ON NORMAL SIMPLE ALGEBRAS*

BY

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1. **Introduction.** Recently published theorems on the direct product of a normal division algebra D of degree m (order m^2) over F and an algebraic field Z of degree (order) r over F have proved to be very important tools for research on division algebras. Of particular value is the use of an integer s called the *index reduction factor* of $D\times Z$. In the present paper new light is thrown on the properties of s by a study of an integer $q=q(Z, D)$ called the *quotient index* of Z and D. This q is the least integer such that the direct product of D and a total matric algebra of degree q contains a sub-field equivalent to Z. It is proved that $r=sq$. The results obtained are also applied to prove an important conjecture of L. E. Dickson made by him in 1926, the so-called *norm condition* that a certain type of algebra be a division algebra.†

2. **Representations of Z by D.** We shall consider algebras over any non-modular field F. Of particular interest will be the two types of algebras *normal division algebras* and *total matric algebras*. The field F itself is a special case of both types.

Definition. *An algebra A is said to be* **associated** *with an algebra B, in symbols*

$$A \simeq B, \tag{1}$$

if A is the direct product

$$A = M \times B, \tag{2}$$

where

$$M \simeq F \tag{3}$$

is a total matric algebra.

Every normal simple algebra A of degree n (order n^2) over F is associated with a normal division algebra D whose degree m is called the *index* of A. In fact $A=M\times D$, $M\simeq F$, and

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† For references to the particular results quoted in the above introduction see the sections following. For applications of my theorem on the index reduction factor see my recent papers in the Bulletin of the American Mathematical Society, these Transactions, Annals of Mathematics, and American Journal of Mathematics, as well as a joint paper by H. Hasse and myself in these Transactions.

$$n = \mu m,$$

where the degree μ of M shall be called the *coindex* of A.

Let Z be an algebraic field of degree (order) r over F and let D be a normal division algebra over F. We shall use the

Definition. *A normal simple algebra $A \simeq D$ will be said to be a representation of Z by D if A contains a sub-field Z_0 equivalent to Z.*

It is well known, from the elementary theory of matrices, that if M is a total matric algebra whose degree is that of Z then M has a sub-field Z_0 equivalent to Z. Hence $A = M \times D$ of coindex r is a representation of Z by D for any D. We may then prove the trivial

THEOREM 1. *There exists a unique algebra*

$$B = H \times D, \ H \simeq F,$$

which is a least representation of Z by D. Its coindex

$$q = q(Z, D),$$

which shall be called the quotient index *of Z and D, is the least coindex of all the representations of Z by D. Every $A \simeq B$ is a representation of Z by D.*

For, as we have seen, there exists at least one representation of Z by D and hence a representation of least coindex. This latter representation is, of course, uniquely determined by D and q. Since B contains a sub-field equivalent to Z so must any $A \simeq B$ contain the same sub-field and be a representation of Z by D.

3. **Algebras commutative with a field.** Let K and K_0 be equivalent sub-fields of a normal simple algebra A. It is well known that there exists a regular quantity y of A such that the equivalence of K and K_0 is given by

$$k \longleftrightarrow k_0 = yky^{-1}$$

for every k of K and k_0 of K_0.

Let C be the set of all quantities of A commutative with every quantity of K. Evidently C is an algebra and we shall say that C is the *sub-algebra of A commutative with K*. Then if K is equivalent to K_0 the algebra C is equivalent to C_0.

For if x is in C then $xk = kx$ for every k of K. Then $(yxy^{-1})\, k_0 = k_0\, (yxy^{-1})$ for every $k_0 = yky^{-1}$ of K_0 so that yxy^{-1} is in C_0. Similarly every x_0 of C_0 defines a quantity $x = y^{-1}x_0y$ in C so that conversely every x_0 of C_0 has the form $x_0 = yxy^{-1}$. Evidently C is equivalent to C_0 under the correspondence $x \longleftrightarrow x_0$.

4. **A set of lemmas.** We shall assume the following three known theorems on normal division algebras D of degree (index) m over F.

LEMMA* 1. *Algebra D has sub-fields of degree m over F.*

LEMMA† 2. *Let Z be an algebraic field of degree r over F. Then $D\times Z \simeq D'$ over Z where D' has index ($=$degree)*

$$m' = \frac{m}{s}$$

such that the index reduction factor s is a divisor of r.

LEMMA 3. *Let $r=se$ in Lemma 2 and let E be a total matric algebra of degree e. Then $D\times E$ contains a sub-algebra D_0 over Z_0 equivalent to D' as over Z.*

By Lemmas 1 and 2 the algebra D' contains a sub-field K of degree m' over Z. The composite L of K and Z has then degree $m'r=m'se=me$ over F. Hence $D\times E$, of degree me, contains a sub-field L_0, equivalent to L, and of degree me. Also L_0 has Z_0 as sub-field.

The sub-algebra C_0 of $D\times E$ commutative with Z_0 contains L_0 and, in fact, D_0. But D_0 is a normal division algebra over Z_0. Hence $C_0=D_0\times G$. If G had order greater than unity it would contain a quantity not in L_0 and commutative with all the quantities of L_0, which is impossible, since L_0 is a maximal sub-field of $D\times E$. Hence $D_0=C_0$ and we have proved

LEMMA 4. *The algebra D_0 of Lemma 3 is in fact the sub-algebra of $D\times E$ commutative with Z_0.*

5. **The principal result.** Let Z_1 be a sub-field of $A\times D$, a normal division algebra over F. If Z is an abstract field equivalent to Z_1 and E is defined as in Lemmas 2, 3, 4, then $D\times E$ contains a sub-field Z_0 equivalent to Z and hence Z_1.

In the algebra $A\times E=M\times(D\times E)$ the sub-algebra commutative with Z_0 is obviously $M\times D_0$. If C is the sub-algebra of A commutative with Z_1 then $C\times E$ is the sub-algebra of $A\times E$ commutative with Z_1. It follows that $C\times E$ is equivalent to $M\times D_0$. Hence $C\times E$ is a normal simple algebra over Z_1, whence $C=H\times D_1$ where H is a total matric algebra and D_1 over Z_1 is equivalent to D_0 as over Z_0. Then $H\times E$ is equivalent to M. We have proved

LEMMA 5. *Let Z be an algebraic field of degree r over F equivalent to a sub-field Z_0 of a normal simple algebra $A=M\times D$ where D has degree (index) m and M has degree μ, index 1. Let the index reduction factor in $D\times Z$ be s, $r=se$. Then*

$$A = H \times (E \times D) \tag{4}$$

* For the proof of Lemma 1 see my *Note on an important theorem on normal division algebras*, Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 649–650.

† For Lemmas 2 and 3 see my Theorems 14 and 18, *On direct products*, these Transactions, vol. 33 (1931), pp. 690–711.

where H and E are total matric algebras, E has degree e, so that e divides μ. The sub-algebra of A commutative with Z_0 is a normal simple algebra $H\times D_0$ over Z_0, where D_0 is a normal division algebra.

Let now $q=q(Z, D)$ be the quotient index of Z and D. By Lemma 3 the algebra $E\times D$ is a representation of Z so that $e\geqq q$. By Lemma 5 with $A\times B$ a least representation of Z by D we have q divisible by e. Hence $q=e$ so that $r=sq$. We have proved our principal result:

THEOREM 2. *Every representation A of an algebraic field Z of order r over F by a normal division algebra D of degree (index) m over F is associated with a least representation B, that is,*

$$A = M \times D = H \times (E \times D) = H \times B,$$

where $M=H\times E$, H, and E are total matric algebras. The quotient index $q=q(Z, D)$, which is the coindex of $B=E\times D$, is a divisor of r, $r=sq$, where $D\times Z=D'$ has degree (index) $m'=m/s$ over Z. If Z_0 in A is equivalent to Z, the sub-algebra of A commutative with Z_0 is a normal simple algebra $H\times D_0$ over Z_0 with D_0 over Z_0 equivalent to D' as over Z.

Thus Theorem 2 is a really simple consequence of the known theorems Lemmas 2 and 3. These lemmas were proved by me as consequences of the uniqueness in the Wedderburn theorem on the structure of simple algebras and so the whole treatment is essentially very elegant and clear.

As a corollary of Theorem 2 with $A=B$ so that H has order unity, we have

THEOREM 3. *A necessary and sufficient condition that a normal simple algebra $A=M\times D$ be a least representation of a sub-field Z by D is that the sub-algebra of A commutative with Z be a division algebra.*

THEOREM 4. *A necessary and sufficient condition that a normal simple algebra A contain sub-fields of degree equal to the degree of A is that A be a least representation of some one of its sub-fields.*

For if A has a sub-field of degree n, the degree of A, then the sub-algebra of A commutative with this field is obviously the field itself and is a division algebra so that A is a least representation. Conversely if A is a least representation of a sub-field Z by D then the algebra D_0 over Z was proved to contain a sub-field L_0 of degree $me=n$, the degree of A.

We also have

THEOREM 5. *Let A be a normal simple algebra of degree n and Z a sub-field of A of degree r so that $n=rt$. Let the sub-algebra of A commutative with Z be a division algebra. Then the index m of A has the value $m=st$, where s is defined in Theorem 2.*

We shall apply the above result to the case $r=p$, a prime, so that $n=pt$, $m=n$ or t, and shall prove an important conjecture of L. E. Dickson.

6. **The norm condition of Dickson.** L. E. Dickson considered normal simple algebras Γ of the following type. He let Γ contain a cyclic sub-field Z of degree p, a prime, over F and hence a quantity j such that $jz=z'j$ for every z of Z where z' is also in Z. He let the algebra Σ in Γ which is commutative with Z be a normal division algebra of degree t over Z so that Γ is composed of all quantities of the form

$$x_1 + x_2 j + \cdots + x_{p-1} j^{p-1}, \quad x_i \text{ in } \Sigma,$$

such that

$$j^p = \gamma = \gamma' \text{ in } \Sigma, \quad j^r x = x^{(r)} j^r \qquad (r = 0, 1, \cdots),$$

where $x^{(r)}$ is in Σ for every x of Σ. Since Σ is a division algebra, Γ is a least representation of Z by the D of Γ and hence Γ has index t or tp by Theorem 5. Thus Γ is either a normal division algebra or the direct product of a normal division algebra by a total matric algebra of degree p. In this latter case we may take Z in this total matric algebra H so that H is a cyclic algebra $H=(1, Z, \Theta)$. By this we imply that H contains a quantity y such that $yz=z'y$, $y^p=1$. But then $y^{-1}=y^{p-1}$, so that $jy^{-1}z=jz^{(p-1)}y^{-1}=zjy^{-1}$ since in fact $z^{(p)}=z$ for every z of Z. Hence jy^{-1} is commutative with every z of Z and is in the algebra of all such quantities Σ. Write $jy^{-1}=X$ in Σ. Then $j=Xy$, $y=X^{-1}j$, and

$$1 = y^p = (X^{-1})(X^{-1})' \cdots (X^{-1})^{(p-1)}\gamma.$$

But $(X^{-1})^{(r)}\cdot(X)^{(r)}=1$, $(X^{(r)})^{-1}=(X^{-1})^{(r)}$ so that

$$X^{(p-1)}\, X^{(p-2)} \cdots X'X\cdot 1$$
$$= X^{(p-1)} \cdots XX^{-1}(X')^{-1} \cdots (X^{(p-1)})^{-1}\gamma = \gamma.$$

Conversely if $\gamma=X^{(p-1)}\cdots X'X$ then $y=X^{-1}j$ evidently has the property $y^p=1$ so that algebra Γ has the cyclic total matric algebra H as sub-algebra and is not a division algebra.

THEOREM 6. *A necessary and sufficient condition that a* Γ *algebra of Dickson be a division algebra is that* γ *be not the norm*

$$X^{(p-1)} \cdots X'X$$

of any quantity X *in* Σ.

We have of course omitted in the statement of Theorem 6 our assumption that Σ is a division algebra, which is taken *here* (but not by Dickson) as a *fundamental part of the definition of* Γ. Professor Dickson conjectured* the above result and proved it for $p=2$, 3 by a computation that it seemed impossible to extend say to $p=5$. He also proved the necessity of the condition. We have here investigated the *structure of* Γ whether or not it is a division algebra and have shown that the above condition is equivalent to the condition that Γ have not or have a total matric sub-algebra.

* *New division algebras*, these Transactions, vol. 28 (1926), pp. 207–234; p. 227.

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