TYPES OF INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH CERTAIN RATIONAL CURVES*

BY

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Introduction

The purpose of this paper is to find and discuss the involutorial transformations belonging to the special complex of lines which meet a rational curve r of order m, and having a pencil of invariant surfaces $|F_n|$ which contain the curve as an $(n-2)$-fold basis element. If $n=2$ the curve r_m is not a basis curve. A pencil of quadrics and any rational curve always lead to a result contained among those found by Montesano.† The remaining admissible cases are as follows:

I. The pencils of surfaces of order n, $n \geqq 3$, which contain a straight line as an $(n-2)$-fold basis element. If $n=3$, in any plane containing the line there is a plane Cremona involution of order seven having for fundamental points four triple points which are not on the line and three double points which are on the line. The space involution is obtained by revolving the plane about the line. This case has already been treated.‡

II. A pencil of cubic surfaces which contain simply (a) a conic, (b) a rational cubic, (c) a rational space quartic, (d) a rational space quintic.

III. A pencil of quartic surfaces which contain doubly (a) a conic, (b) a space cubic.

A pencil of cubic surfaces cannot contain as basis curve a rational curve of higher order than five, because it would necessarily contain all the quadrisecants. A pencil of quartic surfaces cannot contain doubly a basis curve of higher order than three, because it would necessarily contain all the trisecants. Similarly, pencils of surfaces of degree greater than four are inadmissible since they would necessarily contain all the bisecants.

In this paper we shall discuss the transformations defined in II and III. We shall, however, confine ourselves to the case where the residual intersection of any two surfaces of the pencil is not composite, except in II(d) and III(b) where the residual is necessarily composite.

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† Montesano, *Su una classe di trasformazioni razionali ed involutorie dello spazio de genere arbitrario n e di grado* $2n+1$, Giornale di Mathematiche, vol. 31 (1892), pp. 36–50.

‡ Miss E. T. Carroll, *Systems of involutorial birational transformations contained multiply in special linear complexes*, American Journal of Mathematics, vol. 54 (1932).

It can be shown that III(a) can be transformed into II(a) by means of a quadratic involution.

The transformation III(b) is the most interesting of all the cases treated because it has a new kind of singularity. The other transformations have a finite number of parasitic lines, but the transformation III(b) has an infinite number of parasitic lines lying on a ruled surface in addition to ordinary parasitic lines.

Given a rational space curve r_m defined by a homogeneous parameter (λ, μ) and a pencil of surfaces $|F_n|:r_m^{n-2}$. If we make the points of the curve and the surfaces of the pencil projective, then any point $P(y)$ in space will uniquely determine a surface F_n of the pencil, a point $O(z)$ on r_m, and a line PO of the complex of lines which meet r_m. The line PO will cut F_n in P, O^{n-2} and a third point $P'(x)$. We define $P'(x)$ as the image of $P(y)$. Conversely, if we choose P' as the initial point we determine the same surface F_n, point O, and line of the complex. Hence P is the image of P'. The transformation is then involutorial and on a general line of the complex is one pair of points P, P' in involution.

Let the pencil of surfaces $|F_n|:r_m^{n-2}$ be

$$\mu F(x_1, x_2, x_3, x_4) - \lambda F'(x_1, x_2, x_3, x_4) = \mu F(x) - \lambda F'(x) = 0. \tag{1}$$

Call the residual base curve of the pencil γ. Since r_m is rational the coördinates of any point $O(z)$ are

$$x_i = z_i(\lambda, \mu) \qquad (i = 1, 2, 3, 4) \tag{2}$$

where $z_i(\lambda, \mu)$ is homogeneous and of degree m in (λ, μ). Any point on the line joining $P(y)$ to $O(z)$ has coördinates

$$x_i = \rho y_i + \sigma z_i \qquad (i = 1, 2, 3, 4). \tag{3}$$

The value of ρ/σ for $P'(x)$ is given by

$$\mu F(\rho y + \sigma z) - \lambda F'(\rho y + \sigma z) = 0. \tag{4}$$

Since P is on (1) and O is on (2) we find

$$\rho[\mu F(z, y) - \lambda F'(z, y)] + \sigma[\mu F(y, z) - \lambda F'(y, z)] = 0$$

where $F(z, y)$, $F'(z, y)$ are the first polars of $F(y)$, $F'(y)$ with respect to (z), and $F(y, z)$, $F'(y, z)$ are the first polars of $F(z)$, $F'(z)$ with respect to (y). Hence $\rho/\sigma = -R/M$, where

$$R = \mu F(y, z) - \lambda F'(y, z),$$
$$M = \mu F(z, y) - \lambda F'(z, y).$$

The involutorial transformation is therefore expressed by

$$x_i = y_i R - z_i M \qquad (i = 1, 2, 3, 4) \tag{5}$$

where $\lambda/\mu = F(y)/F'(y)$.

If $M(y) = 0$, P and P' coincide, hence $M = 0$ is the equation of the surface of invariant points.

At any point O of r_m each tangent plane of the associated surface F_n intersects F_n in a curve $C_n : O^{n-1}$. The whole C_n is transformed into the point O. Conversely, the image of O is C_n. As the point O describes r_m the C_n generates the surface $R = 0$.

With point O on r_m as vertex draw the cone K of bisecants of r_m. On each generator of K lies one point P', the image of O. Then the F_n associated with O and K intersect in r_m and a residual curve C' which is the part of the image of r_m lying on K and F_n. Since r_m is rational, the equation of K is homogeneous and of degree $(m-1)(m-2)$ in (λ, μ) and of degree $(m-1)$ in (x). The image of r_m lying on the bisecants of r_m is a surface $R' = 0$ and is obtained by eliminating the parameters (λ, μ) between K and F_n. Thus the total image of r_m is $R + R'$.

On each generator of the rational cone with vertex P on γ and standing on r_m is one point P', the image of P. The locus of P' is a curve C''. As P traces γ the curve C'' generates a surface Γ which is the total image of γ. Since any surface of the pencil (1) is invariant then the image of F_n must contain F_n and the images of r_m and γ. Then the equation of Γ is obtained by finding the image of F_n:

$$F_n \sim F_n R R' \Gamma.$$

We shall consider II(c) in detail.

Case II(c)

1. **Equations of the transformation.** We have a pencil of cubic surfaces $|F_3| : r_4$. The residual base curve γ_5 is of order five, genus one, and intersects r_4 in ten points. From (5) the equations of the transformation are

$$I_{29}:\ x_i = y_i R_{28} - z_i M_{17} \qquad (i = 1, 2, 3, 4) \tag{6}$$

where

$$\begin{aligned} R_{28} &= F'(y)F(y, z) - F(y)F'(y, z), \\ M_{17} &= F'(y)F(z, y) - F(y)F'(z, y). \end{aligned} \tag{7}$$

$M_{17} = 0$ is the equation of the surface of invariant points.

2. **Images of the fundamental elements.** The image of $O(z)$ on r_4 lying in the tangent plane of FO, the surface of the pencil associated with O, is a curve $C_3 : O^2$. As O describes r_4 the $C_3 : O^2$ generates the surface R_{28}. In the direction

of the two tangents of $C_3:O^2$ at O the point O is invariant. Hence two sheets of R_{28} and M_{17} have this plane for a common tangent plane. Thus two sheets of R_{28} touch two sheets of M_{17} along r_4. However, the point O for these two sheets is a binode and counts for six in the intersection of R_{28} and M_{17}.

Let L be an arbitrary point on r_4.

The point O has image P' on OL, the residual point of intersection of OL and F_L. As L describes r_4, OL generates a cubic cone K_3, with one point P' on each generator. The locus of P' is then a curve of order 3+the number of times OL is tangent to F_L at O. Given L, the tangent plane at O meets r_4 in two points K_1, K_2; given K, there is a unique point L. This (1, 2) correspondence on r_4 has three coincidences, and the locus of P' is $C_6:O^3$, $p=0$. As O describes r_4 the C_6 generates a surface $R_{21}'=0$.

The equation of R_{21}' may be obtained by eliminating the parameter between the cone $K_3(\lambda, \mu, x)=0$ and $F_3=\mu F(x)-\lambda F'(x)=0$.

Each point P' of C_6 is perspective from O, hence O is invariant in the three directions of the tangents to C_6 at O. Thus three sheets of R_{21}' are tangent respectively to three sheets of M_{17} along r_4.

The tangent line to r_4 at O lies on K_3 and also in the tangent plane to F_O. Hence $C_3:O^2$ and $C_6:O^3$ intersect in one point. As O describes r_4 this point generates a curve δ_{11} which lies on both R_{28} and R_{21}'.

The image of any point P on γ_5 is a curve C_9 which lies on the quartic cone K_4 with vertex P standing on r_4, cuts each generator in one point, and has five branches passing through P. Thus $P\sim C_9:P^5$. As P traces γ_5 the C_9 generates a surface Γ_{35} whose equation may be found by finding the image of any F_3 of the pencil:

$$F_3 \sim F_3 R_{28} R_{21}' \Gamma_{35}.$$

The point P is invariant in the directions of the five tangents of C_9 at P. Hence five sheets of Γ_{35} are tangent respectively to five sheets of M_{17} along γ_5.

3. **Determination of the parasitic lines.** On any F_3 lie twenty-seven lines. Whenever one of these lines passes through the associated point O on r_4 it is parasitic. It is desired to know at how many points O on r_4 a line on F_O passes through O. To do this we map the cubic surface on the plane by means of cubic curves through six basis points 1, 2, 3, 4, 5, 6. In this plane a conic $C_2:1\ 2\sim r_4$ on F_3 and a curve of order seven $C_7:1^2 2^2 3^3 4^3 5^3 6^3\sim\gamma_5$ on F_3. $[C_2, C_7]=10$ points. Therefore $[r_4, \gamma_5]=10$ points, as already indicated.

Consider any line l on any F_3 of the pencil. This line meets any other surface F_3' of the pencil in three points, and since F_3, F_3' intersect in r_4, γ_5, only these three points must be on r_4, γ_5. Hence the lines l may be classified as follows:

A. Lines which meet r_4 three times, do not meet γ_5.

B. Lines meeting r_4 twice, meeting γ_5 once.

C. Lines meeting r_4 once, meeting γ_5 twice.

D. Lines which do not meet r_4, meet γ_5 three times.

We find

2 conics, conics containing 1, or 2, and 3, 4, 5, 6, meet C_2 in three points, do not meet C_7.

Hence there are two lines A on F_3.

4 conics, conics containing 1, 2, and three of 3, 4, 5, 6, meet C_2 twice, meet C_7 once.

6 lines, joins of 3, 4, 5, 6 by pairs, meet C_2 twice, meet C_7 once.

Hence there are $4+6=10$ lines B on F_3.

2 lines, images of 1, 2, meet C_2 once, meet C_7 twice.

8 lines, joins of 1, or 2, to 3, or 4, or 5, or 6, meet C_2 once, meet C_7 twice.

Hence there are $2+8=10$ lines C on F_3.

1 line, join of 1, 2, does not meet C_2, meets C_7 three times.

4 lines, images of 3, 4, 5, 6, do not meet C_2, meet C_7 three times.

Hence there are $1+4=5$ lines D on F_3.

In all there are $2+10+10+5=27$ lines on F_3.

The lines D do not enter the problem since they do not meet r_4.

Given point O on r_4. There are two lines A on F_O. Each line meets r_4 three times, hence there are six points K. Conversely, given a point K, there is one line A, the trisecant of r_4 through K. This line will determine one surface of the pencil, hence one point O. Then there is a (1, 6) correspondence between the points O and K. Since r_4 is rational there are $1+6=7$ coincidences. Hence there are seven parasitic lines which are trisecants of r_4 but do not meet γ_5. These lines are simple on R_{28}, M_{17}; double on R_{21}'; do not lie on Γ_{35}.

Given point O on r_4 there are ten lines B. Each line meets r_4 twice, hence twenty points K. Conversely, given a point K there are five points O. To show this we construct the two cones with common vertex K, K_3 standing on r_4 and K_5 standing on γ_5. These cones intersect in fifteen lines of which ten are lines joining K to the ten points common to r_4, γ_5, hence only five lines B. Each line will determine one point O. There is a (5, 20) correspondence between the points O and K, hence $5+20=25$ coincidences. There are twenty-five parasitic lines which meet r_4 twice and meet γ_5 once. These lines are simple on R_{28}, M_{17}, R_{21}', and Γ_{35}.

Given a point O there are ten lines C. Each line meets r_4 once, hence ten points K. Conversely, given a point K, there are five lines C, the lines joining K to the five apparent double points of Γ_5. Hence there is a (5, 10) correspondence between the points O and K, and $5+10=15$ coincidences. There are fifteen parasitic lines which meet r_4 once and meet γ_5 twice. These lines are simple on R_{28}, M_{17}; double on Γ_{35}; do not lie on R_{21}'.

In all there are forty-seven parasitic lines distributed as follows: All are simple on R_{28} and M_{17}; seven are double on R_{21}', do not lie on Γ_{35}; fifteen do not lie on R_{21}', are double on Γ_{35}; the remaining twenty-five are simple on both R_{21}' and Γ_{35}.

4. **Table of images.** We have the following table:

A general plane $ax_1+bx_2+cx_3+dx_4 \sim S_{29}$. Since S_{29} is linear in R_{28} and M_{17}, the forty-seven parasitic lines are all simple on S_{29}. At each point O of r_4 three sheets of S_{29} have for common tangent plane the tangent plane of F_O.

$$S_1 \sim S_{29}: r_4^{9+3t} \gamma_5^9 47g,$$

where the t in the multiplicity of r_4 means only that the multiplicity is due to contact.

$$r_4 \sim R_{28}: r_4^{9+2t} \gamma_5^9 47g + R_{21}': r_4^{7+2t} \gamma_5^6 25g 7g^2;$$

$$\gamma_5 \sim \Gamma_{35}: r_4^{10+5t} \gamma_5^{11} 25g 15g^2;$$

$$M_{17}: r_4^{5+2t} \gamma_5^5 47g.$$

The Jacobian is

$$J_{112} \equiv R_{28}^2 R_{21}' \Gamma_{35}.$$

CASE II(a)

5. **Equations of the transformation.** Let r_2 be defined as the intersection of the quadric surface $H_2(x)=0$ and the plane π. The pencil (1) is $|F_3|:r_2$. The residual base curve is γ_7, genus five, meets r_2 in six points, and meets π in a seventh point Q not on r_2. From (5) the equations of the transformation are

$$I_{17}: \ x_i = y_i R_{16} + z_i M_{11} \qquad (i = 1, 2, 3, 4), \tag{8}$$

where

$$\begin{aligned} R_{16} &= F'(y)F(y, z) - F(y)F'(y, z), \\ M_{11} &= F'(y)F(z, y) - F(y)F'(z, y). \end{aligned} \tag{9}$$

$M_{11}=0$ is the equation of the surface of invariant points.

6. **Images of the fundamental elements.** The image of $O(z)$ lying in the tangent plane of F_O is a curve $C_3:O^2$ which generates the surface R_{16} as O

describes r_2. The point O is invariant in the directions of the two tangents to C_3 at O, hence two sheets of R_{16} and M_{11} have a common tangent plane at all points of r_2.

The plane π intersects the pencil $|F_3|$ in r_2 and a pencil of lines $|d|$ whose vertex is Q, and which is projective with the points of r_2. Let L be an arbitrary point on r_2. The point O has image P' on OL, the intersection of OL and d_L, the line of $|d|$ associated with L. As L describes r_2, P' generates a conic $C_2:OQO_1O_2O_3$ where O_i $(i=1, 2, 3)$ are the points where d_L passes through L. As O describes r_2, C_2 generates the plane π. Thus the total image of r_2 is $R_{16}+\pi$.

The tangent to r_2 at O is a line OL, $L=0$, and lies in the tangent plane of F_O at O. Thus $C_3:O^2$ and $C_2:OQO_1O_2O_3$ intersect in one point P'. This point is the point of intersection of the tangent of r_2 at O and d_O. As O describes r_2 the point P' generates a curve δ_3 which has a node at Q and touches r_2 at the three points O_1, O_2, O_3. This is Lehmer's nodal cubic,* and lies on R_{16} and π.

The image of a point P on γ_7 is a $C_5:P^3$. As P describes γ_7, C_5 generates a surface Γ_{31}, whose equation is found from the image of any F_3. $F_3 \sim F_3 R_{16} \pi \Gamma_{31}$. Since P is invariant in three directions, three sheets of Γ_{31} are respectively tangent to three sheets of M_{11} along γ_7.

Any plane passing through Q will intersect γ_7 in six other points which lie on a conic.†

7. **Determination of the parasitic lines.** We recall that all lines must meet the basis curves r_2, γ_7 in three points. In the plane $C_2:1\ 2\ 3\ 4 \sim r_2$ on F_3 and $C_7:1^2 2^2 3^2 4^2 5^3 6^3 \sim \gamma_7$ on F_3, hence $[C_2, C_7]=8$ points and C_7 is of genus five.

Now we find that there are three parasitic lines which meet r_2 twice and meet γ_7 once. These are the lines joining Q respectively to O_1, O_2, O_3. They are simple on R_{16}, π, M_{11}, and Γ_{31}. There are twenty-six parasitic lines which meet r_2 once and meet γ_7 twice. These lines are simple on R_{16}, M_{11}; double on Γ_{31}; do not lie on π.

8. **Table of images.** We have the following table:

$$S_1 \sim S_{17}: r_2^{5+3t} \gamma_7^5 29g;$$

$$r_2 \sim R_{16}: r_2^{5+2t} \gamma_7^5 29g + \pi : r_2 3g;$$

$$\gamma_7 \sim \Gamma_{31}: r_2^{8+7t} \gamma_7^9 3g 26g^2;$$

$$M_{11}: r_2^{3+2t} \gamma_7^3 29g.$$

The Jacobian is $J_{64} \equiv R_{16}^2 \pi \Gamma_{31}$.

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* D. N. Lehmer, *Constructive theory of the unicursal cubic by synthetic methods*, these Transactions, vol. 3 (1902), pp. 372–376.

† R. Sturm, *Synthetische Untersuchungen über Flächen dritter Ordnung*, Leipzig, 1867, p. 229.

CASE II(b)

9. **Equations of the transformation.** Let r_3 be a space cubic curve. The pencil (1) is now $|F_3|:r_3$. The residual base curve is γ_6, $p=3$, and intersects r_3 in eight points. From (5) the equations of the transformation are

$$I_{23}:\ x_i = y_i R_{22} - z_i M_{14} \qquad (i = 1, 2, 3, 4), \tag{10}$$

where

$$\begin{aligned} R_{22} &= F'(y)F(y, z) - F(y)F'(y, z),\\ M_{14} &= F'(y)F(z, y) - F(y)F'(z, y). \end{aligned} \tag{11}$$

$M_{14}=0$ is the equation of the surface of invariant points.

The image of r_3 lying in the tangent planes of F_O is R_{22}. Again two sheets of R_{22} and M_{14} are tangent along r_3.

The image of a point O on r_3 which lies on the bisecants of r_3 is a $C_4:O^2$. As O describes r_3 the C_4 generates a surface $R_8'=0$. Two sheets of R_8' are respectively tangent to two sheets of M_{14} along r_3.

The locus of the point common to $C_3:O^2$ and $C_4:O^2$ is a curve δ_7 lying on R_{22} and R_8'.

The image of a point P on γ_6 is a $C_7:P^4$. As P traces γ_6 the C_7 generates a surface Γ_{36}. Four sheets of Γ_{36} are respectively tangent to four sheets of M_{14} along γ_6.

10. **Determination of the parasitic lines.** In the plane $C_2:1\ 2\ 3 \sim r_3$ on F_3 and $C_7:1^2 2^2 3^2 4^3 5^3 6^3\ (p=3) \sim \gamma_6$ on F_3. Hence $[C_2, C_7]=8$ points.

We find that there are thirty-eight parasitic lines distributed as follows: All are simple on R_{22} and M_{14}; sixteen are simple on both R_8' and Γ_{36}; twenty-two do not lie on R_8', are double on Γ_{36}.

11. **Table of images.** We have the following table:

$$S_1 \sim S_{23}: r_3^{7+3t}\gamma_6^7 38g;$$

$$r_3 \sim R_{22}: r_3^{7+2t}\gamma_6^7 38g + R_8': r_3^{3+1t}\gamma_6^3 16g;$$

$$\gamma_6 \sim \Gamma_{36}: r_3^{10+6t}\gamma_6^{11} 16g 22g^2;$$

$$M_{14}: r_3^{4+2t}\gamma_6^4 38g.$$

The Jacobian is $J_{88} \equiv R_{22}{}^2 R_8'\ \Gamma_{36}$.

Case II(b′)

12. If r_3 is a rational plane cubic then the plane π of r_3 factors out of the transformation and (10) reduces to

$$(12)\qquad I_{22}:\ x_i = y_iR_{21} - z_iM_{13} \qquad (i = 1, 2, 3, 4)$$

where R_{21}, M_{13} are given by (11) after factoring out π. $M_{13}=0$ is the equation of the surface of invariant points. $R_{21}=0$ is the total image of r_3. The image of γ_6 is a surface Γ_{42}. There are only thirty-three parasitic lines and they are all simple on R_{21}, M_{13}; and double on Γ_{42}.

$$\begin{aligned}
&S_1 \sim S_{22}: r_3^{6+3t}\gamma_6^7 33g;\\
&r_3 \sim R_{21}: r_3^{6+2t}\gamma_6^7 33g;\\
&\gamma_6 \sim \Gamma_{42}: r_3^{12+6t}\gamma_6^{13} 33g^2;\\
&M_{13}: r_3^{3+2t}\gamma_6^4 33g.
\end{aligned}$$

The Jacobian is $J_{84}\equiv R_{21}^2\ \Gamma_{42}$.

Case II(d)

13. **Equations of the transformation.** The pencil (1) becomes $|F_3|:r_5$. The curve r_5 has one quadrisecant l, hence this line lies on every surface of the pencil. The residual base curve is a space cubic γ_3 which meets r_5 in eight points, but does not meet l. From (5) the equations of the transformation are

$$(13)\qquad I_{35}:\ x_i = y_iR_{34} - z_iM_{20} \qquad (i = 1, 2, 3, 4),$$

where

$$(14)\qquad \begin{aligned}
R_{35} &= F'(y)F(y, z) - F(y)F'(y, z),\\
M_{20} &= F'(y)F(z, y) - F(y)F'(z, y).
\end{aligned}$$

Again $M_{20}=0$ is the equation of the surface of invariant points.

14. **Images of fundamental elements.** The image of a point O on r_5 lying in the tangent plane of F_O at O is a $C_3:O^2$. As O describes r_5 the C_3 generates the surface R_{34}. Two sheets of R_{34} and two sheets of M_{20} have a common tangent plane along r_5.

The image of a point O lying on the bisecants of r_5 is a $C_8:O^4$. As O describes r_5 the C_8 generates a surface R_{40}'. Four sheets of R_{40}' are respectively tangent to four sheets of M_{20} along r_5. The total image of r_5 is $R_{34}+R_{40}'$.

The tangent line to r_5 at O cuts $C_3:O^2$ and $C_8:O^4$ in a common point. As O describes r_5 this point generates a curve δ_{15} which lies on both R_{34} and R_{40}'.

The image of a point Q on l is a $C_7:Q^2$. As Q describes l, the C_7 generates a surface L_4. Two sheets of L_4 are tangent respectively to two sheets of M_{20} along l. The equation of L_4 is found as follows: The plane through l and O intersects F_O in l and a residual conic C_2 which is the part of the image of l which lies on F_O. We obtain the equation of L_4 by eliminating the parameter (λ, μ) between the equations of the plane and F_O.

The image of a point P on γ_3 is a curve $C_{11}:P^6$. As P describes γ_3 the C_{11} generates a surface Γ_{24}. Six sheets of Γ_{24} are tangent respectively to six sheets of M_{20} along γ_3. The equation of Γ_{24} is found from the image of any F_3. Thus: $F_3 \sim F_3 R_{34} R'_{40} L_4 \Gamma_{24}$.

15. **Determination of the parasitic lines.** In the plane a conic $C_2:1 \sim r_5$ $(p=0)$ on F_3; a conic $C'_2:2\ 3\ 4\ 5\ 6 \sim l$ on F_3; a quintic $C_5:1^2 2^2 3^2 4^2 5^2 6^2 \sim \gamma_3$ $(p=0)$ on F_3. Hence $[C_2, C'_2]=4$ points, $[C_2, C_5]=8$ points, $[C'_2, C_5]=0$ points.

In the present problem seven types of lines enter:

A. Lines which meet r_5 three times, do not meet l or γ_3.

B. Lines which meet r_5 twice, do not meet l, meet γ_3 once.

C. Lines which meet r_5 twice, meet l once, do not meet γ_3.

D. Lines which meet r_5 once, do not meet l, meet γ_3 twice.

E. Lines which meet r_5 once, meet l once, meet γ_3 once.

F. Lines which do not meet r_5, meet l once, meet γ_3 twice.

G. The line l itself which meets r_5 four times, does not meet γ_3.

There are no lines C, and lines F do not enter the problem. From the map, and then the number of coincidences on r_5 we find the following: There are eighteen parasitic lines of type A, twenty-four of type B, two of type D, eight of type E.

The map fails to give the number of times l is counted as a parasitic line. We shall determine this in another manner. Denote by Q_i $(i=1, 2, 3, 4)$ the four points common to l and r_5. Now l lies in the tangent plane of F_Q at Q_i, hence is parasitic. But l appears as a parasitic line at each of the four points Q_i independently of the other three. Hence l counts as a parasitic line four times. We shall think of it as four parasitic lines. These lines are simple on R_{34} and M_{20}; triple on R'_{40}; do not lie on L_4 or Γ_{24}.

Thus there are fifty-six parasitic lines distributed as follows: All are simple on R_{34}, M_{20}; eighteen are double on R'_{40}, do not lie on L_4, Γ_{24}; twenty-four are simple on R'_{40}, Γ_{24}, but do not lie on L_4; two are double on Γ_{24} but do not lie on L_4, R'_{40}; eight are simple on L_4, Γ_{24} but do not lie on R'_{40}; four are triple on R'_{40} but do not lie on L_4, Γ_{24}.

16. **Table of images.** We have the following table:

$$S_1 \sim S_{35}: r_5^{11+3t} \gamma_3^{11} l^{11} 56g;$$
$$r_5 \sim R_{34}: r_5^{11+2t} \gamma_3^{11} l^{11} 56g + R'_{40}: r_5^{13+3t} \gamma_3^{12} l^{12} 24g 18g^2 4g^3;$$
$$\gamma_3 \sim \Gamma_{24}: r_5^{7+3t} \gamma_3^{8} l^{7} 24g 8g 2g^2;$$
$$l \sim L_4: r_5^{1+1t} \gamma_3 l^{2} 8g;$$
$$M_{20}: r_5^{6+2t} \gamma_3^{6} l^{6} 56g.$$

The Jacobian is $J_{136} \equiv R_{34}{}^2 R'_{40} L_4 \Gamma_{24}$.

Case III(a)

17. The pencil (1) is now $|F_4|: r_2^2$. The residual base curve is a γ_8 which intersects r_2 in eight points. If we transform $|F_4|$ by a quadratic involution whose fundamental elements are the conic r_2 and a point on γ_8, it transforms into $|F_3|: r_2$, or Case II(a).

Case III(b)

18. **Equations of the transformation.** Given the equations of the space cubic curve r_3 as

$$x_1/x_2 = x_2/x_3 = x_3/x_4 = \lambda/\mu, \tag{15}$$

and let

$$F(x) = 0, \qquad F'(x) = 0 \tag{16}$$

be two quartic surfaces which contain r_3 as a double basis curve. They intersect in r_3 and a residual composite quartic curve which consists of four straight lines l_i, each of which is a bisecant of r_3. Then

$$|F_4|: r_3^2 l_1 l_2 l_3 l_4.$$

A surface of the pencil $\mu F(x) - \lambda F'(x)$ through the point $P(y)$ determines $\lambda/\mu = F(y)/F'(y)$. The coördinates of a point on the line joining $F(y)$ to $O(z) \equiv (F^3, F^2F', FF'^2, F'^3)$ are given by

$$x_i = \rho y_i + \sigma z_i \qquad (i = 1, 2, 3, 4). \tag{17}$$

The residual point of intersection of PO with $F_4(x)$ after making reductions is given by

$$\rho\Delta_1(z, y) + \sigma\Delta_2(z, y) = 0 \tag{18}$$

where $\Delta_i(z, y)$ is the ith polar of $F_4(y)$ with respect to (z).

$\Delta_1(z, y)$ and $\Delta_2(z, y)$ are homogeneous and respectively of degree nineteen and thirty in (y). However, Δ_1 and Δ_2 have a common factor R_{10}' which is of the tenth degree in (y). Hence

$$\Delta_1(y) \equiv M_9(y)R'_{10}(y), \qquad \Delta_2(y) \equiv R_{20}(y)R'_{10}(y).$$

The equations of the transformation are now

$$(19) \qquad I_{21}\colon\ x_i = y_iR_{20} - z_iM_9 \qquad (i = 1, 2, 3, 4).$$

$M_9 = 0$ is the equation of the surface of invariant points.

We shall find that although R_{10}' factors out of the transformation it still plays the most important rôle of any surface in the transformation.

19. **Images of the fundamental elements.** Given any point P on l_i. Any point O on r_3 will determine an associated F_4 and the line OP will cut F_4 in O^2, P, and a third point P' which is the image of P. As O generates r_3 the point P' generates a curve C_5 which lies on the cubic cone K_3 standing on r_3 with vertex P, cuts each generator in one point P', and has two branches passing through P. Then $P \sim C_5{:}P^2$. As P describes l_i the $C_5{:}P^2$ generates a surface of order five which is the total image of l_i.

We shall determine the equation of this surface in an alternate manner. Suppose l_i is the intersection of two planes $u_i(x) = 0$, $v_i(x) = 0$. The pencil of planes

$$(20) \qquad \mu u_i(x) - \lambda v_i(x) = 0$$

is projective with the pencil $|F_4(x)|$. Any point O on r_3 will determine a surface $F_4(x)$ and a plane of (20) passing through O. The plane will cut $F_4(x)$ in l_i and a cubic curve C_3 which is the part of the image of l_i lying on this $F_4(x)$. Thus the whole image of l_i can be obtained by eliminating the parameter (λ, μ) between the pencils (20) and $|F_4(x)|$. Thus

$$(21) \qquad L_{5,i} \equiv F(x)v_i(x) - F'(x)u_i(x) = 0.$$

There are four such surfaces $L_{5,i}$.

The two tangent planes to the associated F_4 at O on r_3 cut the F_4 in two quartic curves, each having a triple point at O, $2C_4{:}O^3$. As O traces r_3 the $2C_4{:}O^3$ generate the surface $\Delta_2(z, y)$.

Any $F_4(x){:}r_3^2$ is ruled and through each point on r_3 pass two generators g, g' of F_4. One generator lies in each of the tangent planes of F_4 at O. Thus both the quartic curves are composite and consist of a cubic and a generator of F_4,

$$2C_4 \equiv C_3g + C'_3g'.$$

The two cubic curves generate the surface R_{20}, while the two generators, g, g', generate the surface R_{10}'.

20. **The surface $R_{10}'=0$.** The cone K_2 standing on r_3 with vertex at a point O on r_3 and the F_4 associated with O intersect in r_3 and two lines, the two generators g, g' of F_4 passing through O. The locus of g, g' is R_{10}'.

From (15) we find three independent quadrics passing through r_3 to be

$$(22)\quad H_1(x) = x_1x_3 - x_2^2 = 0,\ H_2(x) = x_2x_4 - x_3^2 = 0,\ H_3(x) = x_1x_4 - x_2x_3 = 0,$$

and the equation of K_2, vertex $O(\lambda, \mu)$, is

$$(23)\qquad \mu^2H_1(x) + \lambda^2H_2(x) - \lambda\mu H_3(x) = 0,$$

hence

$$(24)\qquad R'_{10} \equiv F'^2(x)H_1(x) + F^2(x)H_2(x) - F(x)F'(x)H_3(x) = 0.$$

The generators g, g' are parasitic lines. Then through each point of r_3 pass two parasitic lines whose locus is $R_{10}'=0$, a ruled surface. It has five sheets passing through r_3. The two sheets of R_{10}' determined by the two generators g, g' of the F_4 associated with the point we shall call the "at" sheets. Now g and g' are bisecants of r_3 hence intersect r_3 in two other points O_1, O_2. At O_1 the associated F_4 has two generators g_1, g_1' which determine the "at" sheets of R_{10}' through O_1. The line g is a generator of R_{10}' but does not lie on the F_4 associated with O_1. Its associated point is O. We shall think of it as coming from point O. Through O_1 pass three such lines g whose origin is at some other point. The three sheets of R_{10}' determined by these generators we shall call the "from" sheets.

21. **Determination of the parasitic lines.** In general neither g nor g' lies on any of the other surfaces of the transformation. We wish to find which of these lines do lie on other surfaces, and any other parasitic lines which may arise.

There are four points on r_3 at which the g and g' of the associated F_4 coincide and thus are contact generators of R_{10}', and also lie on S_{21}, R_{20} and M_9.

At three points of r_3 the associated F_4 is composite, consisting of two quadric surfaces each of which contains r_3. Two generators of each quadric pass through the point. Hence there are four parasitic lines which pass through each of the three points. Two are generators of R_{10}' but the other two are not, as they are not bisecants of r_3. All of the generators of r_{10}' are bisecants of r_3, hence there are six parasitic lines which do not lie on R_{10}'. They are distributed as follows: All are simple on S_{21}, R_{20}, M_9; three lie on each of the surfaces $L_{5,i}$, such that just one is common to $L_{5,j}$, $L_{5,i}$ $(i\neq j)$.

22. **Table of images.** A general plane $S_1 \sim S_{21}$ having nine sheets passing through r_3 such that each of the tangent planes of the associated F_4 is the

common tangent plane of three sheets at all points of r_3. The three remaining sheets are tangent to the three "from" sheets of R_{10}'. The image of r_3 for the latter contact is R_{10}'. There are five sheets passing through each l_i. Six parasitic lines are simple and four double on S_{21}.

$$S_1 \sim S_{21}: \underset{3}{r}^{9+3t+3t+3t'} 4\underset{i}{l}^5 6g4g^2.$$

The surface R_{20} has nine sheets passing through r_3 such that each of the tangent planes of the associated F_4 is the common tangent plane of two sheets at all points of r_3. Three other sheets are tangent to the three "from" sheets of R_{10}'. There are five sheets passing through each l_i. All ten parasitic lines are simple on R_{20}.

$$R_{20}: \underset{3}{r}^{9+2t+2t+3t'} 4\underset{i}{l}^5 6g4g.$$

The surface of invariant points M_9 has four sheets passing through r_3 such that each tangent plane of the associated F_4 is the common tangent plane of two sheets at all points of r_3. There are two sheets passing through each l_i. Six parasitic lines are simple and four double on M_9.

$$M_9: \underset{3}{r}^{4+2t+2t} 4\underset{i}{l}^2 6g4g^2.$$

Any surface $L_{5,i}$ has two sheets passing through r_3 such that the tangent planes of these sheets are the tangent planes of the associated F_4 at all points of r_3. There are two sheets passing through l_i. These two sheets are tangent to the two sheets of M_9 through l_i. These are the sheets determined by the two tangents of $C_5: P^2$ at P. There is just one sheet passing through each of the three remaining lines l_j. There are three simple parasitic lines lying on $L_{5,i}$ distributed respectively on the three $L_{5,j}$.

$$L_{5,i}: \underset{3}{r}^{2+1t+1t} \underset{i}{l}^2 3l_j 3g.$$

The tangent planes of the two "at" sheets of R_{10}' are the tangent planes of the associated F_4 at all points of r_3. The three "from" sheets are tangent to three sheets of S_{21} and R_{20} at all points of r_3. There are two sheets passing through each l_i. Four parasitic lines are double on R_{10}'.

$$R_{10}': \underset{3}{r}^{5+1t+1t} 4\underset{i}{l}^2 4g^2.$$

Collecting,

$$S_1 \sim S_{21}: \underset{3}{r}^{9+3t+3t+3t'} 4\underset{i}{l}^5 6g4g^2;$$

$$r_3 \sim R_{20}: \underset{3}{r}^{9+2t+2t+3t'} 4\underset{i}{l}^5 6g4g + R_{10}': \underset{3}{r}^{5+1t+1t} 4\underset{i}{l}^2 4g^2;$$

$$l_i \sim L_{5,i}: r_3^{2+1t+1t} l_i^2 3l_j 3g;$$

$$M_9 \sim M_9: r_3^{4+1t+1t} 4l_i^2 6g 4g^2;$$

$$S_{21} \sim (R_{20}R'_{10})^9 R_{20}^3 R'_{10} 4L_{5,i}^2 S_1;$$

$$R_{20} \sim (R_{20}R'_{10})^9 R_{20}^2 R'_{10} 4L_{5,i}^5;$$

$$M_9 \sim (R_{20}R'_{10})^4 R_{20} 4L_{5,i}^2 M_9;$$

$$L_{5,i} \sim (R_{20}R'_{10})^2 R_{20} L_{5,i}^2 3L_{5,j};$$

$$R'_{10} \sim (R_{20}R'_{10})^5 R_{20} 4L_{5,i}^2.$$

The Jacobian is $J_{80} \equiv R_{20}^2 R_{10}'^2 L_{5,1} L_{5,2} L_{5,3} L_{5,4}$.

23. **Generalization.** In the preceding cases there has always been a (1, 1) correspondence between the points of r_m and the surfaces of $|F_n|$. Let us assume the correspondence is $(1, k)$. The case where r_m is a straight line has been treated by Miss E. T. Carroll (loc. cit.). Then a general point $P(y)$ will determine just one point $O(z)$ hence one point $P'(x)$, but given $O(z)$ there are k associated surfaces F_n. A general line of the complex through O will cut each surface in a pair of points P, P'. Hence on each line of the complex are k pairs of points in involution. We shall illustrate by working II(c) in detail.

24. **Equations of the transformation.** The coördinates of a point $O(z)$ on r_4 are given by (2)

$$x_i = z_i(\lambda, \mu) \qquad (i = 1, 2, 3, 4),$$

but the pencil of surfaces $|F_3|: r_4$ is written

$$mF(x) - lF'(x) = 0, \tag{25}$$

where $\mu\phi_1(l, m) - \lambda\phi_2(l, m) = 0$, the $\phi_i(l, m)$ being homogeneous forms of degree k in (l, m). Proceeding exactly as before the equations of the transformation are found to be

$$I_{24k+5}: \; x_i = y_i R_{24k+4} - z_i M_{12k+5} \qquad (i = 1, 2, 3, 4), \tag{26}$$

where

$$R_{24k+4} = mF(y, z) - lF'(y, z), \; M_{12k+5} = mF(z, y) - lF'(z, y), \tag{27}$$

and $l/m = F(y)/F'(y)$, $\lambda/\mu = \phi_1(l, m)/\phi_2(l, m)$. $M_{12k+5} = 0$ is the equation of the surface of invariant points.

25. **Images of the fundamental elements.** Associated with a point O on r_4 are k surfaces F_3. In the tangent plane of each F_3 at O lies a $C_3: O^2$ which is the image of O. Thus $O \sim kC_3: O^2$. As O describes r_4 the $kC_3: O^2$ generate the surface R_{24k+4}. The point O is invariant in two directions in each tangent plane,

hence each of the k tangent planes is the common tangent plane of two sheets of R_{24k+4} and M_{12k+5} along r_4.

The image of O which lies on the bisecants of r_4 is a $C_{5k+1}:O^{2k+1}$ which lies on the cubic cone K_3 with vertex O standing on r_4, cuts each generator in k points and has $2k+1$ branches through O. As O describes r_4 the $C_{5k+1}:O^{2k+1}$ generates a surface R'_{18k+3}. There are $2k+1$ sheets of R'_{18k+3} which are tangent respectively to $2k+1$ sheets of M_{12k+5} along r_4.

The locus of the k points common to C_{5k+1} and kC_3 is a curve δ_{10k+1} which lies on both R_{24k+4} and R'_{18k+3}.

The image of a point P on γ_5 is a $C_{8k+1}:P^{4k+1}$ which lies on the quartic cone K_4 with vertex P standing on r_4. As P describes γ_5 the C_{8k+1} generates a surface Γ_{30k+5}. There are $4k+1$ sheets of Γ_{30k+5} which are tangent respectively to $4k+1$ sheets of M_{12k+5} along γ_5.

26. **Determination of the parasitic lines.** The map of the cubic surface on a plane is the same as in §3, hence on any F_3 are two lines A, ten lines B, and ten lines C. However, the number of coincidences is different.

Given point O on r_4 there are two lines A on each of the k associated surfaces F_3. Each line meets r_4 in three points, hence $6k$ points K. Conversely, given a point K there is one line A, the trisecant of r_4 through K. This line determines one point O. There is a $(1, 6k)$ correspondence between the points O and K, hence $1+6k$ coincidences. There are $1+6k$ parasitic lines of type A which are simple on S_{24k+5}, R_{24k+4}, M_{12k+5}; double on R'_{18k+3}; do not lie on Γ_{30k+5}.

Similarly there are $5+20k$ parasitic lines of type B which are simple on all the surfaces S_{24k+5}, R_{24k+4}, M_{12k+5}, R'_{18k+3}, Γ_{30k+5}; and $5+10k$ parasitic lines of type C which are simple on S_{24k+5}, R_{24k+4}, M_{12k+5}, double on Γ_{30k+5}, do not lie on R'_{18k+3}.

27. **Table of images.** We have the following table:

$$S_1 \sim S_{24k+5}: r_4^{8k+1+3t}\gamma_5^{8k+1}(11 + 36k)g;$$

$$r_4 \sim R_{24k+4}: r_4^{8k+1+2t}\gamma_5^{8k+1}(11 + 36k)g$$

$$+ R'_{18k+3}: r_4^{6k+1+2t}\gamma_5^{6k}(5 + 20k)g(1 + 6k)g^2;$$

$$\gamma_5 \sim \Gamma_{30k+5}: r_4^{10k+5t}\gamma_5^{10k+1}(5 + 20k)g(5 + 10k)g^2;$$

$$M_{12k+5}: r_4^{4k+1+2t}\gamma_5^{4k+1}(11 + 36k)g.$$

The Jacobian is $J_4(24k+4) \equiv R^2_{24k+4}\, R'_{18k+3}\, \Gamma_{30k+5}$.

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