THE MOORE-KLINE PROBLEM*

BY

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It has been shown by Moore and Kline‡ that in order that a closed subset M of the euclidean plane be contained in an arc of the plane, it is necessary and sufficient that (1) M be compact, (2) the maximal connected subsets (components) of M be arcs or points, (3) no inner point of any arc of M be a limit point of the complement (in M) of that arc. A closed point set with these properties we shall call a Moore-Kline set (or M. K. set) and we shall say that a topologic space has the Moore-Kline (M. K.) property if every M. K. subset is contained in an arc of that space. Our problem is the characterisation of spaces which have this property, in the universe of generalised continuous curves: i.e., complete, metric, separable, connected, and locally connected spaces.§ The characterisation which we give is, in an equivalent form, also valid for certain non-metric spaces developed by R. L. Moore, and the space of Aronszajn.|| The paper contains an extension to generalised continuous curves of a recent theorem of G. T. Whyburn,¶ with an independent proof.

1. We shall prove for generalised continuous curves C the equivalence of the two following properties:

A: *If b is an end point of an arc m of C, then for every preassigned $\epsilon>0$ there exists a $\delta>0$ such that if y and z are points of $(C-m)\cdot S(b,\delta)$, the set $(C-m)\cdot S(b,\epsilon)$ contains an arc yz.*

B: *If D is an open connected subset of C and ab is an arc of C such that $(ab-a)\subset D$ and $a\subset F(D)$,** then $D-(ab-a)$ is connected.*††

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‡ R. L. Moore and J. R. Kline, *On the most general closed point-set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, vol. 20, p. 218.

§ These, as is now well known, are locally arc-wise connected. See R. L. Moore, Bulletin of the American Mathematical Society, vol. 33 (1927), p. 141 (abstract) and Colloquium Lectures; K. Menger, Monatshefte für Mathematik und Physik, vol. 36 (1929), p. 212; N. Aronszajn, Fundamenta Mathematicae, vol. 15 (1930), p. 232; C. Kuratowski, Fundamenta Mathematicae, vol. 15, p. 306.

|| For references and the relation between these spaces, L. Zippin, *On a problem of N. Aronszajn and an axiom of R. L. Moore*, Bulletin of the American Mathematical Society, vol. 37 (1931), p. 276.

¶ Abstract, Bulletin of the American Mathematical Society, vol. 36, p. 631, No. 329.

** Throughout, $F(X)=\overline{X}-X$, the boundary of X.

†† This will be recognized as an early plane axiom of R. L. Moore.

For, suppose that C has property B. Let $m=a'b$ be any arc of C, where a' and b are its end points, and let ϵ be preassigned. There is a subarc ab of $(a'b)\cdot S(b, \frac{1}{2}\epsilon)$. In $C-a'a$ cover every point of $ab-a$ with a connected neighborhood of diameter less than $\frac{1}{2}\epsilon$. The sum D of these covering sets belongs to $S(b, \epsilon)$, is open and connected, and contains $ab-a$ but not a. Then, by property B, $D-(ab-a)$ is connected, and it is locally connected since it is open in C. Therefore* it is a generalised continuous curve, and is arcwise connected. Let $\delta<\rho(b, F(D))$.† Then if y and z are any two points of $(C-a'b)\cdot S(b, \delta)$ there is an arc yz in $D-(ab-a)$, therefore in $(C-a'b)\cdot S(b, \epsilon)$, and C has property A. On the other hand, if C does not have property B and, therefore, $D-(ab-a)$ as above is not connected, there is a subarc ab' of ab such that b' is a limit point of at least two distinct components of $D-(ab'-a)$. Let $\epsilon<\rho(b', F(D))$, and it is readily seen that C cannot have property A.

THEOREM. *In order that a generalised continuous curve C have the Moore-Kline property it is necessary and sufficient that it have property* A (*or its equivalent,* B).

2. The condition is necessary. For if C does not have property A it must contain an arc m with end point b, say, and there must exist an $\epsilon>0$ such that, for every integer $n>0$, $(C-m)\cdot S(b, 1/n)$ contains a pair of points y_n and z_n for which $(C-m)\cdot S(b, \epsilon)$ contains no arc y_nz_n. But the point set $m+\sum y_n+\sum z_n$ is obviously an M. K. subset of C, and by the M. K. property of C belongs to an arc L. Then it is obvious that the point b belongs to a subarc bx' of $L\cdot S(b, \epsilon)$ which contains infinitely many of the point pairs (y_n, z_n), so that for some integer k there is an arc y_kz_k in $(C-m)\cdot S(b, \epsilon)$.

3. The condition is sufficient. It is clear, from well known theorems, that an M. K. subset M of C has this simple character that the set N of maximal arcs of M is countable and is a null-family.‡ We may therefore write $N=\sum m_n$, where m_n is a maximal arc of M, and we shall call N the arc set of M. It will be advisable to indicate the main thread of our argument. For an arbitrary M. K. subset M of C we find that there exists in C a tree (acyclic continuous curve) T which contains M. We add to T a properly chosen (inductively) countable set of arcs of C and show that in this sum there exists a tree T' containing M and such that no point of $a_1b_1\equiv m_1$ is a branch point§ of T'.

* We are using, as we shall in the sequel without explicit mention, a theorem of P. Alexandroff, *Sur les ensembles de la première classe et les ensembles abstraits*, Comptes Rendus, vol. 178 (1924), p. 185.

† The distance of b from $F(D)$.

‡ A point set is a null-family if not more than a finite number of its components are of diameter greater than a preassigned $\epsilon>0$.

§ A point of order at least three.

Then, inductively, we establish the existence of a tree T^* containing M such that no point whatever of the arc set N of M is a branch point of T^*. Fixing now on two arbitrary end points p and q of T^* we construct a monotonic decreasing sequence of perfect continuous curves† $K_1, K_2, \cdots$, such that each contains M and further such that for every integer j every point of $N_j=\sum_1^j m_n$ is a cut point between p and q of K_j. We are able to conclude that in their infinite product, $\prod^\infty K_n$, every point of N is a cut point between p and q and in consequence that $\overline{N}$ is contained in an arc pq of $\prod^\infty K_n$.‡ Then we are finished if $\overline{N}=M$. While this is not generally the case, this final obstacle is obviated by a very simple device to which we at once proceed.

3.1. Suppose that M^* is an arbitrary M. K. subset of C. It is clear that the set of points H which are end points of maximal arcs or are point components of M^* is a self-compact totally disconnected point set. Suppose that $\overline{N}^*$, where N^* is the arc set of M^*, is not M^*. Then the set of points $M^*-\overline{N}^*$ is totally disconnected and locally self-compact, and contains a countable dense set (h_n). Let t_1 be any arc with end point h_1 which has no point in common with $\overline{N}^*$, and is of diameter less than 1. On t_1 there is a countable set of mutually exclusive arcs of $C-M^*$, converging to h_1: let these be (t_{1j}) and write $t_1' =\sum_j t_{1j}$. If t_{n-1}' has been defined, let $h_{n'}$ be the first point of (h_n) which does not belong to $\sum_1^{n-1} t_i'$. There is an arc t_n with end point $h_{n'}$ which is of diameter less than $1/n$ and has no point in common with $\overline{N}^*+\sum_1^{n-1} t_i'$, and on this there is a countable set of mutually exclusive arcs (t_{nj}) of $C-M^*$ converging to $h_{n'}$: then $t_n' =\sum_j t_{nj}$. It is readily seen that $M=M^*+ \sum_1^\infty t_n'$ is an M. K. set which contains M^* and is such that $\overline{N}=M$, where N is the arc set of M.

4. We deduce a simple consequence of property A. Suppose that ab is an arc of C such that b is a limit point of $C-ab$. Then there exists a sequence of points (b_n') of $C-ab$ converging to b. Then this contains a subsequence (b_n) such that there is an arc $b_nb_{n+1}\subset C-ab$ of diameter less than $1/n$. It follows readily that b is accessible from $C-ab$, and is an inner point of an arc abb'.

4.1. It will be apparent that the simple continuous curves§ have the Moore-Kline property, and it will be suspected that they are in some way specially related to this property. We devote this and the next section to showing that *if C has any local cut point‖ it is a simple continuous curve.* In

† A perfect continuous curve (hereditary continuous curve) is one whose every subcontinuum is a continuous curve.

‡ We have chosen perfect continuous curves K_n to insure that ΠK_n is a continuous curve.

§ The arc, the simple closed curve, the open curve, or the ray. See R. L. Moore, *Concerning simple continuous curves*, these Transactions, vol. 21 (1920), pp. 313–320.

‖ The point x is a local cut point if there exists an open connected set D, and $D-x$ is not connected.

the main body of our argument we shall be able to suppose that C has no local cut point. For, let y denote any local cut point of C, and D an open connected set such that $D-y$ is not connected. There is an arc xyz such that $xy-y$ and $zy-y$ belong to distinct components of $D-y$. If, now, $y \subset \overline{D-xyz}$, there is a sequence of points (v_n) of $D-xyz$ converging to y and such that either $xy-y$ or $zy-y$ (we shall suppose the first, the cases being entirely similar) belongs to a component of $D-y$ which contains no point of $\sum v_n$. Then if (x_n) is an arbitrary sequence of points of $xy-y$ converging to y, for no n can v_n and x_n be arc-joined in $D-yz$. The contradiction with property A is immediate. Therefore there exists a subarc $x'yz'$ of $xyz=xx'yz'z$ such that every point of this arc is a local cut point. We have shown then that all local cut points are points of Menger-Urysohn order two, and that the set of these is open.

4.2. Suppose, in addition, that C has at least one point g which is not a local cut point. There is an arc gy. Let g' be the first point of gy in order $gg'y$ which is a point of $x'yz'$. It is obvious that g' is the point x' or it is the point z': the cases are similar, and we shall say that g' is x'. Then we have the arc $gx'yz'$. There is a point g'' on gx' such that g'' is not a local cut point and every point of $g''x'$ is a local cut point. If g'' is not g it is an inner point of gx', and we readily conclude that there exists an arc hh', $hh' \cdot gg''x' = h+h'$, $h \subset \langle gg'' \rangle$† and $h' \subset \langle g''x' \rangle$, so that the point h' is not of Menger order two, and therefore not a local cut point. Then $g=g''$. In view of §4 and using the argument above, it readily follows that g is not a point of $\overline{C-gx'yz'}$. Therefore g is of Menger order one and is an end point of C. Then if C contains another local non-cut point f it is the arc fg, and if not it is a ray.

Now if every point of C is a local cut point and also a cut point, C is an open curve. But if C contains one non-cut point, it contains a simple closed curve J and C is J. Then in every case C is a simple continuous curve.

4.3. We signalise an immediate consequence of the arguments above. If C contains no local cut points and ab is any arc of C, then $b \subset \overline{C-ab}$ and, by §4, there exists an arc $abb'=ab+bb'$, where $b'b-b \subset C-ab$.

5. We shall have frequent recourse to the following general lemma: *in any complete metric space C, if P is a perfect continuous curve*‡ *and P_n, $n=1, 2, \cdots$, is a null-family of perfect continuous curves whose sequential limiting set H is totally disconnected and such that $P \cdot P_n \neq 0$, then $P+\sum_1^\infty P_n$ is a perfect continuous curve.*

† If q is an arc, $\langle q \rangle$ denotes the arc minus its end points.

‡ It is understood that these are self-compact.

It is fairly obvious that H is a self-compact, totally disconnected subset of P, and that $P+\sum_1^\infty P_n$ is connected and closed. If we let (p_n) be any set of points such that $p_n \subset P\cdot P_n$, then $\sum p_n$ is compact as subset of P, and $\bar{\rho}_n(p_n, P_n)$† converges to zero as n becomes infinite, because (P_n) is a null-family. Since C is a complete metric space, we readily conclude that $\sum P_n$ is also compact, and $P+\sum_1^\infty P_n$ is a compact continuum. If this contains any subcontinuum not a continuous curve, the latter has a subcontinuum of condensation W. Since H is closed and totally disconnected, W contains a continuum of condensation W' such that $W'\cdot H=0$, and there is an integer n' such that $W'\subset P^*=P+\sum_1^{n'} P_n$. Then P^* is not a perfect continuous curve. But this contradicts the easily established fact that the connected sum of a finite set of perfect continuous curves is necessarily a perfect continuous curve.‡

5.1. *Every M. K. subset M of a generalised continuous curve C belongs to a tree of C.*§ Since the set of points which are end points of maximal arcs or are point components of M is a self-compact totally disconnected point set H, and the arc set N of M is a null-family, it is sufficient to know that there exists in C a tree which contains H. This is a simple theorem which we have had occasion to prove for locally compact continuous curves,|| and this proof may be followed with inessential modification. In fact, using connected neighborhoods to replace the more specialised compact continuous curves of that argument, one quickly establishes the existence of a perfect continuous curve on H, and this contains a subcontinuum¶ irreducible about H. But it is obvious that a continuous curve irreducible about H, or more generally about any M. K. set, is necessarily acyclic, that is, a tree.

5.2. If T is a tree irreducible about M, the end points of T must be points of M and every limit point of end points belongs to M. Now no inner point of an arc ab of M belongs to $\overline{M-ab}$, and therefore every limit point of end points of T belongs to H. Then every limit point of branch points of T must also belong to H, and it follows that the set of branch points of T cannot

† The least upper bound to the set of distances $\rho(p_n, p_n')$ where $p_n' \subset P_n$.

‡ It is believed that this is contained somewhere in the literature, but the author cannot place it. The proof, with the use of the Moore-Wilder lemma, for example, follows traditional lines.

§ For locally compact continuous curves, compare Theorem 2, L. Zippin, *On continuous curves and the Jordan curve theorem*, American Journal of Mathematics, vol. 52 (1930), p. 332.

|| *A study of continuous curves and their relation to the Janiszewski-Mullikin theorem*, these Transactions, vol. 31 (1929), p. 745, Theorem 1.

¶ By a theorem of Wilson, used in the references above: if X is a closed subset of a compact continuum K, then K contains a continuum K^* irreducible about X; i.e., no proper subcontinuum of K^* contains all of X.

be dense on any arc of T. In particular, if ab is any maximal arc of M there must exist two inner points f and g, in order $afgb$ such that no point of fg is a branch point of T. Also, the branch points of T on af, if they are not a vacuous or finite set, must form a sequence converging to a; correspondingly, the branch points on gb if not in finite number converge to b. Further, no one of these branch points is of higher than finite order, or it is a limit point of end points (which is not possible). Moreover, if an inner point x of ab is a branch point, and cx is an arc in any of the branches of T at x (distinct from the two containing ax and bx respectively) there is a point c' on $cx-x$ such that $c'x-x$ contains no point of M and no branch point of T.

5.3. This suggests the following "construction." Let ψ be a curve of C consisting of the three arcs ax, bx, cx, where $ax\cdot bx=bx\cdot cx=cx\cdot ax=x$. We wish to show that there is an arc $c'b$ such that $\psi\cdot c'b=c'+b$ and $c'\subset xc$, and such that no point of $c'b$ is at a distance from xb greater than a preassigned $\epsilon>0$. By §4.3 there is an arc bb' such that $\psi\cdot bb'=b$ (compare §4.3). By property A there is an arc yz, no point of which is at a distance greater than ϵ from xb such that $y\subset xc$, $z\subset xb'$, and $\psi\cdot yz=y+z$. We shall say that yz "covers" points of $<xz>$. If the point b cannot be "covered" in this way, there is a point b'' on xb such that b'' cannot be covered but every point of $<xb''>$ can be so covered. There is a $\delta>0$ such that any two points of $(C-axb'')\cdot S(b'', \delta)$ are arc-joined in $(C-axb'')\cdot S(b'', \epsilon)$. There is an arc $y'z'$ which covers the subarc xz' of xb'', such that no point of $y'z'$ is a distance greater than ϵ from xb and such that $z'\subset S(b'', \frac{1}{2}\delta)$. It is immediate that the point b'' can also be covered, and the arc $c'b$, above, exists.

6. Let M^* be an arbitrary M. K. subset of C. By §3.1 there is an M. K. set $M\supset M^*$, such that $M=\overline{N}$ where N is the arc set of M: $N=\sum m_n$ in maximal arcs of M. By §5.1 there is a tree T irreducible about M. By §5.2 there exists on $ab\equiv m_1$ two inner points f and g, in order $afgb$, such that no point of fg is a branch point of T. Then $T-<fg>=T_f+T_g$, where T_f and T_g are trees containing a and b respectively and $\rho(T_f, T_g)=\rho'>0$; and $M=M\cdot T_f+fg+M\cdot T_g$. By §5.2 the branch points of T_g on $<gb>$ form a sequence† of points (q_n) converging to b, and each q_n is of finite order: say j_n+2. With q_n is associated a finite set of branches (Q_{ni}), $i=1, \cdots, j_n$: here a branch is to be understood as the closure of a component of $T-q_n$ containing neither g nor b. Further, there is in Q_{ni} an arc $c_{ni}''q_n$ such that no point of $c_{ni}''q_n-q_n$ is a point of M or branch point of T_g (§5.2). Let $0<\epsilon_1<\min$ (1, $\frac{1}{3}\rho'$).‡ Then by §5.3, and an induction, there exists a set of arcs $(c_{ni}'b)$, $n=1$,

† We assume explicitly that there are infinitely many branch points.

‡ Read "the smaller (smallest) of the two (several) numbers . . . ."

2, $\cdots$, and $i=1, 2, \cdots, j_n$; $c_{ni}'b\cdot ab=b$, $c_{ni}' \subset c_{ni}''q_n$, and $c_{ni}'b \subset S(q_nb, \epsilon_1/2^n)$.† Since the arcs (q_nb) converge to b, the arcs $(c_{ni}'b)$ form a null-family converging to b and $P_b=\sum\sum c_{ni}'b$ is a perfect continuous curve.‡ We note that $ab\cdot P_b=b$ and that $P_b\cdot T_f=0$. There is, on the arc $c_{ni}'q_n$ of $C_{ni}''q_n$, a point c_{ni} such that $c_{ni}q_n\cdot P_b=0$; and $<c_{ni}q_n>$ as subset of $<c_{ni}''q_n>$ contains no point of M or branch point of T_g. It is readily seen that $(T_g-\sum\sum <c_{ni}q_n>)+P_b$ is closed and connected and contains $M\cdot T_g$. As subset of T_g+P_b, which is "perfect," this contains a tree T_b irreducible about $M\cdot T_g$, and it is seen that no point of $gb-b$ is a branch point of T_b. By a precisely similar argument there is a perfect curve $P_a\subset S(af, \epsilon_1)$, $P_a\cdot(ab+T_b)=a$, such that in T_f+P_a there is a tree T_a irreducible about $M\cdot T_f$, and no point of $fa-a$ is a branch point of T_a. Let T_0 designate T, let $T_1=T_a+m_1+T_b$,§ and let $P_1=P_a+m_1+P_b$. Then we have shown that there exists a perfect curve $P_1\subset S(m_1, \epsilon_1)$ such that T_0+P_1 contains a tree T_1 irreducible about M and such that no point of $<m_1>$ is a branch point of T_1.

Let $N_k=\sum_1^k m_j$, and suppose T_{n-1} constructed so that no point of $<N_n>$‖ is a branch point of T_{n-1}. Let $0<\epsilon_n<\min[(\frac{1}{2})^n, \frac{1}{2}\rho(m_n, N_{n-1})>0]$. By the argument above there exists a perfect curve $P_n\subset S(m_n, \epsilon_n)$ such that in $T_{n-1}+P_n$ there exists a tree T_n irreducible about M on which no point of $<m_n>$ is a branch point. From our choice of ϵ_n it is clear that no point of $<N_n>$ is a branch point of T_n. Now since (m_n) is a null-family with H (see §5.2) as its sequential limiting set, it follows that (P_n) is a null-family with H as sequential limiting set, and $T_n+\sum' P_i$ $(n=0, 1, \cdots$, and the prime indicates that the summation is over values of $i>n)$ is a perfect continuous curve K_n. It is seen that $K_n\supset K_{n+1}\supset M$. Then $\prod^\infty K_n$ contains a tree irreducible about M: let this be T'. Now we know that $(<N_k>)\cdot H=0$, and from our construction $(<N_k>)\cdot P_j$ is 0 if $j>k$. Then it follows that no point of $<N_k>$ can be a branch point of K_j, $j>k$. *Then no point of* $<N>$ *is a branch point of* T'.

7. We interrupt the course of argument to establish a needed consequence of both property A and the assumption that C is without local cut point. We prove A′: *if x is an end point of an arc m of M, then for every preassigned $\epsilon>0$ there exists a $\delta>0$ such that if y and z are points of $(C-N)\cdot S(x, \delta)$ they may be arc-joined in $(C-N)\cdot S(x, \epsilon)$.*

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† For $S(X, \epsilon)$ read the set of points whose distance from X is less than ϵ.

‡ We shall sometimes call these perfect curves, and sometimes "perfect."

§ It is remembered that $m_1\equiv ab$.

‖ By an extension of terminology, whenever X is a point set whose components are arcs, $<X>$ will denote the set of open arcs.

Let x denote an end point of an arbitrary arc $m \equiv x''x$ of M. There is an open connected subset D^* of C containing x, of diameter less than a preassigned ϵ, and such that $x''x \cdot F(D^*) = x'$, where x' is some inner point of $x''x$. Let H^* denote the set of points of M which belong to $F(D^*)$ or which are contained in D^* and belong to an arc of M having at least one point in common with $F(D^*)$. Let $H'' = H^* - x'x$. Now x' is not a limit point of H'' since $H'' \subset M - x''x'x$. If t is any point of $\overline{H}'' - H''$, $t \subset M \cdot D^*$ and we readily conclude that infinitely many of the arcs of M are of diameter greater than $\frac{1}{2}\rho\{t, F(D^*)\}$. Since this is impossible, we have that H'' is closed. It follows that $D^* - D^* \cdot H''$ is open, and contains an open component $D \supset xx' - x'$; then $D \cdot H'' = 0$. Suppose that m' is any maximal arc of M distinct from $x''x'x$ which has a point z in D. If m' does not belong to D it has a point z' on $F(D)$ such that $zz' - z \subset D \subset D^*$. Now if z' is a point of $F(D^*)$ it follows from definition that $zz' \subset H''$, which is impossible since $H'' \cdot D = 0$. Then z' is in D^* and since D is a component of $D^* - D^* \cdot H''$ we conclude that $z' \subset H''$. But it is clear that in this case also $zz' \subset H''$. Then our contradiction shows that if $m' \cdot D \neq 0$, $m' \subset D$. Now let $B'' = N \cdot D$, where N is the arc set of M. We have shown that if $B'' \cdot m' \neq 0$, $B'' \supset m'$. It is clear that $B'' \supset xx' - x'$.

We have to show that $D - B''$ is connected.† Otherwise B'' contains a subset B which is closed relative to D and is such that every point of B is a limit point of at least two distinct components of $D - B$. Suppose that $m' \equiv yy'y''$ is an arc of B'' such that the inner point y' is a point of B. Then if some point w, in order $yy'wy''$ say, is not a point of B there is a first point w' in order $ww'y'$ such that w' is a point of B. It follows at once that the arc $yy'w'$ does not have property A at the point w'. We may conclude that if B contains an inner point of an arc m'' of M it contains the entire arc. Analogously, if B contains an inner point of $x'x$ it contains $xx' - x'$.

Now let N'' be the set of those end points of the arc set N which are at the same time points of B: we have shown that this is not vacuous if B is not vacuous. Let t be a point of N''. Since t is a point of D, there is a neighborhood U of t such that $\overline{U} \subset D$. Now although B is closed only relative to D, $\overline{U} \cdot D$ is closed absolutely (that is, in C). Since $B \subset N$, it follows that $\overline{U} \cdot N''$ is closed absolutely and $\overline{U} \cdot N''$ is self-compact and countable. Therefore it contains an isolated point t^*. Finally, if t^* is an end point of an arc of B this arc will not have property A at the point t^*, and if t^* is an isolated point in B it is necessarily a local cut point of C. This establishes A$'$.

8. We shall come at once to the principal argument of this paper and as-

† This establishes the local connectedness if we then regard D as an arbitrary neighborhood of any point of it.

sume, reserving §9 for its proof, that there exists in C a tree T^* irreducible about M and such that *no point whatever of N is a branch point of T^*.*† Then let p and q be arbitrary end points of T^*. We shall proceed to construct an arc, which we designate by $p(N)q$, which has the end points p and q and which contains N.

Suppose that T^* contains a ψ-curve, $\psi = px+cx+qx$, and that c is an end point of a maximal arc of N which is contained in cx. If c is an end point of T^*, there is a point c' and an arc cc' such that $cc'\cdot T^* = c$. If c is not an end point, let cc' designate an arc of T^*, $cc'\cdot cx = c$. Let an $\epsilon > 0$ be preassigned. We wish to show that there exists an arc st, $st \subset S(cx, \epsilon)\cdot\{C-(N+cx)\}$, $s \subset <cc'>$, and $t \subset <pq>$. If, for any n, $m_n\cdot cx \neq 0$, $m_n \subset cx$ or some point of m_n is a branch point of T^*. If $m_n\cdot c \neq 0$, $m_n \subset cx$ or c is not an end point of a maximal arc of N belonging to cx. Since x is a branch point it is surely not a point of N. Therefore, for any n, if $m_n\cdot cx \neq 0$, $m_n \subset cx$. Then there exists an ϵ' such that if, for some j, $m_j\cdot S(cx, \epsilon') \neq 0$, and also $m_j\cdot cx = 0$, then $\delta(m_j) < \frac{1}{4}\epsilon$.

Now by §5.3 there is an arc $s't'$ in $(C-cx)\cdot S(cx, \epsilon')$, $s' \subset <cc'>$ and $t' \subset <pq>$. If s' or t' is a point of N, it is clear that there exists an arc ss' and an arc tt', where s is a point of $<s'c>\cdot(C-N)$, t is a point of $<xt'>\cdot(C-N)$, such that the arc st in $ss'+s't'+t't$ belongs to $S(cx, \epsilon')$. Suppose that m is any arc of N such that $st\cdot m \neq 0$. Then certainly m does not belong to cx, and $m\cdot cx = 0$. Also, $m\cdot S(cx, \epsilon') \neq 0$. Then $m \subset S(cx, \epsilon)$. Now if (m_k) denotes the set of arcs of (m_n), where k ranges over some particularised subset of the positive integers, which have points in common with st, then $st + \sum(m_k) \subset S(cx, \epsilon)$. Since (m_k) is a null-family, and each arc of it has a point on st, the sequential limiting set also belongs to st. Then there exists an open connected set D^*, and $st+\sum m_k = \overline{st+\sum m_k} \subset D^* \subset S(cx, \epsilon)\cdot(C-cx)$. We must show, finally, that there is an arc st in $D^*\cdot(C-N)$.

But this is the proof of §7, with slightest modification. One begins at the third line of the second paragraph of that section, reading "Let H^* $\cdot\cdot\cdot$." Then let $H'' = H^*$, and omit the next line which relates to a set which does not, in our new argument, exist. Now $\overline{(st+\sum m_k)}\cdot H'' = 0$. Let D be the component containing $\overline{st+\sum m_k}$, and $D\cdot H'' = 0$. In the next line omit *distinct from $x''x'x$*, and omit the last line of this and the last line of the next paragraph (these relating to $x''x'x$).

8.1. Now if pq is the arc pq of T^* it is clear at once that either $m_1 \subset pq$ or $m_1\cdot pq = 0$. If $N_1 \equiv m_1 \subset pq$, let $p(N_1)q = pq$ and $T_1^* = T_0^* \equiv T^*$. If $m_1\cdot pq = 0$ there is in T^* the ψ-curve $\psi_1 = px_1 + c_1x_1 + qx_1$ where c_1 is the point a_1 or the point b_1 so that $c_1x_1 \supset m_1$; clearly x_1 is not a point of N (being branch point of

† Compare §6, where this is shown for $<N>$.

T^*). If c_1 is an end point of T^*, there is a point c and an arc cc_1 such that $cc_1 \cdot T^* = c_1$ and $\delta(cc_1) < 1$ (compare §4.3).† If c_1 is not an end point of T^*, let cc_1 designate an arc of T^* such that $cc_1 \cdot c_1x_1 = c_1$. From §8 we may conclude that for any preassigned $\epsilon > 0$, there is an arc s_1t_1 such that (1) $s_1t_1 \subset S(c_1x_1, \epsilon) \cdot (C - \{N + c_1x_1\})$. We may suppose, further, that (2) s_1 is the only point which s_1t_1 has in common with that component of $(T^* + c_1c) - c_1$ which contains c; then the arc s_1c_1 of $T^* + c_1c$ has c_1 only in common with c_1x_1 and s_1 only in common with s_1t_1; and (3) t_1 is the only point of s_1t_1 which belongs to the sum of the two components of $T^* - x_1$ which contain p or q respectively; for definiteness we suppose that this is the component containing q, and there is in T^* an arc t_1x_1', where x_1' (which may be t_1) is a point of $< x_1q >$.

8.2. It is fairly intuitive that if y_1z_1 is any arc of pq with x_1 as inner point, then it is possible to choose the ϵ above so that $x_1' \subset < y_1x_1 > + < x_1z_1 >$. Rigorously: there is an ϵ' such that $px_1q \cdot S(x_1, \epsilon') \subset < y_1z_1 >$ and a δ such that if t is any point of $T^* \cdot S(x_1, \delta)$ then the arc tx_1 of T^* belongs to $S(x, \epsilon')$. Let y' and z' be points in order $y_1y' \subset x_1z' = z_1$ such that $y'x_1z' \subset S(x_1, \delta)$. Now $T^* - (< y'x_1 > + < x_1z' >)$ contains an at most finite number of components, $x_0, x_1, \cdots, x_n$, which have any point in $T^* \cdot \{C - S(x_1, \delta)\}$; one of these, say X_0, is a tree containing c_1x_1, and $X_0 \cdot \overline{X}_i = 0$, $i = 1, \cdots, n$. Let $\epsilon < \rho(X_0, \sum_1^n \overline{X}_i)$. Then we shall suppose the ϵ of the previous paragraph to have been so chosen that $0 < \epsilon < 1$, and $\delta(x_1x_1') < 1$, and we shall designate it by ϵ_1.

8.3. Since $s_1t_1 \cdot c_1x_1 = 0$, it is clear that at most a finite number of the components of $T^* - c_1x_1$ have points in common with s_1t_1. Then it is not difficult to define an arc, which we designate by $s_1(*)t_1$, and which has the following detailed structure: $s_1(*)t_1 = s_1t_1' s_1' t_1'' \cdots s_1^{k_1}t_1$, where s_1t_1', $s_1't_1''$, $\cdots$, $s_1^{k_1}t_1$ are non-degenerate arcs of s_1t_1 with end points only on T_0^*,‡ while $t_1's_1'$, $t_1''s_1''$, $\cdots$, $t_1^{k_1}s_1^{k_1}$ are subarcs (or points) of T_0^* corresponding to different components of $T_0^* - c_1x_1$.§ Let $p(N_1)q$ denote the arc $px_1c_1s_1(*)t_1x_1'q$, where $px_1 + x_1'q \subset p(N_0)q$. Let T_1^* be a continuum of $T_0^* + s_1(*)t_1$ irreducible about $M + p(N_1)q$. Then T_1^* is necessarily a tree because the components of $M + p(N_1)q$ are arcs or points.‖ It is readily seen that T_1^* is irreducible about M, that it has no point of N as branch point, and that every point of $N_1 \equiv m_1$ separates p and q.

8.4. We suppose, for induction, that c_kx_k, s_kt_k, $s_k(*)t_k$, $p(N_k)q$, and T_k^* have been defined for all $k \leqq n-1$ so that (1_k) $c_kx_k \subset T^*$; (2_k) $s_kt_k \subset S(c_kx_k,$

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† It will be appreciated that, in this case, T^* and m_1 are locally identical at c_1.

‡ To prepare for an induction we write T_0^* for T^* and $p(N_0)q$ for the arc pq of T^*.

§ This will be recognised as traditional "procedure" from "first point on . . . to last point," etc.

‖ One recalls that $N \cdot s_1t_1 = 0$.

$1/k$); (3_k) T_k^* is irreducible about M and has no point of N for branch point; (4_k) $N_k=\sum_1^k m_i\subset p(N_k)q$ which is the arc pq of T_k^*; (5_k) for $i<k$, $c_ix_i\cdot c_kx_k=0$, *or* $=x_k$, *or* there is a j, $i<j<k$, such that $c_ix_i\cdot c_kx_k\subset x_jx_j'$ (of T_{j-1}^*);† (6_k) (a) $\delta(x_kx_k')<1/k$, (b) $x_kx_k'\cdot T_k^*\subset T^*$;† ($7_k$) if x is any point of $p(N_k)q$ which is a branch point of T_k^*, then x is an inner point of an arc y_1xy_2 of $p(N_k)q$ such that if $(C-T^*)\cdot y_ix\neq 0$, $i=1, 2$, then $(y_ix-x)\cdot M=0$, and y_ix-x contains no branch point of T_k^*. The proof of (7_1) is an easy consequence of the structure of $s_1(*)t_1$: we shall give the details in their proper place (see §8.8).

8.5. Let t be any point of T_{n-1}^* which does not belong to $(pN_{n-1})q$. There is an arc tt' of T_{n-1}^* such that $tt'\cdot p(N_{n-1})q=t'$. Now suppose that t does not belong to T^*. Then tt' contains an arc α such that $\alpha\cdot T^*=0$. Now if α does not belong to T_{n-2}^* it has a subarc β and $\beta\cdot T_{n-2}^*=0$. Since $\beta\subset\alpha\subset T_{n-1}^*$, it follows that $\beta\subset p(N_{n-1})q$. But $tt'\cdot p(N_{n-1})q=t'$. Then there must exist an integer j, $1\leqq j\leqq n-2$, such that $\alpha\subset T_i^*$, $i\geqq j$, but not in T_{j-1}^*, where $T_0^*=T^*$. Then α has a subarc γ such that $\gamma\cdot T_{j-1}^*=0$, and $\gamma\subset p(N_j)q$. Since γ does not belong to $p(N_{n-1})q$ there is an integer k, $j<k\leqq n-1$, such that $\gamma\subset p(N_{k-1})q$ but not in $p(N_k)q$. Since $px_k+x_k'q$ of $p(N_{k-1})q=px_kx_k'q$ is contained in $p(N_k)q$, it follows that $\gamma\subset x_kx_k'$ of T_{k-1}^*. But $\gamma\subset\alpha\subset T_k^*$, and it follows from (6b) of §8.4 that $\gamma\subset T^*$. The contradiction shows that $tt'\subset T^*$. It shows also that every branch point of T_{n-1}^* which belongs to $p(N_{n-1})q$ is a point of T^*, for if t' be such a point we can obviously find for it an arc corresponding to tt' above.

8.6. Now, either m_n belongs to $p(N_{n-1})q$ or $m_n\cdot p(N_{n-1})q=0$. In the first case, let $p(N_n)q=p(N_{n-1})q$ and $T_n^*=T_{n-1}^*$; the sets c_nx_n, etc., are vacuous. In the second case, there is in T_{n-1}^* the ψ-curve, $\psi_n=px_n+c_nx_n+qx_n$, where c_n is the end point a_n or b_n of m_n so that $c_nx_n\supset m_n$. We have seen above that $c_nx_n\subset T^*$ and this is (1_n) (of §8.4). It is obvious that $c_nx_n\cdot c_{n-1}x_{n-1}$ is vacuous or it is the point x_n. Let τ be a subarc of $c_ix_i\cdot c_nx_n$, $i<n-1$. Then τ does not belong to $p(N_{n-1})q$, but $\tau\subset p(N_i)q$. Then there is a k, $i<k<n-1$, such that $\tau\subset p(N_{k-1})q$ but is not contained in $p(N_k)q$. Then $\tau\subset x_kx_k'$ of T_{k-1}^*, and we see that (5_n) holds.

8.7. Let $G(x_n)$ be the component of $T_{n-1}^*-N_{n-1}$ which contains x_n, and $\rho_n=\rho(T_{n-1}^*-G(x_n),\ c_nx_n)>0$. Let ϵ_n be a number greater than zero such that (1′) $\epsilon_n<1/n$, (2′) $\epsilon_n<\frac{1}{4}\rho_n$, and (3′) $\epsilon_n<\frac{1}{2}\epsilon_{n-1}$. We have a fourth restriction (4′) to impose upon ϵ_n, but it will be convenient to suppose this made and postpone for a moment its consideration. Now if the point c_n of c_nx_n is an end point of T_{n-1}^*, let c^nc_n be an arc, $\delta(c^nc_n)<1/n$, such that $c^nc_n\cdot T_{n-1}^*=c_n$.‡

† It is to be borne in mind that x_ix_i' denotes *always* the arc of T^*.

‡ For this, and the next line, one follows §8.1 replacing 1 by n.

Otherwise, let $c^n c_n$ designate an arc of T^*_{n-1} such that $c^n c_n \cdot c_n x_n = c_n$. Let $s_n t_n$ be an arc, defined in complete analogy with $s_1 t_1$, such that (1) $s_n t_n \subset S(c_n x_n, \epsilon_n) \cdot (C - \{N + c_n x_n\})$, and (2) and (3) are parallel to (2) and (3) of §8.1. Then we may suppose that $s_n(*)t_n = s_n t_n' s_n'' t_n'' \cdots s_n^{k_n} t_n$; $p(N_n)q = p x_n c_n s_n(*) t_n x_n' q$, where $px_n + x_n' q \subset p(N_{n-1})q$; and T_n^* irreducible about $M + p(N_n)q$, have all been defined. Then it is clear at once that (2_n), (3_n), and (4_n) all hold. We have established (1_n) and (5_n) in §8.6, so that there remain (6_n) and (7_n) to complete our induction.

8.8. Let $y_1 x_n y_2$ be the arc of (7_{n-1}).† Clearly we may suppose that $\delta(y_1 x_n y_2) < 1/n$. Then, as we have seen in §8.2, we may choose ϵ_n such that (4′) $x_n' \subset \langle y_1 x_n y_2 \rangle$, where the argument of that section permits us to express this choice quite formally. Then $(6_n\mathrm{a})$ is immediately verified. Since x_n' is not x_n we shall say for definiteness that $x_n' \subset \langle x_n y_2 \rangle$. Now if $x_n x_n' \cdot (C - T^*) = 0$, $(6_n\mathrm{b})$ is verified at once. But if $x_n x_n' \cdot (C - T^*) \neq 0$, then no point of $\langle x_n x_n' \rangle$ is a branch point of T^*_{n-1} or a point of M. In this event, since $T_n^* \supset p(N_n)q$ and is irreducible about M it follows that T_n^* contains no point of $\langle x_n x_n' \rangle$. Then $(6_n\mathrm{b})$ holds.

To verify (7_n) one bears in mind §8.5, that $T^*_{n-1} - p(N_{n-1})q \subset T^*$, and writes for $p(N_n)q$ its detailed structure: $p(N_n)q = p x_n c_n s_n t_n' s_n' t_n'' s_n'' \cdots t_n^{k_n} s_n^{k_n} t_n x_n' q$. From the analogy with $s_1 t_1$, no point of $\langle s_n t_n' \rangle + \langle s_n' t_n'' \rangle + \cdots + \langle s_n^{k_n} t_n \rangle$ is a branch point of T_n^*. On the other hand, $x_n c_n + t_n' s_n' + t_n'' s_n'' + t_n^{k_n} s_n^{k_n} \subset T^*$. Again, $px_n + x_n' q \subset P(N_{n-1})q$. It will be clear that we need only discuss points of $c_n s_n + t_n x_n'$, because these arcs are, in a sense, ambiguous. Thus, if c_n was an end point of T^*_{n-1} the point $s_n \subset c^n c_n$ of $C - (T^*_{n-1} - c_n)$, and no point of $c_n s_n$ can be a branch point of T_n^*. If c_n was not an end point of T^*_{n-1}, the arc $c_n s_n$ is an arc of T^*_{n-1} and belongs to T^*. If t_n is not the point x_n', the arc $t_n x_n'$ is a non-degenerate arc of points of T^*. But if t_n is x_n', no point of $s_n^{k_n} x_n' = s_n^{k_n} t_n x_n'$ is a branch point of T_n^*. Then our induction is complete.

8.9. Accordingly, we suppose T_n^*, $s_n t_n$, $s_n(*)t_n$, $c_n x_n$, to have been defined for all values of $n = 0, 1, 2, \cdots$. It may have happened, but is immaterial to the argument, that for some values of n, $s_n t_n$, etc., are vacuous: T_n^* is always defined. We have seen that $c_n x_n \subset T^*$. Now if more than a finite number of these are of diameter greater than a preassigned $\epsilon > 0$, it follows from well known theorems of Wilder on trees that there must exist an arc X of T^* of diameter at least $\frac{1}{2}\epsilon$ which belongs to infinitely many of these. But if $X \subset c_i x_i \cdot c_k x_k$, $i < k$, then, for some $j > i$, $X \subset x_j x_j'$. But $d(x_j x_j') < 1/j$, and it is immediate that there is an n such that X does not belong to $c_i x_i$, $i > n$. Then $(c_n x_n)$ is a null-family, and it has the sequential limiting set H of (m_n). Then

† § 8.4, for $k = n-1$: x_n is a branch point of T^*_{n-1} and $x_n \subset p(N_{n-1})q$.

(s_nt_n) is a null-family with sequential limiting set H. To apply our lemma quite rigorously we let (s_it_i) denote those arcs of (s_nt_n) which have a point in common with T^* and (s_jt_j) the set which do not. For the latter it is clear from the preceding section that $\delta(c_js_j)<1/j$, so that the set $(c_js_jt_j)$ is a null-family with limiting set H. Then $\Gamma_n = T_n^* + \sum' s_it_i + \sum'' c_js_jt_j$ is a perfect continuous curve, $n=0, 1, 2, \cdots$, and $\Gamma_n \supset \Gamma_{n-1} \supset M$. Then $\prod^\infty \Gamma_n$ contains a continuous curve Γ irreducible about M. Now m_n separates T_n^* between p and q. Since no point of m_n is a branch point of T_n^*, $T_n^* - m_n$ is the sum of two components P and Q containing p and q respectively, and $\rho(P, Q) = \rho'' > 0$. Then for every k, $k>n$, ρ_k of §8.7 will be less than ρ'', and from restrictions $2'$ and $3'$ on ϵ_k it follows by customary arguments that every point of m_n separates p and q in Γ_n: therefore in Γ. Now Γ contains an arc pq which we designate by $p(N)q$, and $p(N)q \supset m_n$ for every n. Then $p(N)q \supset N$, and consequently $p(N)q \supset \overline{N} = M \supset M^*$, and M^* is our original and arbitrary Moore-Kline subset of C.

9. Then our principal theorem is completely proved when we have justified the assumption of §8 that there exists in C a tree T^* irreducible about M and such that *no point of N is a branch point of T^**. We have seen in §6 that there is a tree T' irreducible about M which has no point of $<N> = \sum <m_n>$ for a branch point. If no point of (a_n) or (b_n) is a branch point of T', then T' is the tree T^*. We may suppose, possibly by a reordering of the arcs and a relettering of the end points of one of them, that the point b_1 of m_1 is a branch point of T'; it will be convenient to let b designate the point b_1. Now $T' - <m_1> = T + T^0$, where T and T^0 are trees, b is at least an ordinary point of T and $T^0 \supset a_1$ (or, possibly T^0 is a_1). It will be convenient to let the term *a branch of* T designate the closure of a component of $T-b$.† Let (B_n), $n=1, 2, \cdots$, denote these branches.‡ It is well to have in mind the discussion of §5.2: we shall conclude from it that if bx is any arc of T with end point b, since $bx\cdot(C-M)\neq 0$, bx contains a sequence of arcs converging to b which have on them no point of M and no branch point of T.

Choose an ϵ, $0<\epsilon<\min \{1, \delta(B_1), \delta(B_2), \frac{1}{3}\rho(T, T^0)\}$. Let t denote an end point, distinct from b, of B_1, and let t' be a point of the arc bt such that $bt' \subset S(b, \epsilon)$. Now at most a finite number of the branches of T are of diameter greater than $\epsilon/3$. Then there exists a set L_1 which is the sum of a finite set of arcs such that (1) $L_1 \subset S(b, \epsilon)\cdot(C-N)$,§ (2) each arc of L_1 has an end point

† Strictly, b may not be a branch point, and generally there are other branches; for the moment only these sets will concern us.

‡ Although we have assumed explicitly that these are in infinite number, we make the convention that when a set B_k does not exist, then it is the point b.

§ By §7.

on bt' and an end point on some branch B_i, $i>1$, whose diameter exceeds $\epsilon/3$, (3) each branch B_i of diameter greater than $\epsilon/3$ has at least one point in common with L_1. Let $B_1, B_{n_2}, \cdots, B_{n_k}$ denote the branches which have a point in common with L_1: they are necessarily finite in number since $L_1 \cdot b=0$. Then we may express T as the sum of two trees, $T=J+J'$, where $J=\sum B_j$, $j=1, n_2, \cdots, n_k$, so that b is of order k on J, and $J'=\sum' B_i$ summed over all values of i not equal to j above. Then $J' \cdot L_1=0$, $J' \cdot J=b$, and $\delta(J')<2\epsilon/3$ since $\delta(B_i)<\epsilon/3$ if B_i is a branch of J'. It will be seen that there exists in J a set of k arcs, Z_{1j}, which contain no point of M and no branch point of J and no point of L_1, $Z_{11} \subset bt$, $Z_{1j} \subset B_j (j=1, n_2, \cdots, n_k)$, such that $(J-\sum <Z_{1j}>)+L_1=J''+K_1$ where the tree J'' contains b and is of diameter less than $\epsilon/3$ while the perfect curve $K_1 \supset L_1+t$. Then $J_1=J'+J''$ is a tree, and $\delta(J_1)<\epsilon$. Let x_1 and y_1 be the end points of Z_{11} in order bx_1y_1t. We have the following relations: (1) $M \cdot T \supset M \cdot J_1+M \cdot K_1$, (2) $J_1 \cdot K_1=0$, (3) $J_1 \cdot x_1y_1=x_1$, (4) $x_1y_1 \cdot K_1=y_1$. There exists an open connected set U_1 such that $U_1 \cdot (x_1y_1+K_1)=0$, and $S(b, \epsilon) \supset U_1 \supset J_1-x_1$.

9.1. Suppose we have defined

$$I_n \equiv K_1+y_1x_1+K_2+y_2x_2+\cdots+y_{n-1}x_{n-1}+K_n+y_nx_n+J_n$$

where two sets on the right are without common point unless they are adjacent, and in this case one of them is an arc and the only common point is the juxtaposed end point; and the open connected set U_n, $J_n-x_n \subset U_n \subset \overline{U}_n \subset S(b, \epsilon_n)$, where $\epsilon_n<(1/n)$. Further: K_i is "perfect," J_n is a tree containing b, y_ix_i is an arc of bt' containing no point of M or branch point of T', and there is the order $bx_ny_nx_{n-1} \cdots x_1y_1t't$. Finally, $M \cdot T=M \cdot I_n$. Let $0<\epsilon_{n+1}<(1/n+1)$, and $\overline{S}(b, \epsilon_{n+1}) \subset U_n$. Then we can define in $S(b, \epsilon_{n+1})$ a set L_n which is the sum of a finite set of arcs each with an end point on bx_n, and an arc $x_{n+1}y_{n+1}$ of bx_n (in order: $bx_{n+1}y_{n+1}x_n$), and a tree J_{n+1}, and an open connected set U_{n+1}, *and* we can define in U_n a perfect curve K_{n+1}, *such that* replacing J_n by $K_{n+1}+J_{n+1}$ and U_n by U_{n+1}, the resulting sets I_{n+1} and U_{n+1} complete the induction.†

Now $bt+\sum_1^\infty K_n$ is "perfect," so that $b+\sum x_ny_n+\sum K_n$ is "perfect" and this contains a tree Y irreducible about $M \cdot T$. It is clear that b is an end point of Y. It should be clear also that $Y+m_1+T^0$ is a tree irreducible about M with no point of $<N>$ as branch point, *and on which the point $b \equiv b_1$ is not a branch point.* Now $P=bt'+\sum_1^\infty L_n$ is "perfect," $P \subset S(b, \epsilon)$, and $Y \subset T'+P$. Let X denote $\sum x_ny_n$.

If a_1 is an end point of T', $T^0=a_1$. Then let $T_1'=Y+m_1$. If a_1 is an ordinary

† The step by step details should be obvious from the preceding paragraphs, and we pretend merely to have given these a precise form.

point of T', it is an end point of T^0 and there exists a set of arcs of T^0 which we designate by X', such that these contain no point of M or branch point of T^0, and such that a_1 is a point component of $T^0-<X'>$. Then let $T_1'=T^0+m_1+Y$, $P_1=P$, and $X_1=X'+m_1+X$. If a_1 is a branch point of T', therefore at least an ordinary point of T^0, there exists, by precisely the argument above, a perfect curve $P'\subset S(a_1, \epsilon)$, and a set of arcs X' such that in T^0+P' there is a tree Y' irreducible about $M\cdot T^0$ and (1) a_1 is an end point of Y', (2) X' contains no point of M or branch point of Y' and a_1 is a point component of $Y'-<X'>$, (3) no point of $Y'\cdot <N>$ is a branch point of Y'. Let $P_1=P+P'$, $X_1=X+m_1+X'$, and $T_1'=Y+m_1+Y'\subset T'+P_1$: $P_1\subset S(m_1\cdot\epsilon)$. Then T_1' is irreducible about M, has no point of $<N>$ *and no point of* m_1 as branch point, and a_1 and b_1 are point components of $T_1'-<X_1>$.

9.2. Suppose T_{n-1}' defined. Then let $G(m_n)$ denote the component of $T_{n-1}'-\sum_1^{n-1}<X_i>$ which contains m_n. Let $\rho_n=\rho(T_{n-1}'-G(m_n), m_n)>0$. Choose ϵ_n, $0<\epsilon_n<\min(\frac{1}{4}\rho_n, \epsilon/n)$. By precisely the arguments above there exists a perfect curve $P_n\subset S(m_n, \epsilon_n)$ and a set of arcs X_n such that in $T_{n-1}'+P_n$ there is a tree T_n' irreducible about M which has no point of $<N>$ and no point of $N_n=\sum_1^n m_i$ as branch point, and the points of E_n where E_n denotes $\sum_1^n(a_i+b_i)$ are point components of $T_n'-\sum_1^n<X_i>$, the latter set containing no point of M or branch point of T_n'. It is not difficult to see that $\Gamma_n=T_n'+\sum_n^\infty P_i\supset\Gamma_{n+1}\supset M$ is "perfect" and contains no point of N_n as branch point. Then by familiar arguments (§§6, 8.9) there is in $\prod_1^\infty\Gamma_n$ a tree T^* irreducible about M with no point whatever of N as branch point.

We have justified our assumption in §8, and completed the proof of the Moore-Kline Theorem. It is not a difficult consequence of the self-compactness of a Moore-Kline set that the equivalence of the Moore-Kline property and property B can be extended to the non-metric spaces of Moore (see his Colloquium Lectures).

10. If C is an arbitrary generalised continuous curve having the Moore-Kline property, and is not a simple continuous curve, and if M is an arbitrary Moore-Kline subset of C, then in order that two points x and y of C shall be end points of an arc xy of C containing M, *it is necessary and sufficient that* (1) x and y are not end points of the same arc of M (unless, trivially, M *is* an arc xy), (2) neither x nor y is an inner point of any arc of M, (3) neither x nor y is an end point of a maximal arc of M and at the same time a limit point of the complement in M of that arc. Whether x and y belong to M or not, $M+x+y$ is an M. K. set. Let us see, first, that if z is a point of an M. K. set M' of C and is not a point of any arc of M', then there exists an arc zq, where q is not determined, and $zq\supset M'$. For, there is some arc $ab\supset M'$. If z

is an inner point of this arc, then by precisely the argument of the first paragraphs of §9, there exists a tree T^*† containing M' which has z as end point. Then, calling p the point z, it follows from §8 that there is an arc $zq \supset M'$.

To return to x and y. There is some arc $vw \supset M$. Since x and y are not end points of the same arc of M, there exist open connected sets D_x and D_y containing x and y respectively, $D_x \cdot D_y = 0$, and $M \subset D_x + D_y$. If either x or y is an end point of an arc of M one of them is, and let us assume that x is such a point and is end point of a maximal arc xz of M. Now $D_x - (xz - z)$ is an M. K. space, and from (3) above $M \cdot D_x - (xz - z)$ is an M. K. set, and z is not an end point of an arc of this set. There is in $D_x - (xz - z)$ an arc zq which contains $D_x \cdot M - (xz - z)$, and $qz + zx$ is an arc $qx \supset M \cdot D_x$. Similarly there is an arc $q'y$ in D_y, $q'y \supset M \cdot D_y$. Now $C - \{(xq - q) + (yq' - q')\}$ is a generalised continuous curve (one recalls §7, for example) and contains an arc qq'. Then $xq + qq' + q'y$ is the desired arc xy.

It can be proved that if M is any M. K. set of C and is homeomorphic with a given set M^* of a line segment a^*b^*, there exists in C an arc ab containing M and preserving that ordering of the points of M which is induced upon them by the homeomorphism.‡ We shall not give this proof, for which the methods of the paper and the following additional property of C suffice: *if xyz is any arc of C, every inner point y is accessible from $C - xyz$* (we have seen this for the end points). With a proof of this last, which seems curious and of interest in itself, we shall conclude the problem.

On the hypothesis of this section, it is clear that y is a limit point of $C - xyz$. There exists an open connected set D_y, *of arbitrarily small diameter*, such that $D_y \cdot xyz = \langle x'yz' \rangle$, where x' and y' are inner points of xyz, and $\overline{D}_y \cdot xyz = x'yz'$. It may happen that $\langle x'yz' \rangle$ separates D_y, but in this case every point of $\langle x'yz' \rangle$ and in particular the point y, is a limit point of every component of $D_y - \langle x'yz' \rangle$: otherwise property A cannot hold (compare §7). But then $(D_y - \langle x'yz' \rangle) + y$ is connected. Therefore $y + (C - xyz)$ is connected and locally connected, and arcwise connected. Then some arc y^*y exists, $y^*y \cdot xyz = y$.

11. With modifications which we shall give, the methods of this paper yield the following theorem due, for locally compact continuous curves, to G. T. Whyburn: *In order that every self-compact totally disconnected subset H of a generalised continuous curve C belong to an arc L of C, it is sufficient that no point of C be a local cut point.*

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† For the properties of T^*, see §9.

‡ For euclidean spaces E_n, as special case of C, compare the argument given by the author in a paper to appear in the American Journal of Mathematics, *Generalisation of a theorem due to C. M. Cleveland.*

We shall give the form of the proof, and dwell only on that part of it which differs a little from the arguments we have already met with.

If R is any countable point set of C, and D an open connected subset, then $D-D\cdot R$ is arcwise connected (this is a simple form of the argument in §7). There is a tree T irreducible about H, and a countable set Q of points (q_n) of H which is dense in H. Let a and b be arbitrary end points of T. If $q_1 \subset ab \subset T$, let $T_1=T$ and $a(1)b=ab$. Otherwise there is the ψ-curve, $\psi_1=ax_1+q_1x_1+bx_1$. Let an ϵ be preassigned. There is in $S(q_1x_1)\cdot\{C-(Q+x_1)\}$ an arc st with $s\subset <x_1q_1>$ and t the only point on X_1, by which we designate the sum of those two components of $T-x_1$ which contain a or b respectively. There is in T an arc tx_1', where x_1' is on $<ax_1>$ or $<x_1b>$; we shall suppose it on x_1b for definiteness. If p is any point of sq_1-q_1, we shall say that it is covered by an arc s_pt_p if $s_p\subset <px_1>$ and $t_p\subset <pq_1>$, $s_pt_p\cdot X_1=0$, and $s_pt_p\cdot(Q+x_1)=0$.

It is clear that for every point p such a covering arc exists, of arbitrarily small diameter. The arc q_1s is homeomorphic with the linear interval $0\leqq r\leqq 1$, and we let q_1 correspond to $r=0$. The subset of q_1s corresponding to the points $1/2^{n-1}\geqq r\geqq 1/2^n$ we cover by a finite set P_n of arcs of diameter less than $(1/2^n)\epsilon$. Then in $T+\sum P_n$ there is an arc $ax_1q_1x_1'b$, and this is $a(1)b$.† There is in $T+\sum P_n$ a tree T_1 irreducible about $H+a(1)b$.

It will be observed that $a(1)b$ has been constructed to contain x_1. And this has been done so that it may be clear how, when $a(n)b$ has been constructed to contain $Q_n=\sum_1^n q_i$, $a(n+1)b$ can be constructed to contain Q_{n+1}. Suppose $a(n)b$ and T_n defined. If $q_{n+1}\subset a(n)b$, let $a(n+1)b=a(n)b$. If not, consider $q_{n+1}x_{n+1}$. If $x_{n+1}\cdot Q_n=0$, let $G(q_{n+1})$‡ be the component of T_n-Q_n containing $x_{n+1}q_{n+1}$; if $x_{n+1}\subset Q_n$,§ let this be the component of $T_n-(Q_n-x_{n+1})$. The inevitable induction follows closely §9.2, and the proof that the customary sets Γ_n are perfect continuous curves follows the arguments of §8.9. It seems to us that the remainder of this proof should be clear.

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† It is clear that $q_1x_1+x_1x_1'+x_1't+ts+\Sigma P_n$ is a cyclicly connected compact continuous curve containing q_1; if q_1 is not an end point there is a simpler construction available, but in general we should need the one above also.

‡ It is well, here, to recall §9.2.

§ In this proof there is no "reduction" of the order of branch points, of §9.