOREM 2b. *Any convex polygon which contains all the points* a_i *also contains the zeros of every characteristic polynomial.*

For suppose t_k, a zero of a characteristic polynomial, were to lie outside of this polygon K. Not being a point a_i, the point t_k would have to be a root of equation (F), i.e., a root of minus the conjugate imaginary of (F):

$$\sum_{j=1}^{p} \frac{\alpha_j}{\bar{a}_j - \bar{t}_k} + \sum_{j=1}^{n-1} \frac{1}{\bar{z}_j' - \bar{t}_k} = 0. \tag{2.1}$$

By the Gauss-Lucas theorem,‡ the points z_j' as zeros of $S_n'(z)$ lie in the smallest convex polygon enclosing the points z_j, the zeros of $S_n(z)$. Since the latter points, by Theorem 2a, lie in K, the points z_j' also lie in K. The quantities $(z_j' - t_k)$ and $(a_j - t_k)$ are thus vectors from t_k to points in K and, therefore, lie in the angle ϕ subtended at t_k by K. The vectors $(\bar{z}_j' - \bar{t}_k)^{-1}$ and $\alpha_j(\bar{a}_j - t_k)^{-1}$ are also drawn within the angle ϕ. As t_k is by hypothesis outside of K, $\phi < 180°$ and hence the left-hand side of expression (2.1) cannot sum to zero. This result contradicts the assumption that t_k is a zero of equation (F).

An immediate generalization of Theorems 2a and 2b is

THEOREM 3. *Any convex region K containing all the points* a_i *also contains the zeros of every Stieltjes and every characteristic polynomial.*

In particular, if K is a circle with its center at the origin, this theorem leads to the following corollary:

If $|a_i| \leqq A$, *all i, then also* $|z_j| \leqq A$ *and* $|t_k| \leqq A$, *all j and k.*

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* E. B. Van Vleck, Bulletin of the American Mathematical Society, vol. 4 (1898), p. 438.

† The theorem was first proved in the case $p=3$ by Bôcher (*Ueber die Reihenentwickelungen der Potentialtheorie*, Leipzig, 1894, pp. 215–218) as a problem in the equilibrium of particles, a method which carries over at once to the general case. Klein states the general theorem (*Differentialgleichungen*, p. 208) crediting it to Bôcher. The theorem was discovered independently by Pólya (Comptes Rendus, 1912, p. 767). See also J. L. Walsh, Tôhoku Mathematical Journal, vol. 23 (1924), pp. 312–317.

‡ Gauss, Werke, vol. 3, p. 112; Lucas, Comptes Rendus, 1868, and Journal de l'École Polytechnique, vol. 46 (1879), p. 8.

CASE $\gamma \neq 0$

3. We shall begin by proving

THEOREM 4. *If all the points a_i lie in a circle C of radius r, the zeros of every Stieltjes polynomial and of every characteristic polynomial lie in the concentric circle C' of radius $r' = r$ sec γ.*

In order to prove the first part of this theorem, let us suppose that z_1 is the z_j farthest away from the center of circle C. Then Γ, the circle through z_1 and concentric with C, will contain in or on its circumference all of the points z_j. Let us denote by T the tangent to Γ at z_1. If z_1 were to lie outside of C', the circle C would subtend at z_1 an angle $\phi < \pi - 2\gamma$. The vector $(\bar{a}_k - \bar{z}_1)^{-1}$ would lie in this angle ϕ. The vector $\bar{\alpha}_k(\bar{a}_k - \bar{z}_1)^{-1}$ would therefore lie in the angle got through enlarging ϕ by γ on both its sides, that is to say, the vector $\bar{\alpha}_k(\bar{a}_k - \bar{z}_1)^{-1}$ would lie on the same side of the line T as circle C. The last statement, being also true of the vector $2(\bar{z}_j - \bar{z}_1)^{-1}$, would show that

$$\sum_{k=1}^{p} \frac{\bar{\alpha}_j}{\bar{a}_j - \bar{z}_1} + \sum_{j=2}^{n} \frac{2}{\bar{z}_j - \bar{z}_1} \neq 0,$$

in contradiction to the hypothesis that z_1 is a zero of (S).

Similarly, if t_1, a zero of a characteristic polynomial, were to lie outside of C', we should draw a circle Γ through t_1 concentric with C and denote by T the tangent to Γ at t_1. By the above reasoning, the vector $\bar{\alpha}_j(\bar{a}_j - \bar{t}_1)^{-1}$ would lie on the same side of T as the circle C. The same would hold of the vectors $(\bar{z}_j' - \bar{t}_1)^{-1}$. For, by the Gauss-Lucas theorem, the points z_j' are situated in any convex region enclosing all the z_j, which, according to the first part of our theorem, lie in the circle C'. Thus, t_1 cannot lie outside of C', if it is to be a zero of (F).

By specializing the circle C to have its center at the origin, we deduce that, *if* $|a_i| \leqq A$, *all* i, *then* $|z_j| \leqq A$ sec γ *and* $|t_k| \leqq A$ sec γ, *all* j *and* k.

The circle C' of Theorem 4 gives us the smallest convex region which, for all n and p, will enclose the zeros of every Stieltjes and every characteristic polynomial. For, when $n = 1$ the theorem coincides with our previous results, which were best approximations.*

4. Let us now consider how we may extend Theorem 4 to an arbitrary convex region K.

By the *covering function* of such a region let us mean a function $k(\lambda)$ such that the inequality

$$|z - \lambda| \leqq k(\lambda) \tag{4.1}$$

is satisfied for all values of λ by and only by the points z of K.

* M. Marden, these Transactions, vol. 32 (1930), p. 658–660.

One covering function $k(\lambda)$ of a given convex region K may be always chosen as the maximum value of $|z-\lambda|$ for points z in K. For, clearly through this choice of the function $k(\lambda)$ inequality (4.1) will be satisfied by all points z in K for each value of λ, and by no point Z outside of K. For, otherwise, there would exist a line which would separate the point Z from the region K, and hence there would exist a circle with center at some point Λ and radius κ which would contain K but not the point Z. As $k(\Lambda) \leqq \kappa$,

$$|Z - \Lambda| > k(\Lambda);$$

i.e., for $\lambda = \Lambda$, Z would not satisfy inequality (4.1).

Chosen in this way, the covering function of the circle with center at α and radius ρ, for example, is

$$k(\lambda) = |\lambda - \alpha| + \rho.$$

A given convex region has in general, however, more than one covering function. For example, another covering function for the circle with center at α and with radius ρ is

$$k(\lambda) = \rho \quad \text{for } \lambda = \alpha, \qquad k(\lambda) = \infty \quad \text{for } \lambda \neq \alpha. \tag{4.2}$$

Clearly, no covering function of a convex region may ever be negative. Conversely, if $k(\lambda)$ is any real, non-negative function, the totality of points z which for all λ satisfy the inequality

$$|z - \lambda| \leqq k(\lambda), \tag{4.3}$$

if any such points z exist, form a convex region K of which $k(\lambda)$ would be a covering function. For, region K would consist of the common points of the circles (4.3), and the common points of any number of convex regions also form a convex region.

A further important property of covering functions is that, if $k_1(\lambda) \geqq k_2(\lambda)$, the region K_1 with $k_1(\lambda)$ as covering function contains the region K_2 with $k_2(\lambda)$ as covering function. This is because any z which satisfies the inequality

$$|z - \lambda| \leqq k_2(\lambda)$$

for all values of λ must also satisfy the inequality

$$|z - \lambda| \leqq k_1(\lambda)$$

for all values of λ.

To return to our problem, let us suppose the points a_j all to lie in the convex region K with $k(\lambda)$ as covering function. For each value of λ the points a_j will therefore lie in the circle

$$|z - \lambda| \leqq k(\lambda)$$

and hence the points z_j and t_k will lie in the circle

$$|z - \lambda| \leqq k(\lambda) \sec \gamma.$$

The last is equivalent to saying that the points z_j and t_k will be situated in a convex region K' which has $k'(\lambda) = k(\lambda) \sec \gamma$ as a covering function and which, since $k'(\lambda) \geqq k(\lambda)$, will enclose K.

Thus we are led to a generalization of Theorems 3 and 4.

THEOREM 5. *If all the points a_j lie in a convex region K with $k(\lambda)$ as covering function, the zeros of every Stieltjes polynomial and of every characteristic polynomial will lie in a convex region K' which will contain K and which will have as a covering function* $k'(\lambda) = k(\lambda) \sec \gamma$.

5. What is this region K' in the case that K is an ellipse, rectangle or straight-line segment?

If K is an ellipse with center, vertices, and foci at the points $(0, 0)$, $(\pm a, 0)$, and $(\pm c, 0)$ respectively, it may be regarded as the envelope of the circle with center at λ and radius $a(1+c^{-2}\lambda^2)^{1/2}$ as λ describes the y-axis. One covering function of K is therefore

$$(5.1) \qquad \begin{aligned} k(\lambda) &= a(1 + c^{-2}\lambda^2)^{1/2} && \text{for } \mathfrak{R}(\lambda) = 0, \\ k(\lambda) &= \infty && \text{for } \mathfrak{R}(\lambda) \neq 0. \end{aligned}$$

Consequently

$$\begin{aligned} k'(\lambda) &= (a \sec \gamma)\ (1 + c^{-2}\lambda^2)^{1/2} && \text{for } \mathfrak{R}(\lambda) = 0, \\ k'(\lambda) &= \infty && \text{for } \mathfrak{R}(\lambda) \neq 0, \end{aligned}$$

from which follows that K' is a confocal ellipse with major axis $2a \sec \gamma$. Results concerning the zeros of equations (S) and (F) being independent of the choice of coördinate axes, we may state the following generalization of Theorem 4.

THEOREM 6a. *If all the points a_j lie in or on a given ellipse with a major axis of $2a$, the zeros of every Stieltjes polynomial and of every characteristic polynomial lie in or on the confocal ellipse with the major axis* $2a \sec \gamma$.

A line segment being a limiting case of an ellipse, we may obtain from the preceding theorem a generalization of Theorems 1a and 1b.

THEOREM 6b. *If all the points a_j lie on the line segment AB, the zeros of every Stieltjes and every characteristic polynomial lie in or on the ellipse which has the points A and B as foci and which has a major axis of length* $\overline{AB} \sec \gamma$.

In particular, the choice of A and B as the points $z = \pm a$ of the real axis gives the ellipse of Theorem 6b axes of length $2a \sec \gamma$ and $2a \tan \gamma$. Consequently, *if the a_i are real and $|a_i| \leqq a$, all i, then for all j and k*

$$|\Re(z_j)| \leqq a \sec \gamma; \quad |\Im(z_j)| \leqq a \tan \gamma; \quad |\Re(t_k)| \leqq a \sec \gamma; \quad |\Im(t_k)| \leqq a \tan \gamma.$$

This corollary reduces to Theorems 1a and 1b on setting $\gamma = 0$.

Let us next study the case that K is the rectangle $ABCD$, where the side AB joins the points $(\pm a, b)$, and the side BC, the points $(a, \pm b)$. Such a rectangle could be considered as composed of the common points of the four regions K_1, K_2, K_3 and K_4 with the covering functions $k_1(\lambda)$, $k_2(\lambda)$, $k_3(\lambda)$ and $k_4(\lambda)$ respectively, where

$$\begin{aligned} k_1(\lambda) &= [(\lambda + a)^2 + b^2]^{1/2} && \text{for points } \lambda \text{ on the positive real axis,} \\ k_1(\lambda) &= \infty && \text{for all other } \lambda; \end{aligned}$$

$$\begin{aligned} k_2(\lambda) &= [(\lambda - a)^2 + b^2]^{1/2} && \text{for points } \lambda \text{ on the negative real axis,} \\ k_2(\lambda) &= \infty && \text{for all other } \lambda; \end{aligned}$$

$$\begin{aligned} k_3(\lambda) &= [(\lambda + b)^2 + a^2]^{1/2} && \text{for points } \lambda \text{ on the positive imaginary axis,} \\ k_3(\lambda) &= \infty && \text{for all other } \lambda; \end{aligned}$$

$$\begin{aligned} k_4(\lambda) &= [(\lambda - b)^2 + a^2]^{1/2} && \text{for points } \lambda \text{ on the negative imaginary axis,} \\ k_4(\lambda) &= \infty && \text{for all other } \lambda. \end{aligned}$$

The region K_1, for instance, consists of the interior of the circumscribing circle Γ (of radius $c = (a^2 + b^2)^{1/2}$) from which has been removed the sector $AEDFA$.

The region K' will hence be composed of the common points of the regions K_1', K_2', K_3' and K_4' whose covering functions are respectively $k_1'(\lambda)$, $k_2'(\lambda)$, $k_3'(\lambda)$ and $k_4'(\lambda)$, where $k_j'(\lambda) = k_j(\lambda) \sec \gamma$.

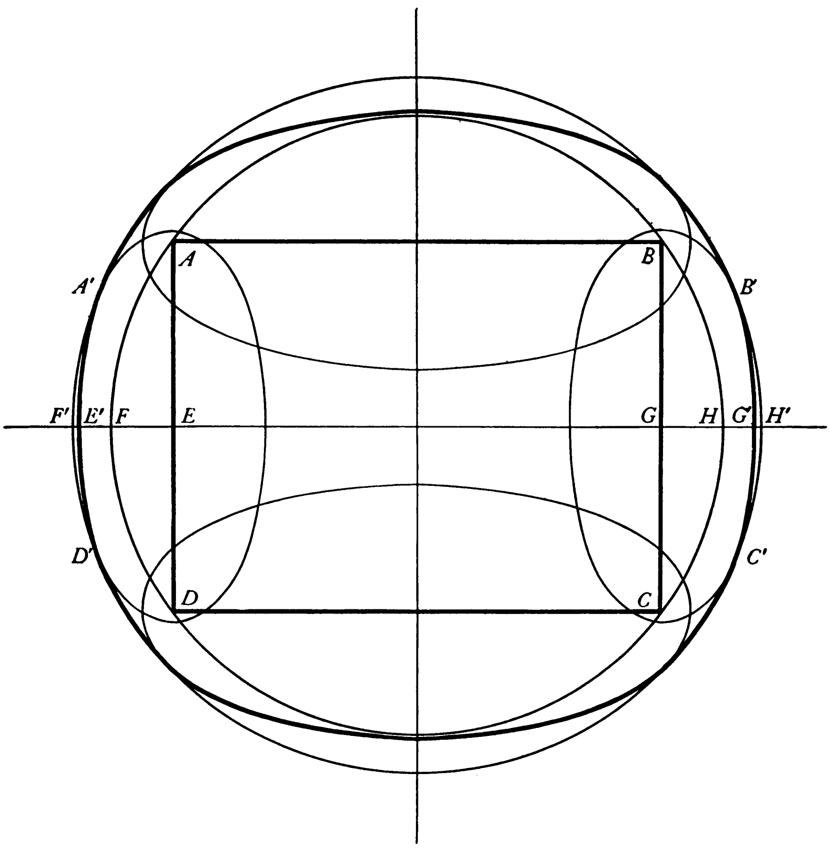
Compared with function (5.1), the functions $k_j'(\lambda)$ suggest how, for instance, the region K_1' may be constructed. The ellipse which has the points A and D as foci and the length $2b \sec \gamma$ as major axis is, at the points A' and D', tangent to the circle Γ' which is concentric with Γ and is of radius $c \sec \gamma$. The region K_1' will then consist of the points remaining in the circle Γ' after the sector $A'E'D'F'A'$ has been removed. Since the regions K_2', K_3' and K_4' are similar to K_1' in structure, the common points of the four form as K' the region bounded by the outer heavy line in the accompanying figure.

THEOREM 6c. *If all points a_j lie in a given rectangle $ABCD$, the zeros of every Stieltjes and every characteristic polynomial lie in the region K' enclosed by the outer heavy line in the accompanying figure.*

On calculating the axes of the four ellipses, we may deduce from the above theorem, that *if* $|\mathfrak{R}(a_i)| \leqq a$ and $|\mathfrak{I}(a_i)| \leqq b$ *for all i, then for all j and k*

$$|\mathfrak{R}(z_j)| \leqq A, \quad |\mathfrak{I}(z_j)| \leqq B;$$
$$|\mathfrak{R}(t_k)| \leqq A, \quad |\mathfrak{I}(t_k)| \leqq B,$$

where A is the larger of the quantities a $\sec\gamma$ *and* $a+b\tan\gamma$ *and B is the larger of the quantities b* $\sec\gamma$ *and* $b+a\tan\gamma$.



These results suggest the nature of the region K' in the case that K is an arbitrary convex polygon.

6. In addition to the hypotheses of §1, let us now assume the coefficient of dw/dz in equation (E) to represent the ratio of two real polynomials. Under these conditions a smaller zero-containing region is possible than that afforded by Theorem 5. This smaller region is described in the following theorem.

THEOREM 7. *If the a_j and the corresponding α_j are real or appear in conjugate imaginary pairs, the zeros of every Stieltjes and of every characteristic polynomial will lie in the smallest convex region which encloses all of the ellipses having a_j and $\bar{a}_j$ as foci and* $|a_j-\bar{a}_j|$ sec (arg α_j) *as major axis.**

For the purpose of establishing this theorem, we shall first prove that *the ray from any point w through the point*

$$\frac{e^{-i\gamma}}{1-\bar{w}}+\frac{e^{i\gamma}}{-1-\bar{w}}$$

always meets the ellipse $x^2\cos^2\gamma+y^2\cot^2\gamma=1$.

This lemma generalizes the intuitively evident fact that, if R is the resultant of the two forces at w due to unit particles at $z=\pm 1$ attracting according to the inverse distance law, then R produced must intersect the segment joining the points $z=\pm 1$.

Let α and β be the angles made with the positive real axis by the lines joining $z=-1$ and $z=1$ respectively to point $w=u+iv$. Then $M=e^{-i\gamma}(1-\bar{w})^{-1}$ and $N=e^{i\gamma}(-1-\bar{w})^{-1}$ may be represented by vectors drawn from point w to the points

$$m = w+\frac{1}{r_1}e^{(\pi+\alpha-\gamma)i}, \qquad n = w+\frac{1}{r^2}e^{(\pi+\beta+\gamma)i}$$

where r_1 and r_2 are the distances of w from the points $z=-1$ and $z=1$ respectively. As the mid-point of the segment joining m and n is

$$s = w-\frac{1}{2r_1r_2}(r_2e^{(\alpha-\gamma)i}+r_1e^{(\beta+\gamma)i}),$$

the slope of the line ws is tan arg $(s-w)$, or

$$\begin{aligned}\lambda &= \frac{r_2\sin(\alpha-\gamma)+r_1\sin(\beta+\gamma)}{r_2\cos(\alpha-\gamma)+r_1\cos(\beta+\gamma)}\\ &= \frac{[r_1^2+r_2^2]v\cos\gamma+[(r_1^2-r_2^2)u-(r_1^2+r_2^2)]\sin\gamma}{[(r_1^2+r_2^2)u+(r_2^2-r_1^2)]\cos\gamma+[r_2^2-r_1^2]v\sin\gamma}\\ &= \frac{v(u^2+v^2+1)\cos\gamma+(u^2-v^2-1)\sin\gamma}{u(u^2+v^2-1)\cos\gamma-2uv\sin\gamma}.\end{aligned}$$

From this it follows that

* Insofar as it concerns the zeros of Stieltjes polynomials, the theorem was first discovered by Charles Vuille in his doctoral thesis, Ecole Polytechnique Fédéral de Zurich, 1916. His proof is based, however, upon a long computation covering pp. 62–75 of his thesis.

$$\lambda^2 + 1 = \frac{r_1^2 r_2^2 [u^2 \cos^2 \gamma + (v \cos \gamma - \sin \gamma)^2]}{[u(u^2 + v^2 - 1) \cos \gamma - 2uv \sin \gamma]^2}$$

and

$$(v - \lambda u)^2 + 1 = \frac{r_1^2 r_2^2 u^2}{[u(u^2 + v^2 - 1) \cos \gamma - 2uv \sin \gamma]^2} .$$

Now, the abscissas of points of intersection of the line $y - v = \lambda(x - u)$ with the ellipse are the roots of the equation

$$x^2(\cos^2 \gamma + \lambda^2 \cot^2 \gamma) + 2\lambda x(v - \lambda u) + (v - \lambda u)^2 \cot^2 \gamma = 1,$$

whose discriminant is

$$\Delta = \{ [\lambda^2 + 1] - [(v - \lambda u)^2 + 1] \cos^2 \gamma \} \cot^2 \gamma$$

$$= \frac{r_1^2 r_2^2 (v \cos \gamma - \sin \gamma)^2 \cot^2 \gamma}{[u(u^2 + v^2 - 1) \cos \gamma - 2uv \sin \gamma]^2} .$$

This discriminant being non-negative, the ray ws always meets the given ellipse.

The lemma just proved may be stated in a more general form; namely, that *the ray from any point w to the point*

$$\frac{\bar{\alpha}_j}{\bar{a}_j - \bar{w}} + \frac{\alpha_j}{a_j - \bar{w}}$$

always meets the ellipse E_j having a_j and $\bar{a}_j$ as foci and $|a_j - \bar{a}_j| \sec (\arg \alpha_j)$ *as major axis*. In this form, as we shall now see, the lemma leads almost immediately to a proof of Theorem 7.

If z_1, a zero of some Stieltjes polynomial, were to lie outside of K, the smallest convex region enclosing all the ellipses E_j, a line L could be drawn through it which would not cut K. We could suppose this line L to have on it, or on the same side of it as K, all of the remaining z_j. For, were this not the case, we could move L parallel to itself away from K until it met the last z_j, which we would rename as z_1. The vectors $2(\bar{z}_j - \bar{z}_1)^{-1}$ and $[\bar{\alpha}_k(\bar{a}_k - \bar{z}_1)^{-1} + \alpha_k(a_k - \bar{z}_1)^{-1}]$ would then lie on the same side of L as K and hence could not sum up to zero. This result contradicts the hypothesis that z_1 is a zero of system (S).

Similarly, if t_1, a zero of some characteristic polynomial, were to lie outside of K, a line L could be drawn through it not cutting K. According to the Gauss-Lucas theorem the z_j' lie in the same convex region as the z_j and therefore lie in K. Consequently, the vectors $(\bar{z}_j' - \bar{t}_1)^{-1}$ and $[\bar{\alpha}_k(\bar{a}_k - \bar{t}_1)^{-1} + \alpha_k(a_k - \bar{t}_1)^{-1}]$ lie on the same side of L and cannot sum up to zero. This is contrary to the supposition that t_1 is a zero of partial fraction (F).

The above theorem is a generalization of Theorems 3 and 6b and to some extent an analogue to Jensen's theorem on the distribution of the non-real zeros of a real polynomial.

7. Theorem 5 may also be applied to the study of the mapping properties of the quotient of two linearly independent solutions w_1, w_2 of equation (E), when the second solution is a Stieltjes polynomial $S_n(z)$.

Let $\eta = w_1/w_2$. Then, since

$$\eta' = \frac{1}{w_2^2}(w_1' w_2 - w_1 w_2') = \frac{1}{w_2^2}\left[\exp\left(- \int \sum \frac{\alpha_j}{z - a_j} dz \right)\right],$$

$$\eta = \int \frac{(z - a_1)^{\alpha_1}(z - a_2)^{\alpha_2} \cdots (z - a_p)^{\alpha_p}}{S_n(z)^2} dz.$$

The function η becomes infinite at the zeros of $S_n(z)$ and possibly at some of the points a_j. Agreeing not to count the latter points, we may determine the number of zeros of $S_n(z)$ in a given region S as follows. The function η maps the region S upon a region Σ, which will, in general, be spread over several sheets of a Riemann surface. If the point at infinity on each sheet be considered as distinct from that on any other sheet, the number of zeros of $S_n(z)$ in S will be equal to the number of points at infinity in Σ.*

These considerations lead us to our final theorem.

THEOREM 8. *Let K be a convex region with $k(\lambda)$ as covering function, and S any finite region outside of the convex region K' which has $k'(\lambda) = k(\lambda)$* sec γ *as covering function. Let w_2 be an arbitrary Stieltjes polynomial, and w_1 any solution of equation* (E) *which is linearly independent of w_2. Then, if all the singular points a_j lie in K, the region S is mapped by the function $\eta = w_1/w_2$ upon a region which does not contain any point at infinity.*

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* Cf. Klein, *Differentialgleichungen*, p. 213.

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