ON THE EXISTENCE OF ACYCLIC CURVES SATISFYING CERTAIN CONDITIONS WITH RESPECT TO A GIVEN CONTINUOUS CURVE*

BY

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Part I of this paper has to do with connected sets of cut points† of a given continuous curve. It is shown that, in the plane, any two points belonging to a connected set K of cut points of a given continuous curve M lie together in an arc which is common to K and a point set consisting of the boundaries of a finite number of complementary domains of M. G. T. Whyburn‡ calls attention to the fact that from his results it follows that K is arcwise connected. To show that any two points of K can be joined by an arc which is common to K and a point set consisting of the boundaries of a finite number of complementary domains of M, is the object of Part I. Part II has to do with a totally disconnected closed subset K of a given plane continuous curve M no subset of which disconnects M. R. L. Moore§ has shown that, in the plane, any two points not belonging to a bounded continuous curve can be joined by an arc which does not disconnect the continuous curve. The object of Part II of this paper is to show that if, in the plane, M is a bounded continuous curve which contains no domain, and K is a closed and totally disconnected subset of M, such that no subset of K disconnects M, then there exists an acyclic continuous curve T, containing K, such that (1) all the end points|| of T belong to K and (2) the point set $M \cdot T$ is totally disconnected and $M-T$ is connected.

I wish to acknowledge my indebtedness to Professor R. L. Moore, not only for the suggestions of the problems treated, but also for his many helpful criticisms in their solutions. To him is due, in a large measure, the success

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† A point P is said to be a cut point of a continuum M if $M-P$ can be expressed as the sum of two mutually separated sets.

‡ *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), p. 176.

§ *Concerning paths that do not separate a given continuous curve*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 745–753.

|| The term *end point* will be used in the sense as defined by R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358, i.e., a point P of a continuous curve M is an end point of M provided it is true that if t is an arc of M having P as one of its extremities, then $M-(t-P)$ contains no connected subset which contains P.

of these investigations. His stimulating personality has been a source of constant encouragement to me in the study of mathematics.

Part I

Lemma I. *If K is a connected set of points belonging to a continuous curve M, and J is the outer boundary of a complementary domain D of M, and M_1 is a component of $M-J$ which lies within J, then $K\cdot\overline{M}_1$ is connected.*

By a theorem of R. L. Moore,† J contains exactly one limit point P of M_1 and thus $\overline{M}_1=M_1+P$. The closed point sets M_1+P and $M-M_1$ have only the point P in common and their sum is M. Hence, unless $K=K\cdot\overline{M}_1$, then $K-P$ is the sum of the two mutually separated point sets $K\cdot M_1$ and $K(M-M_1)$. Therefore $K\cdot M_1+P$ is connected. But $K\cdot M_1+P=K\cdot\overline{M}_1$. Hence $K\cdot\overline{M}_1$ is connected.

Lemma II. *If X is a point belonging to the boundary of a complementary domain D of a continuous curve M, then there do not exist infinitely many simple closed curves of M, each of which encloses X and is the outer boundary of a complementary domain of M.*

Lemma II follows from the fact that the complementary domains of M form a contracting sequence,‡ and the fact that no two complementary domains of M have the same outer boundary.§

Lemma III. *Suppose* (1) *K is a connected set of points belonging to a continuous curve M,* (2) *$D_1, D_2, D_3, \cdots$ are all the bounded complementary domains of M such that for each i, $\overline{D}_i$ contains a point of K,* (3) *for each i, J_i is the outer boundary of D_i, and* (4) *$J_1^*, J_2^*, J_3^*, \cdots$ is a finite or infinite subsequence of the sequence $J_1, J_2, J_3, \cdots$. Then the set K^* consisting of all points X of K such that X is not interior to J_i^* for any value of i is connected, and if all but a countable number of the points of K are cut points of M, and no point of K^* is interior to J_i for any value of i, and K^* contains more than one point, then K^* is a subset of the boundary of the unbounded complementary domain of M.*

Suppose there exists a subsequence $J_1^*, J_2^*, J_3^*, \cdots$ of the sequence $J_1, J_2, J_3, \cdots$ for which the set K^* consisting of all points X of K, such that

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† *Concerning paths that do not separate a given continuous curve*, loc. cit., Theorem 6.

‡ If H is a sequence of point sets and for each positive number e only a finite number of point sets of the set H are of diameter greater than e, then H is said to be *a contracting sequence* of point sets. See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 81–88. See also R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923).

§ See R. L. Moore, *Concerning paths that do not separate a given continuous curve*, loc. cit., Theorem 1.

X is not interior to J_i^* for any value of i, is not connected. The set K^* can be expressed as the sum of two mutually separated sets K_1 and K_2. It follows from Lemma II that there exists a subsequence $J_1^{**}, J_2^{**}, J_3^{**}, \cdots$ of the sequence $J_1^*, J_2^*, J_3^*, \cdots$ such that (1) if X is any point lying on or within some curve of the sequence $J_1^*, J_2^*, J_3^*, \cdots$, then X lies on or within some curve of the sequence $J_1^{**}, J_2^{**}, J_3^{**}, \cdots$, (2) for $m \neq n$, J_m^{**} contains no point interior to J_n^{**}. For each positive integer n, and for each point X of K_1 belonging to J_n^{**}, add to K_1 every point Y of K such that Y belongs to some component of $M-J_n^{**}$ which lies within J_n^{**} and has X as a limit point. Let K_1^* denote the resulting set. For each positive integer n and for each point X of K_2 belonging to J_n^{**} add to K_2 every point Y of K such that Y belongs to some component of $M-J_n^{**}$ which lies within J_n^{**} and has X as a limit point. Let K_2^* denote the resulting set. Since K is the sum of the sets K_1^* and K_2^*, one of these sets contains a limit point of the other. We shall consider the case where K_1^* contains a limit point of K_2^*. Let P denote one such limit point. The point P belongs to K_1, for suppose it does not. There exists a positive integer t such that J_t^{**} encloses P. Only a finite number of components of $M-J_t^{**}$ are of diameter greater than a positive number,† and all the points of K belonging to that component of $M-J_t^{**}$ which contains P belong also to K_1^*. It follows that P is not a limit point of K_2^* contrary to hypothesis. Hence P belongs to K_1. It follows then that P is the sequential limit point of a sequence $P_1, P_2, P_3, \cdots$ of points belonging to the set $K_2^*-K_2$ such that for $i \neq j$, P_i and P_j belong to different components of $M-(J_1^{**}+J_2^{**}+J_3^{**}+\cdots)$. For each positive integer n let Z_n denote the limit point of that component of $M-(J_1^{**}+J_2^{**}+J_3^{**}+\cdots)$ which contains P_n. Since (1) for each n the point Z_n belongs to K_2, (2) the curves J_1^{**}, $J_2^{**}, J_3^{**}, \cdots$ form a contracting sequence,‡ and (3) not more than a finite number of components of $M-J_i^{**}$, for any value of i, are of diameter greater than a positive number, it follows that P is a limit point of K_2 and belongs to K_1, contrary to the assumption that K_1 and K_2 are mutually separated. Hence K^* is connected and the supposition that a subsequence $J_1^*, J_2^*, J_3^*, \cdots$ of the sequence $J_1, J_2, J_3, \cdots$ exists for which the set of all points X of K such that X is not interior to J_i^* for any value of i is not connected, has led to a contradiction.

Let K^{**} denote the set of all points X of K such that X is not interior to J_i for any value of i. Suppose K^{**} contains more than one point. If Q denotes

† See W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, (2), vol. 28 (1927), Theorem 1.

‡ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

a point of K^{**} which does not belong to the boundary B^* of the unbounded complementary domain D^* of M, let C_q denote a circle with center at Q, such that C_q encloses no point of D^*. By a theorem of G. T. Whyburn,† the set of all cut points X of M such that X lies on some simple closed curve of M is a countable set. But every cut point of M is on the boundary of some complementary domain‡ of M, and therefore either belongs to B^* or lies on or within J_i for some value of i. Hence, since K^{**} is connected and therefore contains uncountably many points within C_q, there are points of K^{**} interior to C_q which belong also to $K-K^{**}$, which is impossible. Therefore K^{**} is a subset of B^*.

Theorem I. *If B is the boundary and J the outer boundary of a bounded complementary domain D of a continuous curve M, and K is a connected set of cut points of M such that K contains a point in common with J, any point of K which belongs to $B-J$ can be joined to some point of J by an arc common to K and to B.*

Let P denote a point of K belonging to $B-J$, and let M_1 denote that component of $M-J$ which contains P. Let Q denote the limit point§ of M_1 belonging to J and let K^* denote the set $K\cdot\overline{M}_1$. Since K is connected, Q belongs to K. By Lemma I, K^* is connected. Since M_1+Q is a continuous curve, and $B(M_1+Q)$ is the boundary of the unbounded complementary domain of M_1+Q, by Lemma III the set $B\cdot K^*$ is connected. Hence, by a theorem of R. L. Wilder,|| P can be joined to Q by an arc belonging to B and to K^*.

Theorem II. *If J is the outer boundary of a bounded complementary domain D of a continuous curve M, and K is a connected set of cut points of M containing a point in common with J, then any point of K within J can be joined to some point of J by an arc which is common to K and to a point set consisting of the boundaries of a finite number of complementary domains of M.*

Let P denote a point of K within J. By Lemma II, if G denotes the collection of all simple closed curves of M each of which encloses P and is the outer boundary of some complementary domain of M, then G is finite. Let J_1, J_2, $J_3, \cdots, J_n$ denote the curves of the collection G, where for each positive integer $i(i<n)$, J_i is a subset of the point set consisting of J_{i+1} together with

† *Concerning continua in the plane*, these Transactions, vol. 29 (1927), pp. 369–400, Theorem 29.

‡ See R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213.

§ See R. L. Moore, *Concerning paths that do not separate a given continuous curve*, loc. cit., Theorem 6.

|| *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7, pp. 340–377, Theorem 20.

its interior. Suppose J_m is J. For each positive integer $k(k<m)$, let D_k denote that complementary domain of M whose outer boundary is J_k, let M_k denote that component of $M-J_k$ which contains P, and let Q_k denote the point of J_k which is a limit point of M_k. Suppose there exists a positive integer $q(q<m)$ such that Q_{q+1} does not belong to the boundary of D_q. Let K_{q+1} denote the set of all points X of K such that (a) X belongs to $K\cdot\overline{M}_{q+1}$, (b) the point X is not interior to the outer boundary of any complementary domain of M (except D_{q+1}) whose boundary contains a point of $K\cdot\overline{M}_{q+1}$. Since J_{q+1} is the outer boundary of D_{q+1}, every cut point of M which belongs to M_{q+1} is also a cut point of M_{q+1}. By Lemmas I and III together with the fact that $\overline{M}_{q+1}$ is a continuous curve, K_{q+1} is connected. But it contains Q_q and Q_{q+1} and is a subset of the boundary of D_{q+1}. By Wilder's theorem† there exists for each positive integer $i(i<m)$ an arc from Q_i to Q_{i+1} which is common to K and the boundary of D_{i+1}. Hence there exists an arc from P to a point of J which is common to K and a point set consisting of the boundaries of a finite number of complementary domains of M.

THEOREM III. *If K is a connected set of cut points of a continuous curve M, and X and Y are two points of K, then there exists an arc from X to Y which is common to K and a point set consisting of the sum of the boundaries of a finite number of complementary domains of M.*

Suppose X belongs to the boundary B^* of the unbounded complementary domain‡ D^* of M. If Y belongs to B^* the proof of the theorem follows from Lemma III together with Wilder's theorem.§ Suppose Y does not belong to B^*. By Lemma II if G denotes the collection of all simple closed curves of M each of which encloses Y and is the outer boundary of some complementary domain of M, then G is finite. There exists a simple closed curve J^* of the collection G such that the point set consisting of the sum of all the simple closed curves of G contains no point exterior to J^*. By Theorem II the point Y can be joined to a point Y_1 of J^* by an arc YY_1, and by Lemma III together with Wilder's theorem|| Y_1 can be joined to X by an arc Y_1X such that each of the arcs YY_1 and Y_1X is common to K and a point set consisting of the sum of the boundaries of a finite number of complementary domains of M. From the sum of the arcs YY_1 and Y_1X there exists an arc from X to Y which satisfies the conditions of Theorem III.

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† *Concerning continuous curves*, loc. cit.

‡ Every other case may be reduced to this one by an inversion.

§ *Concerning continuous curves*, loc. cit.

|| Ibid.

Part II

Definition. An arc XY will be said to have property α with respect to a continuous curve M provided it satisfies the following conditions: (1) the arc XY contains no cut point of M, (2) the common part of arc XY and M is totally disconnected, (3) if the boundary B of a complementary domain D of M contains two points U and V in common with arc XY, then segment UV is a subset of D, (4) segment XY of the arc XY contains no point common to the boundaries B_1 and B_2 of two distinct complementary domains D_1 and D_2 of M which is also a boundary point of some complementary domain of the point set $D_1+D_2+B_1+B_2$.

It is clear that if XY is an arc having property α with respect to a continuous curve M, and YZ is an arc which contains only the point Y in common with M or with arc XY, and if XZ, the sum of the arcs XY and YZ, satisfies properties (3) and (4) of the preceding definition, then arc XZ has property α with respect to M.

Theorem IV. *If M is a bounded continuous curve, and K is a closed and totally disconnected subset of M whose omission leaves M connected and such that K belongs to the boundary B of a complementary domain D of M, then there exists an acyclic continuous curve whose end points are identical with the set K and which is a subset of $D+K$.*

Since B is a continuous curve,† by a theorem of R. L. Moore‡ there exists a continuum N containing K and which is a subset of $D+K$. Let N_1 denote the continuum formed by adding to N all the bounded complementary domains of N. Since K is closed and totally disconnected, it follows that there exists a circle C which is a subset of $D-N_1$. Let T denote an inversion about C, let M^* and $N_1{}^*$ denote the images of M and N_1 respectively under T, and let $M_1{}^*$ denote the continuum formed by adding to M^* all the bounded complementary domains of M^*. It is clear that (a) M_1^*-K is connected, (b) the set of points common to M_1^* and N_1^* is K, (c) neither M_1^* nor N_1^* separates the plane. Therefore by a theorem of R. L. Moore§ there exists a simple closed curve J which encloses M_1^*-K but encloses no point of N_1^*-K and contains K but no point $(N_1^*+M_1^*)-K$. Let P denote the center of C. If J does not contain P let J^* denote J, and if J contains P let J^* denote a

† See Marie Torharst, *Über den Rand der einfach zusammenhängenden ebenen Gebiete*, Mathematische Zeitschrift, vol. 9 (1921), pp. 45–65.

‡ *Some separation theorems*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 711–716, Theorem I.

§ *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476, Theorem 2.

simple closed curve which contains K but not P and is a subset of J plus a circle which encloses P but which neither contains nor encloses any point of $N_1^*+M_1^*$. H. M. Gehman† has shown that if EF is an arc containing a totally disconnected closed point set L, then L is identical with the end points of an acyclic continuous curve which contains only the set L in common with the arc EF. From this it follows that there exists an acyclic continuous curve W whose end points are identical with K and which is a subset of K plus the exterior of J^*. The image W^* of W under the inverse of T satisfies the conditions of Theorem IV.

THEOREM V. *If M is a bounded continuous curve, and K is a closed point set, and K^* is the set of all points X such that X belongs either to K or to a point set containing a point of K and consisting of a complementary domain of M together with its boundary, then K^* is closed.*

Suppose P is a limit point of K^* which does not belong to K^*. The point P belongs to M and since P does not belong to K^* it is not a boundary point of any complementary domain D of M such that D is a subset of K^*. Since P does not belong to K, there exists a circle C with center at P such that C neither contains nor encloses any point of K. Let C^* denote a circle with center at P and of diameter one-half that of C. It follows that there are infinitely many complementary domains of M each of which contains a point exterior to C and a point interior to C^*. This is impossible since the complementary domains of M form a contracting sequence.‡ Hence the suppositon that P is a limit point of K^* which does not belong to K^* has led to a contradiction. Therefore K^* is closed.

THEOREM VI. *Suppose K is a closed and totally disconnected point set, and M is a closed and bounded point set containing K such that* (a) *the sum of all the components of M which are not single points can be expressed as the sum of a countable number of continuous curves $C_1, C_2, C_3, \cdots$, not more than a finite number of which are of diameter greater than a positive number,* (b) *the set of all points X such that X belongs to at least two curves of the sequence $C_1, C_2, C_3, \cdots$ is a subset of K. Then each component of M is a continuous curve and not more than a finite number of components of M are of diameter greater than a positive number.*

From the fact that a continuum which is not a continuous curve fails to be connected im kleinen at a continuum of points,‡ and the fact that K is

† *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), pp. 553–568, Theorem 6.

‡ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

closed and totally disconnected, it follows that each component of M is a continuous curve. Suppose there exists a positive number e and an infinite sequence $M_1, M_2, M_3, \cdots$ of components of M each of which is of diameter greater than e. There exists a subcontinuum M^* of M and a subsequence $M_1^*, M_2^*, M_3^*, \cdots$ of the sequence $M_1, M_2, M_3, \cdots$ having M^* as sequential limiting set.† There exists a point P belonging to M^*-K, and a positive integer n, such that, if C and C^* are circles with centers at P and of diameters e/n and $e/(2n)$ respectively, then neither C nor C^* contains or encloses any point of K. Since there are infinitely many components of M each containing a point exterior to C and a point interior to C^*, there are infinitely many curves of the sequence $C_1, C_2, C_3, \cdots$ each of which is of diameter greater than $e/(4n)$ contrary to hypothesis. Hence the supposition that there exists a positive number e and an infinite sequence $M_1, M_2, M_3, \cdots$ of components of M each of which is of diameter greater than e has led to a contradiction.

THEOREM VII. *If M is a bounded continuous curve which contains no domain, and T is an acyclic continuous curve having the properties* (a) *T contains no cut point of M,* (b) *the set of all points common to T and M is totally disconnected,* (c) *if D is any complementary domain of M, any two points of the set $T\cdot\bar{D}$ can be joined by an arc of T which except for end points is a subset of D, then $M-T$ is connected.*

Let D denote a complementary domain of the continuous curve $M+T$ whose boundary B contains a point of T. Let D_M denote the complementary domain of M which contains D, and let B_M denote its boundary. Let M_1 denote the set $B-T$. Suppose M_1 is not connected. If T contains only one point A in common with B_M the point A is a cut point of B_M, and by a theorem of R. L. Moore‡ A is a cut point of M contrary to hypothesis. Hence there exists by property (c) a complementary domain D_1 of $M+T$ distinct from D such that D_1 is a subset of D_M. Let J denote the outer boundary of D with respect to D_1. By a theorem of R. L. Moore§ J is a simple closed curve. The curve J contains a point of $T-M$ and a point of $M-T$. Suppose there exists a component L of M_1 which contains no point of J. By a theorem of R. L. Moore‖ the component L^* of $B-J$ which contains L has exactly one limit point Q in J. Since T contains no cut point of M the point Q does not belong to $T\cdot M$. If Q belongs to T let Q_1 denote a point of L and let Q_2 de-

† See R. G. Lubben, *Concerning limiting sets in abstract spaces*, these Transactions, vol. 30, pp. 668–685.

‡ *Concerning the common boundary of two domains*, loc. cit.

§ *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922).

‖ *Concerning paths that do not separate a given continuous curve*, loc. cit., Theorem 6.

note a point of M_1 J. There exists an arc† of M from Q_1 to Q_2. If Q belongs to M_1, let Q_1 denote a point of T belonging to L^* and let Q_2 denote a point of T belonging to J. There exists an arc of T from Q_1 to Q_2. In either case the supposition that there exists a component of M_1 containing no point of J contradicts the fact that J is the outer boundary of D with respect to D_1. Hence there exist arcs a_1 with end points A_1 and A_2 and b_1 with end points B_1 and B_2 such that arcs a_1 and b_1 belong to different components of $J \cdot M_1$. Let s_1 and s_2 denote the two arcs of J from A_1 to B_1. Both s_1 and s_2 contain points of T. There exists an arc‡ t of T from a point in s_1 to a point in s_2. Let t_1 denote a subarc of t having one end point C_1 in s_1 and one end point C_2 in s_2 and containing only the points C_1 and C_2 in common with J. Let t_2 denote an arc with C_1 and C_2 as end points and which except for C_1 and C_2 is a subset of D. Let J_1 denote the simple closed curve formed from the sum of the arcs t_1 and t_2. The curves J and J_1 have only the points C_1 and C_2 in common. Since the segment t_2 is a subset of D and the segment t_1 contains no point in common with D, the point set $J-J_1$ is not a subset of a single complementary domain of J_1. Hence a_1 and a_2 belong to different complementary domains of J_1. Of the two points C_1 and C_2, if one belongs to $T-M$ the other is a cut point of M contrary to hypothesis. If C_1 and C_2 both belong to M, by property (c) they do not both belong to the boundary of any complementary domain of M except D_M. Hence $J-(C_1+C_2)$ is a subset of D_M. There exists an arc t_3 in D_M having one end point in t_1 and one end point in t_2 and which, except for end points, contains no point in common with J_1. The point set J_1+t_3 contains a simple closed curve enclosing a point of M and having in common with the set C_1+C_2 exactly one point. This is impossible since neither C_1 nor C_2 is a cut point of M. Thus the supposition that M_1 is not connected has led to a contradiction.

Suppose X and Y are points belonging to different components of $M-T$. There exists an arc XY which contains no point of T. Let Y_1 denote the first point in the order from X to Y which XY has in common with that component M_2 of $M-T$ which contains Y. Since M is a continuous curve there exists a first point X_1 in the order from Y_1 to X which the interval Y_1X of XY has in common with $M-M_2$. Hence X_1 and Y_1 belong to the boundary of the same complementary domain of $M+T$ but to different components of $M-T$, which is impossible. Therefore the supposition that X and Y belong to different components of $M-T$ has led to a contradiction and Theorem VII is established.

† See R. L. Moore, *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917).

‡ Ibid.

THEOREM VIII. *Suppose* (a) *M is a continuous curve which contains no domain*, (b) *A, B and C are three distinct points*, (c) *a_1 and a_2 are two arcs with end points A, B and B, C respectively such that a_1 and a_2 each have property α with respect to M*, (d) *B does not belong to M. Then there exists an arc from A to C which is a subset of a point set consisting of the sum of the arcs a_1 and a_2 together with a single complementary domain of M, such that arc AC has property α with respect to M.*

The case where A and C both belong to a point set consisting of a single complementary domain of M together with its boundary, or where A belongs to arc a_2, is trivial. If A belongs to a complementary domain D of M let H denote $\bar{D}$. If A belongs to M let G_1 denote the collection of all complementary domains of M such that A belongs to the boundary of each domain of the collection G_1, and let H denote the point set consisting of the sum of all the domains of the collection G_1 together with their boundaries. Since the complementary domains of M form a contracting sequence,† H is closed. If a_2 contains a point of H let E denote the first such point in the order from C to B. Then E belongs to the boundary B_1 of a domain D_1 of the collection G_1. There exists an arc from A to E which, except for end points, is a subset of D_1. Arc AE plus the interval EC of a_2 is an arc from A to C which satisfies the conditions of the theorem. If a_2 contains no point of H, let G_2 denote the collection of all the complementary domains of M such that each domain of the collection G_2 either contains a point of a_2 or has a point of a_2 on its boundary. Let K denote the point set consisting of the sum of all the domains of the collection G_2 together with their boundaries. By Theorem V the set K is closed. Since a_2 contains no point in common with H, the point A does not belong to K. Let F denote the first point in the order from A to B which a_1 has in common with K. If F belongs to a_2 then the sum of the intervals AF and FC of a_1 and a_2 respectively is an arc from A to C satisfying the conditions of the theorem. If F does not belong to a_2 it belongs to the boundary B_2 of a domain D_2 of the collection G_2. Let V denote the first point in the order from C to B which a_2 has in common with B_2+D_2. There exists an arc from F to V which except for end points is a subset of D_2. The sum of the intervals AF and VC of a_1 and a_2 respectively together with arc FV is an arc from A to C satisfying the conditions of the theorem and the proof is complete.

THEOREM IX. *If P is not a cut point of a continuous curve M and e is a positive number, there exists a circle C enclosing P and of diameter less than e such that the set of all points of M, each at a distance from P greater than e, lie in a connected subset of $M-C$.*

† See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

Let M_1 denote the set of all points of M each at a distance from P greater than or equal to e. Let G denote a collection of connected open subsets of M no one of which contains P but such that if X is any point of M_1 then X belongs to some subset g_X of the collection G. There exists a finite subcollection G_1 of G which contains all the points of M_1. Since $M-P$ is connected and any two points belonging to a connected open subset of a continuous curve can be joined by an arc† lying wholly in the open subset it follows that there exists a finite set A of arcs, each lying within $M-P$, and such that the point set M_2 consisting of M_1 plus the arcs of the set A is connected. Since M_2 does not contain P there exists a circle enclosing P which neither contains nor encloses any point of M_2 and the theorem is established.

THEOREM X. *If P is not a cut point of a bounded continuous curve M which contains no domain and e is a positive number, there exists a positive number d_e such that if A and B are any two points, each at a distance from P less than d_e, and $M-(A+B)$ is connected, then A and B are the extremities of an arc which has property α with respect to M and which is of diameter less than e.*

By Theorem IX, if e is a positive number there exists a positive number d_e such that the set of all points of M each at a distance from P greater than e lie in a connected subset M_1 of M such that no point of M_1 is at a distance from P less than d_e. Let C and C_1 denote circles with P as center and of diameters e and d_e respectively. Let I and I_1 denote the interiors of C and C_1 respectively, and let M_1^* denote that component of $M-I_1$ which contains M_1. Let S denote the set of all points, let S_1 denote $S-I$, and let S_2 denote the set of all points X such that X belongs to a complementary domain of $M+C$ whose boundary is a subset of M_1^*+C. Let K denote the set $M_1^*+S_1+S_2$ and let D denote the complementary domain of K. Clearly P is in D. Suppose P_1 and P_2 are two points of D which belong to a complementary domain D_1 of M but which can not be joined by an arc lying in $I\cdot D_1$. Let a_1 denote an arc from P_1 to P_2 which is a subset of D and let a_2 denote an arc from P_1 to P_2 which is a subset of D_1. If a_2 is a subset of $C+I$ let a_n denote a_2. If a_2 is not a subset of $C+I$ let E_1 denote the first point in the order from P_1 to P_2 which a_2 has in common with a_1 such that interval P_1E_1 of a_2 contains a point exterior to C and let F_1 denote the last point in the order from P_1 to E_1 which $P_1E_1-E_1$ has in common with a_1. Let J_1 denote the simple closed curve which is the sum of the intervals E_1F_1 of a_1 and E_1F_1 of a_2. If M_1^* is interior to J_1 there exists a complementary domain of J_1+C common to I and the exterior of J_1 such that its boundary J_1^* contains the interval E_1F_1 of a_1. If M_1^* is a subset of the exterior of J_1 there exists a complementary domain of J_1+C common

† See R. L. Moore, *Concerning continuous curves in the plane*, loc. cit., Theorem 1.

to I and the interior of J_1 such that its boundary J_1^* contains the interval E_1F_1 of a_1. In either case let t_1 denote the arc from P_1 to E_1 which is the sum of the interval P_1F_1 of a_2 and the arc E_1F_1 of J_1^* which contains a point of C. Let W_1 denote the last point in the order from P_1 to P_2 which a_2 has in common with t_1. Let a_3 denote the sum of the intervals P_1W_1 and W_1P_2 of t_1 and a_2 respectively. Arc a_3 contains no point of M. If a_3 is a subset of $C+I$ let a_n denote a_3. If a_3 is not a subset of $C+I$ let E_2 denote the first point in the order from P_1 to P_2 which a_3 has in common with a_1 such that interval P_1E_2 of a_3 contains a point exterior to C, and let F_2 denote the last point in the order from P_1 to E_2 which $P_1E_2-E_2$ has in common with a_1. Let J_2 denote the simple closed curve which is the sum of the intervals E_2F_2 of a_1 and E_2F_2 of a_3. If M_1^* is interior to J_2 there exists a complementary domain of J_2-C common to I and the exterior of J_2 such that its boundary J_2^* contains the interval E_2F_2 of a_1. If M_1^* is exterior to J_2 there exists a complementary domain of J_2+C common to I and the interior of J_2 such that its boundary J_2^* contains the interval E_2F_2 of a_1. In either case let t_2 denote the arc from P_1 to E_2 which is the sum of the interval P_1F_2 of a_3 and the arc E_2F_2 of J_2^* which contains a point of C. Let W_2 denote the last point in the order from P_1 to P_2 which a_3 has in common with t_2. Let a_4 denote the sum of the intervals P_1W_2 and W_2P_2 of t_2 and a_3 respectively. Arc a_4 contains no point of M. If a_4 is a subset of $C+I$ let a_n denote a_4. If a_4 contains a point exterior to C it is clear that after a finite number of operations just described one may obtain an arc a_n from P_1 to P_2 which is a subset of $C+I$ and which contains no point of M. It follows that there exists an arc a_m from P_1 to P_2 which is a subset of $I\cdot D_1$ contrary to the assumption that P_1 and P_2 are points of D_1 which can not be joined by an arc of $I\cdot D_1$.

If K is designated by a point and each point of $S-K$ is designated by a point, the set of elements thus obtained is an upper semi-continuous collection† of elements filling up the plane and is in one-to-one continuous correspondence T with the surface H of a sphere. For each point set Q in S let $T(Q)$ designate its image under T. Since any two points of D belonging to a complementary domain D_i of M can be joined by an arc of $I\cdot D_i$, $T(D\cdot D_i)$

† See R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428. A collection G of continua is said to be an *upper semi-continuous* collection if for each element g of the collection G and each positive number e there exists a positive number d such that if x is any element of G at a lower distance from g less than d then the upper distance of x from g is less than e. If M is a point set and P is a point, then by $l(PM)$ is meant the lower bound of the distance from P to all the different points of M. If M and N are two point sets, then by $l(MN)$ is meant the lower bound of the values $l(PN)$ for all points P of M, while by $u(MN)$ is meant the upper bound of these values for all points P of M. The point set M is said to be at the upper distance $u(MN)$ from the point set N and is said to be at the lower distance $l(MN)$ from N.

is a complementary domain of $T(K+M)$. If AB is an arc in D such that $T(AB)$ has property α with respect to $T(K+M)$, then AB has property α with respect to M. By a slight modification of a theorem by R. L. Moore† it may easily be seen that any two points belonging to $H-T(K)$ whose omission leaves $T(K+M)$ connected may be joined by an arc lying in $H-T(K)$ and which has property α with respect to M. Therefore Theorem X is established.

THEOREM XI. *If M is a bounded continuous curve which contains no domain, and A and B are two points such that $M-(A+B)$ is connected, there exists an arc from A to B which has property α with respect to M.*

If neither A nor B belongs to M then R. L. Moore‡ has shown how to construct an arc from A to B which does not disconnect M. From the nature of his construction, this arc has property α with respect to M. If M contains no domain and $M-(A+B)$ is connected it follows from a slight modification of his construction that there exists an arc from A to B which has property α with respect to M.

THEOREM XII. *If M is a bounded continuous curve which contains no domain and K is a totally disconnected closed subset of M, there exists an acyclic continuous curve T containing K, and such that* (1) *all the end points of T belong to K, and* (2) *the point set $M \cdot T$ is totally disconnected and $M-T$ is connected.*

By Theorem IV, for each complementary domain D_i of M whose boundary δ_i contains two or more points of K, there exists an acyclic continuous curve C_i whose end points are identical with the points of the set $K \cdot \delta_i$ and lying except for end points wholly within D_i. Let K_1 denote the point set $K+C_1+C_2+C_3+\cdots$. That K_1 is closed may be proved with the use of the fact that the complementary domains of M form a contracting sequence.§ By Theorem VI each component of K_1 is a continuous curve and not more than a finite number of components of K_1 are of diameter greater than a positive number. Suppose K_1 contains a simple closed curve J. Since $J \cdot M = J \cdot K$ and $M-K$ is connected, then $M-J$ is a subset of one of the complementary domains of J and hence J is a subset of a point set consisting of a single complementary domain of M plus its boundary which is impossible. Hence K_1 contains no simple closed curve.

By Theorems IX and X together with the fact that a closed and bounded

† *Concerning paths that do not separate a given continuous curve*, loc. cit.

‡ Ibid.

§ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

point set has the Borel-Lebesgue property, there exists a sequence of sets of circular regions $G_1, G_2, G_3, \cdots$ having the properties that for each positive integer i, (1)G_i is finite and covers K and each region of G_i contains at least one point of K, (2) each region of the set G_i is of diameter less than $1/i$, (3) if g is a region of the set $G_i(i>1)$, $\bar{g}$ lies interior to some region of the set G_{i-1} and has the property that if X_1 and X_2 are any two points of g whose omission does not disconnect M then there exists an arc from X_1 to X_2 having property α with respect to M and lying interior to each region of the set G_{i-1} which contains $\bar{g}$.

For each positive integer i let N_{1i} denote the point set obtained by adding to K_1 all the regions of the set G_i together with their boundaries. If for all values of i the set N_{1i} is connected, let T denote the set of points common to the sets $N_{11}, N_{12}, N_{13}, \cdots$. If for some value of i the set N_{1i} is not connected let m denote the smallest positive integer such that N_{1m} is not connected. The point set N_{1m} has only a finite number n of components. Let L_1 denote one of them. If m is greater than 3 there exists a component L_2 of N_{1m} distinct from L_1 and two regions g_s and g_t of the set G_{m-2} having in common a point P not belonging to M such that g_s contains a point X_1 of the set $L_1 \cdot K_1$ and g_t contains a point X_2 of the set $L_2 \cdot K_1$. By property (3) of the sets of circular regions, there exist arcs PX_1 and PX_2 each having property α with respect to M and each lying in a single region of the set G_{m-3}. By Theorem VIII there exists an arc X_1X_2 having property α with respect to M and which is a subset of a point set consisting of the sum of the arcs PX_1 and PX_2 together with a single complementary domain of M. For $m=1, 2, 3$ there exists an arc X_1X_2 from a point X_1 of $L_1 \cdot K_1$ to a point X_2 of $K_1 - L_1$ which has property α with respect to M.

If k is a component of K_1 then for each point Z of k add to k all points W such that W lies within or on the boundary of some complementary domain of M which contains Z or has Z on its boundary. Let H_k denote the set thus obtained and let H denote the point set obtained by adding together all points of all the sets H_k for all components k of K_1. Since each point of H at a distance from every point of K greater than a positive number e lies either within or on the boundary of a complementary domain of M of diameter greater than e it follows that H is closed. From the fact that K is closed and totally disconnected and the fact that every infinite set of point sets whose sum is bounded has a limiting set,† it follows that the nondegenerate‡ sets H_k form a contracting sequence.

Let H^* denote the point set obtained by adding together all points of all

† See R. G. Lubben, *Concerning limiting sets in abstract spaces,* loc. cit.

‡ A point set containing but a single point is said to be degenerate.

the sets H_k for all components k of K_1 such that k is a subset of L_1. Since the components of K_1 and the complementary domains of M form contracting sequences and each component of K_1 contains a point of K, therefore H^* is closed. The point set $H^* \cdot K_1$ is a subset of L_1. Let E_1 denote the last point which the arc X_1X_2 has in common with H^*, and let E_2 denote the first point which the interval E_1X_2 of X_1X_2 has in common with the set $\overline{H-H^*}$. If $E_1=E_2$ then E_1 does not belong to K_1. For suppose it does. Then since E_1 belongs to M it belongs also to K. Hence E_1 does not belong to any set H_k for any component k of K_1 such that K is a subset of K_1-L_1. Since E_1 belongs to K it is interior to some region of the set G_m belonging to L_1 and there exists a positive number e such that the distance from E_1 to any point of K_1-L_1 is greater than e. Therefore since E_1 belongs to $\overline{H-H^*}$ there are infinitely many sets H_k, each of diameter greater than $e/2$, which is impossible. Hence the supposition that $E_1=E_2$ and E_1 belongs to K_1 has led to a contradiction. Thus if $E_1=E_2$, then E_1 belongs to the boundaries of two distinct complementary domains D_1 and D_2 of M such that the boundary of D_1 contains a point of K belonging to L_1 and the boundary of D_2 contains a point of K belonging to the set $N_{1m}-L_1$. There exists in $\overline{D}_1$ an arc Y_1E_1 and in $\overline{D}_2$ an arc Z_1E_2 such that (a) the point Y_1 belongs to K_1 and also to $K_1 \cdot D_1$ if D_1 contains a point of K_1, (b) the point Z_1 belongs to K_1 and also to $K_1 \cdot D_2$ if D_2 contains a point of K_1, (c) segments Y_1E_1 and Z_1E_2 have no point in common with K_1 or with M. If $E_1 \neq E_2$ and E_1 belongs to K_1, let Y_1 denote E_1. If $E_1 \neq E_2$ and E_1 does not belong to K_1, it belongs to the boundary B_1 of a complementary domain D_1 of M such that B_1 contains a point of $K_1 \cdot L_1$. There exists an arc Y_1E_1 such that (a) the point Y_1 belongs to K_1 and also to K_1-K if D_1 contains a point of K_1, (b) segment Y_1E_1 contains no point in common with K_1 or with M. If $E_1 \neq E_2$ and E_2 belongs to K_1, let Z_1 denote E_2. If $E_1 \neq E_2$ and E_2 does not belong to K_1, it belongs to the boundary B_2 of a complementary domain D_2 of M such that B_2 contains a point of K_1-L_1. There exists an arc Z_1E_2 such that (a) the point Z_1 belongs to K_1 and also to K_1-K if D_2 contains a point of K_1, (b) segment Z_1E_2 contains no point in common with K_1 or with M. Let a_{11} denote the arc which is the sum of the arcs Y_1E_1 and Z_1E_2 and the interval E_1E_2 of the arc X_1X_2. Then (1) the arc a_{11} has property α with respect to M, (2) for m greater than 3 the set of points common to arc a_{11} and M which does not belong to K_1 is a subset of a point set consisting of the sum of two regions of the set G_{m-3}, (3) the end points Y_1 and Z_1 of arc a_{11}, which are the only points common to a_{11} and to K_1, belong to different components of N_{1m}. The number of components of $N_{1m}+a_{11}$ is at least one less than the number of components of N_{1m}. If $N_{1m}+a_{11}$ is not connected, by treating the sets K_1+a_{11} and $N_{1m}+a_{11}$ in the same manner as that in which the sets K_1 and N_{1m} respectively were treated one may obtain an arc a_{12} which satisfies with respect to M, G_{m-3}, K_1+a_{11}, and $N_{1m}+a_{11}$, the same properties that arc a_{11} satisfies with respect to M, G_{m-3}, K_1 and N_{1m} respectively. The number of components of the set $N_{1m}+a_{11}+a_{12}$ is at least two less than the number of components of the set N_{1m}. It is clear that by the addition to K_1 of a finite set A_1 of arcs $a_{11}, a_{12}, a_{13}, \cdots, a_{1k}$ $(k<n)$, each arc a_{1j} of the set A_1 satisfying with respect to M, G_{m-3}, $K_1+\sum_{i=1}^{j-1}a_{1i}$ and $N_{1m}+\sum_{i=1}^{j-1}a_{1i}$ the same properties that arc $a_{i(j-1)}$ satisfies with respect to M, G_{m-3}, $K_1+\sum_{i=1}^{j-2}a_{1i}$ and $N_{1m}+\sum_{i=1}^{j-2}a_{1i}$ respectively, one may obtain a point set K_2 which has the following properties: (1) K_2 is closed and contains no simple closed curve, (2) if P and Q are any two points of K_2 belonging to the boundary of a complementary domain D of M then P and Q are the extremities of an arc belonging to K_2 such that segment PQ is a subset of D, (3) the set K_2+N_{1m} is connected, (4) the set $M\cdot K_2$ is totally disconnected, (5) each component of K_2 is an acyclic continuous curve, and (6) K_2 contains no cut point of M. That K_2 has properties (1) and (2) is clear from the facts (a) A_1 is finite, (b) K_1+a_{11} has properties (1) and (2), (c) if $K_1+\sum_{i=1}^{j-1}a_{1i}(j<k-1)$ has properties (1) and (2) then $K_1+\sum_{i=1}^{j}a_{1i}$ has them also. The point set K_2 has property (3) since the number of components of N_{1m} is n and the number of components of $N_{1m}+\sum_{i=1}^{r}a_{1i}$ $(r<k-1)$ is at least one less than the number of components of $N_{1m}+\sum_{i=1}^{r-1}a_{1i}$. The set of points common to M and the set A_1 of arcs is closed and totally disconnected, hence K_2 has property (4). By Theorem VI the set K_2 has property (5), and since no arc of the set A_1 contains a cut point of M the set K_2 has the property (6). Add K_2 to each of the sets N_{11}, N_{12}; $N_{13}, \cdots$ and denote the resulting sets by $N_{21}, N_{22}, N_{23}, \cdots$ respectively. If for each value of i, N_{2i} is connected, let T denote the set of all points common to the sets $N_{21}, N_{22}, N_{23}, \cdots$. If, for some value of i, N_{2i} is not connected, let n denote the smallest positive integer such that N_{2n} is not connected. Let r denote the number of components of N_{2n}. By the addition to K_2 of a finite set A_2 of arcs $a_{21}, a_{22}, a_{23}, \cdots, a_{2j}(j<r)$, each arc a_{2k} of the set A_2 satisfying with respect to M, G_{n-3}, $K_2+\sum_{i=1}^{k-1}a_{2i}$ and $N_{2n}+\sum_{i=1}^{k-1}a_{2i}$ $(k<r)$ the same properties that arc a_{11} satisfies with respect to M, G_{m-3}, K_1 and N_{1m} respectively, one may obtain a point set K_3 which satisfies the same properties with respect to M, G_{n-3}, N_{2n} that K_2 satisfies with respect to M, G_{m-3} and N_{1m} respectively. Suppose the process is continued indefinitely. Let K^* denote the set $K_1+K_2+K_3+\cdots$. The set K^* is connected and the set $M\cdot K^*$ is closed. For suppose W denotes a limit point of $M\cdot K^*$ which does not belong to $M\cdot K^*$. The point W does not belong to K, and since for any positive number e there exists a positive integer t such that no arc of the set $A_j(j>t)$ contains a point of $M-K_{t-1}$ at a distance from every point of K

greater than e, it follows that W belongs to K_{t-1} and hence to $M \cdot K^*$ contrary to hypothesis. Therefore $M \cdot K^*$ is closed. Since for each positive integer i the set K_i contains in common with M only a totally disconnected set of points, the point set $M \cdot K^*$ is totally disconnected. From the fact that each arc of the set A_i for all values of i has property α with respect to M and the fact that $M-K$ is connected it follows that $M \cdot K^*$ contains no cut point of M.

If β_i is the boundary of a complementary domain D_i of M and the set of points $\overline{D}_i \cdot K^*$ is not an acyclic continuous curve whose end points are identical with the set $\beta_i \cdot K^*$, there exists by Theorem IV an acyclic continuous curve C_i whose end points are identical with the set $\beta_i \cdot K^*$ and which, except for end points, is a subset of D_i. Let T_1 denote the set formed from K^* by replacing the set $D_i \cdot K^*$ by C_i for each value of i. The point set T_1 is closed and connected and has the property that if X and Y are any two points of T_1 belonging to the boundary of a complementary domain D of M then X and Y are the extremities of an arc belonging to T_1 such that segment XY is a subset of D. That T_1 is a continuous curve may be seen from the fact that any point of T_1 not belonging to the totally disconnected set $M \cdot T_1$ lies in some complementary domain of M, together with the fact that a continuous curve can not fail to be connected im kleinen at only a totally disconnected set of points.†

Suppose there exists an end point P of T_1 which does not belong to K. Then P belongs to M and hence to K^* and there exists a positive integer r such that P belongs to K_r. By a theorem of G. T. Whyburn‡ and the fact that each nondegenerate component of K_r is an acyclic continuous curve, all of whose end points belong to K, it follows that there exists an arc c_r lying in $K_r - K$ whose end points belong to M and such that c_r contains P as an interior point. For each segment c_i of $c_r - M$ whose end points belong to M let b_i denote a segment of $T_1 - M$ such that the end points of b_i are the end points of c_i. The point set consisting of the sum of the segments b_i together with the set $M \cdot c_r$ is an arc lying in T_1 and containing P as interior point which contradicts the assumption that P is an end point of T_1. Hence the end points of T_1 all belong to K.

Suppose J is a simple closed curve lying in T_1. The curve J contains a point of $M-K$, otherwise J is a subset of a point set consisting of a single complementary domain D of M together with its boundary, contrary to the fact that $\overline{D} \cdot T_1$ contains no simple closed curve. The curve J contains a point of K. For suppose it does not. There exists a positive integer s such that $M \cdot J$

† See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

‡ *Concerning continua in the plane*, loc. cit.

is a subset of $M \cdot K_s$. Since J separates M there exists a point of $M-K_s$ interior to J and a point of $M-K_s$ exterior to J. For each segment c_i of $J-M$ whose end points belong to M let b_i denote a segment lying in K_s-M and such that the end points of b_i are the end points of c_i. The point set consisting of the sum of the segments b_i together with the set $M \cdot J$ is a connected set of points which is a subset of a single component k of K_s. Hence k separates M. Since K_s contains no cut point of M and has the properties (1) K_s contains no simple closed curve, (2) $M \cdot K_s$ is totally disconnected, (3) any two points of K_s belonging to the boundary B of a complementary domain D of M are the end points of a segment lying in $D \cdot K_s$, therefore any component of K_s is an acyclic continuous curve which satisfies with respect to M the same properties that T satisfies with respect to M in Theorem VII. Hence k does not separate M and the supposition that J contains no point of K has led to a contradiction.

For each positive integer i let S_i denote the sum of the regions of the set G_i together with their boundaries. That the set of all junction points† of T_1 not belonging to S_i is finite may be shown as follows. Since T_1 is a continuous curve, for $j>i$ there are not more than a finite number of components of T_1-S_j each containing a point of T_1-S_i. Since all the end points of T_1 belong to K and any simple closed curve of T_1 contains a point of K, then each component of T_1-S_j together with its limit point, which contains a point of T_1-S_i, is an acyclic continuous curve whose end points, finite in number, all belong to S_i. Let L denote the set of all junction points of T_1. For each positive integer i there are only a finite number of components of the set $T_1-(L+S_{i+1})$ each of which contains a point of T_1-S_i. Suppose there exist a positive integer m and a component r of $T_1-(L+S_{m+1})$ such that for each positive integer n there exists a simple closed curve of T_1 which contains r but which contains no point of $M-S_n$. Since each simple closed curve of T_1 contains a point of $M-K$ it follows that there exists a sequence $J_1, J_2, J_3, \cdots$ of simple closed curves of T_1 and a subsequence $S_1^*, S_2^*, S_3^*, \cdots$ of the sequence $S_1, S_2, S_3, \cdots$ such that for each i, J_i contains r and at least one point of $M-S_{i+1}^*$ but contains no point of $M-S_i^*$. Let $J_1^*, J_2^*, J_3^*, \cdots$ denote a subsequence of the sequence $J_1, J_2, J_3, \cdots$ having a sequential limiting set N^*. Since for each i the curve J_i^* contains r the set N^* separates the plane. Suppose N^* contains

† See R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proceedings of the National Academy of Sciences, vol. 14 (1928). If P is a point of a continuous curve N and K is a domain containing P such that P is a cut point of the component of $N \cdot K$ which contains P, and furthermore there exist three arcs PA_1, PA_2, and PA_3 which lie in N and have only the point P in common, then P is said to be a *junction point* of N. The continuum $PA_1+PA_2+PA_3$ is called a *triod* and the point P is its *emanation point*.

a point Q belonging to $M-K$. There exists a positive integer m such that Q does not belong to S_m. It follows that Q belongs to all but a finite number of the simple closed curves of the sequence $J_1, J_2, J_3, \cdots$. This is impossible, hence N^* contains no point of $M-K$. Since N^* is connected im kleinen at every point not belonging to K, then N^* is a continuous curve.† Therefore by a theorem of R. L. Moore,‡ N^* contains a simple closed curve. This is impossible since N^* contains no point of $M-K$ and the supposition that for each positive integer n there exists a simple closed curve of T_1 which contains r but which contains no point of $M-S_n$ has led to a contradiction. Hence if r is a component of $T_1-(L+S_m)$ there exists a positive integer n such that every simple closed curve of T_1 which contains r contains a point of $M-S_n$. It follows then that there exists a subsequence $S_1^{**}, S_2^{**}, S_3^{**}, \cdots$ of the sequence $S_1, S_2, S_3, \cdots$ such that for each positive integer i, if J is a simple closed curve of T_1 which is not a subset of S_i, then J contains a point of $M-S_i^{**}$. Suppose there exists a component of $T_1-(L+S_2)$ which is not a subset of S_1 but which is a subset of a simple closed curve J lying in T_1. Let P denote a point of J belonging to $M-S_1^{**}$. If P is not a junction point of T_1 there exists a component t of $J-(L+K)$ containing P. If P is a junction point of T_1 it is a limit point of the set $M\cdot J$. For suppose there exists a segment s_1 of $(J-M)+P$ which contains P. Since P does not belong to K there exists a positive integer e such that P belongs to K_e but not to K_{e-1}. Thus P is an interior point of some arc of the set A_{e-1}. Since for any complementary domain D of M the set $\overline{D}\cdot T_1$ is an acyclic continuous curve whose end points are identical with the set $T_1(\overline{D}-D)$ it follows that the two components of s_1-P belong to different complementary domains D_1 and D_2 of M and that P is both a junction point of T_1 and a boundary point of a complementary domain of the point set $\overline{D}_1+\overline{D}_2$. This is impossible since P is interior to some arc of the set A_{e-1}, each arc of which has property α with respect to M and the supposition that there exists a segment of $(J-M)+P$ which contains P has led to a contradiction. It follows then that since $L-S_1^{**}$ is a finite set of points there exists a point Q belonging to $M\cdot J-S_1^{**}$ which is not a junction point of T_1. Let t denote the component of $J-(L+K)$ which contains Q. The component t is either a segment of the curve J or the curve minus a single point. Subtract t from T_1. Let T_1^* denote the point set $T_1-\sum_{i=1}^{i_1}t_i$ where for each positive integer $k(k\leqq i_1)$, (1) t_k is either a segment of a simple closed curve or a simple closed curve minus a point such that the curve is a subset of $T_1-\sum_{i=1}^{k-1}t_i$ and contains a component of the set $T_1-(L+S_2)$, (2) t_k

† See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

‡ *Concerning continuous curves in the plane*, loc. cit.

contains a point of $M-S_1^{**}$, (3) if t_k is a segment its end points belong to the point set consisting of K together with the junction points of $T_1-\sum_{i=1}^{k-1}t_i$ and if t_k is not a segment then t_k plus a single point of K is a simple closed curve, (4) t_k contains no point of K nor any junction point of $T_1-\sum_{i=1}^{k-1}t_i$, and further such that the point set T_1^* contains no simple closed curve which is not a subset of S_1. In general let T_j^* denote the point set $T_{j-1}^*-\sum_{i=1}^{i_j}t_i$, where for each positive integer $k(k\leqq i_j)$, (1) t_k is either a segment of a simple closed curve or a simple closed curve minus a point such that the curve is a subset of $T_{j-1}^*-\sum_{i=1}^{k-1}t_i$ and contains a component of the set $T_1-(L+S_{j+1})$, (2) t_k contains a point of $M-S_j^{**}$, (3) if t_k is a segment its end points belong to the point set consisting of K together with the junction points of $T_{j-1}^*-\sum_{i=1}^{k-1}t_i$ and if t_k is not a segment then t_k plus a single point of K is a simple closed curve, (4) t_k contains no point of K nor any junction point of $T_{j-1}^*-\sum_{i=1}^{k-1}t_i$, and further such that the point set T_j^* contains no simple closed curve which is not a subset of S_j.

Let T denote the set of points common to the sets T_1^*, T_2^*, T_3^*, $\cdots$. The continuum T contains no simple closed curve. Since (a) for each positive integer i the set $T_i^*-T_{i+1}^*$ consists of a finite number of components, each of which is a subset of S_{i}, (b) the continuous curve T_1 has the property that every simple closed curve lying in T_1 contains a point of K, it follows that T is connected im kleinen at every point of $T-K$ and hence† T is a continuous curve. Since $M\cdot T$ is a subset of $M\cdot K^*$ the set $M\cdot T$ is totally disconnected and contains no cut point of M.

Suppose there exist two points X and Y which belong to the boundary B of a complementary domain D of M such that X and Y are not the extremities of any arc belonging to T and lying, except for end points, within D. There exists an arc XY lying within $\overline{D}\cdot T_1$ such that segment XY is a subset of D. There exist two positive integers d and e such that $T_d^*-\sum_{i=1}^{e}t_i$ contains segment XY and $T_d^*-\sum_{i=1}^{e+1}t_i$ does not. Hence $(T_d^*-\sum_{i=1}^{e}t_i)-(T_d^*-\sum_{i=1}^{e+1}t_i)$ contains a junction point of $T_d^*-\sum_{i=1}^{e}t_i$ contrary to the fact that for any two positive integers r and s $(T_r^*-\sum_{i=1}^{s}t_i)-(T_r^*-\sum_{i=1}^{s+1}t_i)$ contains no junction point of $T_r^*-\sum_{i=1}^{s}t_i$. Hence the supposition that X and Y are not the extremities of any arc belonging to T and lying, except for end points, within D has led to a contradiction. It follows then by Theorem VII that $M-T$ is connected.

Suppose there exists an end point P of T which does not belong to K. Since P is not an end point of T_1 and for any positive integer r there are only a finite number of components of the set $T_1-(L+S_{r+1})$ each of which con-

† See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

tains a point of $T_1 - S_r$ it follows that there exist two positive integers d and e such that P is an end point of $T_d^* - \sum_{i=1}^{e+1} t_i$ but not an end point of $T_d^* - \sum_{i=1}^{e} t_i$. Since P does not belong to K, $(T_d^* - \sum_{i=1}^{e} t_i) - (T_d^* - \sum_{i=1}^{e+1} t_i)$ is a segment lying in a simple closed curve which is a subset of $T_d^* - \sum_{i=1}^{e} t_i$. Hence the point P is a junction point of $T_d^* - \sum_{i=1}^{e} t_i$ and an end point of $T_d^* - \sum_{i=1}^{e+1} t_i$. This contradicts the fact that no junction point of a continuous curve N is an end point of $N - t$, where t is a segment lying in N. Hence all the end points of T belong to K and Theorem XII has been established.

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