THE TOTAL VARIATION OF $g(x+h)-g(x)$

BY

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1. There is a fundamental theorem in the theory of Lebesgue integration that if $f(x)$ be any function integrable Lebesgue, over the interval (a, b), then the integral

$$\int_a^b |f(x+h)-f(x)|\,dx \tag{1}$$

tends to zero with h. This theorem is usually proved by approximating to $f(x)$ in terms of continuous functions, for which the property is obvious.

The integral (1) is the total variation in (a, b) of the function $F(x+h)-F(x)$, where

$$F(x) = \int^x f(t)dt$$

and the quoted property is equivalent to the following statement:

(I) *If $F(x)$ be any absolutely continuous function in (a, b), the total variation of*

$$F(x+h) - F(x)$$

tends to zero with h.

This result no longer holds if we substitute for $F(x)$ any non-absolutely continuous function $G(x)$ of bounded variation. Indeed, it provides a necessary and sufficient condition for absolute continuity.† This fact may be first rendered plausible by taking the simplest case of a discontinuous function of bounded variation, and bearing in mind that a general continuous function of bounded variation is always a limit of simple discontinuous ones. If we assume for instance

$$g(x) = \alpha \text{ for } x \leqq c,$$

$$g(x) = \beta \text{ for } c < x,$$

* Presented to the Society, October 29, 1932; received by the editors April 21, 1932.

† This result was proved, by an entirely different method, by A. Plessner, Journal für Mathematik, vol. 160 (1929), pp. 26–32. A study of the total variation of $g(x+h)-g(x)$ when $g(x)$ is constant in the complementary intervals of Cantor's set is contained in an article by Hille and Tamarkin, American Mathematical Monthly, vol. 36 (1929), pp. 255–264. The existence of these two papers was called to our attention after the present paper had been completed. We notice also an erroneous statement in a paper by J. M. Whittaker, Proceedings of the Edinburgh Mathematical Society, (2), vol. 1 (1929), p. 232 (Lemma 1).

we have, for $h>0$,

$$g(x+h)-g(x)=0 \qquad \text{for } x\leqq c-h \text{ and } x>c,$$
$$g(x+h)-g(x)=\beta-\alpha \text{ for } c-h<x\leqq c$$

and over any interval (a, b) containing c and $c+h$ internally, the total variation of $g(x+h)-g(x)$ is the sum of the absolute values of its jumps, viz.

$$2|\beta-\alpha| = 2\left|\int_a^b dg(x)\right| = \left|\int_a^b dg(x)\right| + \left|\int_a^b dg(x+h)\right|.$$

In this special case we may then write, for every h, and every interval (a, b), $a\neq c$, $b\neq c$,

$$\int_a^b |d[g(x+h)-g(x)]| = \int_a^b |dg(x)| + \int_a^b |dg(x+h)|. \tag{2}$$

Actually this is the relation which we shall show (§3) to hold for any singular function $g(x)$ of bounded variation, if not for all h, at any rate for almost all. By a singular function we mean a function of bounded variation whose derivative is zero almost everywhere, of which the above discontinuous $g(x)$ provides a very special example.

An arbitrary function of bounded variation is the sum of an absolutely continuous function and a singular function of bounded variation, uniquely defined and called the singular part of the original function. Thus the proof of the above statement will carry with it, as an immediate consequence of I, the following result:

(II) *If $g(x)$ be any function of bounded variation with singular part $\gamma(x)$ continuous at the end points a, b,* we have*

$$\begin{aligned}\overline{\lim_{h\to 0}}\int_a^b |d[g(x+h)-g(x)]| &= \overline{\lim_{h\to 0}}\int_a^b |d[\gamma(x+h)-\gamma(x)]| \\ &= 2\int_a^b |d\gamma(x)|.\end{aligned}\tag{3}$$

This statement is of course only a rough corollary of the result for singular functions and absolutely continuous functions. More precise consequences to be born in mind are

(i) the left-hand equality in (3) holds not merely for h tending to 0 continuously, but for h tending to 0 through any discontinuous sequence;

* As an unessential restriction reducing $\int_{a+0}^{b+0}|d\gamma|$ to $\int_a^b|d\gamma|$.

(ii) the right-hand equality in (3) holds similarly for discontinuous approach of h to 0, if a certain set of measure 0 be avoided by h. To obtain any intermediary or the lower value of the considered limits, we have to take subsequences of this set of measure 0.

Examples will be constructed (§§4–6) to show what various possibilities exist with regard to the exceptional set of values of h. It will be seen that this set may be more than countable, or countable, or entirely absent. In particular therefore the limits in (3) may be unique limits. On the other hand, the examples will also show that the lower limits corresponding to the upper limits in (3) may in other cases have any lesser non-negative value. The case in which the lower limit is 0 would seem to have particular interest, and it might be useful to examine in greater detail the particular sequences of h_n tending to 0, for which this limit is obtained. It seems a question for instance whether a non-absolutely continuous function $g(x)$ exists for which $h_n = 1/n$ would provide a sequence of this kind, or more generally, any other special sequence for which h_n/h_{n+1} is bounded.

2. We use the following simple lemmas.

LEMMA 1. *Given two functions $g_1(x)$, $g_2(x)$ of bounded variation, of which $g_1(x)$ has derivative zero, $g_1'(x) = 0$, on the set complementary to a given set H_1 in (a, b). Then we have*

$$\int_a^b |d[g_1(x) - g_2(x)]| = \int_a^b |dg_1| + \int_a^b |dg_2| - 2\theta \int_{H_1} |dg_2|, \tag{4}$$

where $0 \leqq \theta \leqq 1$.*

We have

$$\int_{CH_1} |dg_2| - \int_{CH_1} |dg_1| \leqq \int_{CH_1} |d[g_2(x) - g_1(x)]|$$
$$\leqq \int_{CH_1} |dg_2| + \int_{CH_1} |dg_1|.$$

Since

$$\int_{CH_1} |dg_1| = 0,$$

we have

* This lemma has been stated initially in a less general form. The authors are indebted to Dr. S. Saks for the suggestion of extending their proof to the present, essentially more general, situation. We refer to de la Vallée Poussin, *Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire*, Paris, 1916, pp. 90–95, as to the notion of the total variation over a set, and as to some formulas which we are using here.

$$\int_{CH_1} |d[g_2 - g_1]| = \int_{CH_1} |dg_2|.$$

On the other hand, over the set H_1 itself,

$$-\int_{H_1} |dg_2| \leqq \int_{H_1} |d[g_2 - g_1]| - \int_{H_1} |dg_1| \leqq \int_{H_1} |dg_2|.$$

Hence, by addition,

$$\int_a^b |dg_2| - 2\int_{H_1} |dg_2| \leqq \int_a^b |d[g_2 - g_1]| - \int_a^b |dg_1| \leqq \int_a |dg_2|$$

giving (4) as required.

LEMMA 2. *Let E_h be a variable set depending on a parameter h, and such that each point x belongs to E_h at most for a set of measure* 0 *of values of h*. Then for each function $g(x)$ of bounded variation,*

$$\int_{E_h} |dg| = 0$$

for almost all values of h. (More precisely, the exceptional values of h form a set of measure 0 *depending on g.)*

This is immediate from the theory of change of order of integration in a repeated Stieltjes integral.† Let $G(x)$ be the indefinite total variation of $g(x)$:

$$G(x) = \int^x |dg|$$

and $E(x, h)$ the characteristic function of E_h, equal to 1 in E_h, and 0 elsewhere, for each value of h. We have then

$$V(h) = \int_{E_h} |dg| = \int_{-\infty}^{\infty} E(x, h)dG(x).$$

Integrating this with respect to h, we get

$$\int_{-\infty}^{\infty} V(h)dh = \int_{-\infty}^{\infty} dh \int_{-\infty}^{\infty} E(x, h)dG(x) = \int_{-\infty}^{\infty} dG(x) \int_{-\infty}^{\infty} E(x, h)dh,$$

* In other words, E_h is the section y=constant h, of a plane set E whose sections x=constant all have measure 0. Such a set E has plane measure 0, and it is well known that E_h must then have measure 0 for almost all h. The present Lemma states that E_h has also measure 0 with respect to any function of bounded variation, for almost all h, and is an immediate adaptation of the classical result.

† See, e.g., L. C. Young, *The Theory of Integration*, Cambridge Tracts, No. 21, p. 41 (Theorem IV).

where, by the hypothesis,

$$\int_{-\infty}^{\infty} E(x, h)dh = 0 \text{ for each } x,$$

as the measure of the set of values of h for which x belongs to E_h. Thus

$$\int_{-\infty}^{\infty} V(h)dh = 0,$$

and hence the non-negative function $V(h)$ vanishes except at most in a set of measure 0.

In our application of this Lemma, the set E_h will be the h-translation of a fixed set E_0 of measure 0, that is, the set of points x such that $x-h$ belongs to E_0. It has the characteristic function

$$E_h(x) = E_0(x + h).$$

For fixed x this still represents a set of measure 0 in h, and so the hypothesis of the Lemma is fulfilled. Thus we have

COROLLARY. *Given any function of bounded variation $g(x)$, and any set E_0 of measure* 0, *the total variation of g over the h-translation of E_0 vanishes except at most for a set of values of h of measure* 0.

3. We have now immediately our

THEOREM. *If $g(x)$ be a function of bounded variation constant in the complementary intervals of a closed set H of measure* 0, *we have, for almost all h,*

$$\int_a^b | d[g(x + h) - g(x)] | = \int_a^b | dg(x + h) | + \int_a^b | dg(x) | . \tag{5}$$

For, $g(x+h)$ is then constant in the complementary intervals of the h-translation H_h of H, so by Lemma 1, for

$$g_2(x) = g(x + h), \; g_1(x) = g(x),$$

the two sides of (5) differ by at most

$$2 \int_{H_h} | dg |$$

and by Lemma 2, in the Corollary form, this vanishes for almost all h.

In constructing examples, it is simplest to consider periodic functions with interval of periodicity (a, b), and assume for instance $a=0$, $b=1$. Then (5) takes the form

$$\int_0^1 | d[g(x + h) - g(x)] | = 2 \int_0^1 | dg(x) | . \tag{5'}$$

In that case also it suffices to consider only positive values of h, since $g(x+h)-g(x)$ is then still periodic with the same period as g, and so

$$\int_0^1 |d[g(x+h)-g(x)]| = \int_0^1 |d[g(x)-g(x-h)]| .$$

Furthermore we need only consider monotonic functions, with $\int_0^1 dg = 1$.

4. The classical example of a singular function of bounded variation is that of the monotone function constant in the complementary intervals of Cantor's typical ternary set* and representing an even mass distribution over this set, in the interval (0, 1). If x be expressed as a ternary fraction

$$x = \cdot\alpha_1\alpha_2\cdots\alpha_i\cdots = \sum_{i=1}^{\infty}\alpha_i 3^{-i},\ \alpha_i = 0,\ 1 \text{ or } 2, \tag{6}$$

and α_n be the first (if any) of its digits equal to 1, the function is defined at x to have the value

$$g(x) = \sum_{i=1}^{n-1}\alpha_i 2^{-i-1} + \alpha_n 2^{-n}, \tag{7}$$

or if there be no digit 1 in (6), then

$$g(x) = \sum_{i=1}^{\infty}\alpha_i\, 2^{-i-1}. \tag{7'}$$

This function is a particular example of a class of functions possessing the following property; this is most easily described by introducing the expression "ternary interval of order n" to designate specifically the open intervals of length $1/3^n$ whose left-hand end points are the terminating ternary fractions of at most n digits, all even:

$$\cdot\alpha_1\alpha_2\cdots\alpha_n\,0\,0\,0\cdots,\ \alpha_i = 0 \text{ or } 2.$$

Each such interval is contained in exactly one of lower order, and contains exactly 2^{r-n} intervals of higher order $r>n$. The intervals of given order are of course mutually exclusive and non-abutting.

PROPERTY (A). *For each ternary interval* (α, β) *of order* n, *and each ternary interval* (α', β') *of order* $(n+l)$ *contained in it,*

$$\left|\frac{g(\beta')-g(\alpha')}{g(\beta)-g(\alpha)}\right| \leqq \delta$$

where l is a positive integer and δ a constant less than 1, *independent of the particular interval.*

* The set of all non-terminating ternary fractions with even digits only, together with the terminating fractions whose last digit at most is odd.

Such functions share with the ordinary measuring function $m(x)=x$ a property relative to fractions with prescribed digits which we shall use in the form of

LEMMA 3. *If $g(x)$ be any function, constant in the complementary intervals of Cantor's ternary set H_0, and possessing the property* (A), *and if E be a subset of H_0 in which* (*in the ternary scale*)

$$x = \cdot\alpha_1\alpha_2 \cdots \alpha_i \cdots \tag{6}$$

has fixed prescribed digits (0 *or* 2) *for an infinity of indices,*

$$\alpha_{n_1}, \alpha_{n_2}, \cdots, \alpha_{n_i}, \cdots, n_1 < n_2 < \cdots \to \infty$$

(*the other digits being arbitrary,* 0 *or* 2), *then*

$$\int_E |dg| = 0.$$

Consider the ternary intervals of order n_k and left-hand end points

$\cdot\beta_1\beta_2 \cdots \beta_{n_1-1}\alpha_{n_1}\beta_{n_1+1} \cdots \beta_{n_2-1}\alpha_{n_2}\beta_{n_2+1} \cdots \beta_{n_k-1}\alpha_{n_k}$, α_{n_i} as above, $\beta_i = 0$ or 2,

and let σ_k denote their sum-set. Then $E \subset \sigma_k$ for each k (actually $E=\lim_{k\to\infty}\sigma_k$). Also

$$\int_{\sigma_k} |dg| \leqq \delta \int_{\sigma_{k-l}} |dg| . \tag{8}$$

This is because each interval of σ_k is in a different ternary interval of order n_k-l contained in σ_{k-l}, and by property (A) contributes not more than δ times the variation over this interval to the total variation over σ_k. Thus if $[k/l]$ be the integral part of k/l,

$$\int_E |dg| \leqq \int_{\sigma_k} |dg| \leqq \delta^{[k/l]} \int_0^1 |dg| \to 0 \text{ with } 1/k$$

since $\delta<1$.

This lemma is applied in conjunction with another relating purely to the ternary set H_0:

LEMMA 4. *To each non-terminating ternary fraction $h = \cdot a_1a_2 \cdots a_i \cdots$ there corresponds an infinite sequence of indices*

$$n_1 < n_2 < \cdots < n_i < \cdots \to \infty$$

and two specifications for the whole sequence of digits $\{c_{n_i}\}$ (0 *or* 2), *which a ternary fraction $x = \cdot c_1c_2 \cdots c_i \cdots$ must certainly satisfy if it is to belong to both H_0 and its h-translation H_h.*

This means that the common part H_0H_h is the sum of two sets of the kind considered in Lemma 3. What we prove precisely is as follows:

(i) If h has an infinite number of odd digits

$$a_{n_i} = 1,\ n_1 < n_2 < \cdots < n_i < \cdots \to \infty$$

then the corresponding digits of x must be alternately 0 and 2, i.e.

$$c_{n_1} = c_{n_3} = c_{n_5} = \cdots = 0 \text{ or } 2,$$
$$c_{n_2} = c_{n_4} = c_{n_6} = \cdots = 2 \text{ or } 0.$$

(ii) If h has only a finite number of odd digits,

$$a_i \neq 1 \text{ for } i > N,$$

and (r_i) are the indices for which $a_i = 0$, (s_i) those for which $a_i = 2$, after the Nth, then in x we must have either

$$c_{r_1} = c_{r_2} = \cdots = c_{r_i} = \cdots = 0$$

or

$$c_{s_1} = c_{s_2} = \cdots = c_{s_i} = \cdots = 2.$$

We deduce this from the following obvious facts, true for any h, terminating or not:

Given in the ternary scale

$$h = \cdot a_1a_2 \cdots a_i \cdots \qquad (a_i = 0,\ 1 \text{ or } 2),$$
$$x = \cdot b_1b_2 \cdots b_i \cdots \qquad (b_i = 0 \text{ or } 2),$$
$$x + h = \cdot c_1c_2 \cdots c_i \cdots \pmod 1 \qquad (c_i = 0 \text{ or } 2),$$

we have

(a) $$\text{if } a_{n_1} = a_{n_2} = 1,\ a_i \neq 1 \text{ for } n_1 < i < n_2,$$

then either

$$b_{n_1} = 0, \quad b_{n_2} = 2 \ (\text{and } c_{n_1} = 2,\ c_{n_2} = 0),$$

or the same with n_1 and n_2 interchanged;

(b) $$\text{if } \ a_r = b_r = 0\ (= c_r),\ a_s = b_s = 2\ (= c_s),$$

then $a_i = 1$ for some index between r and s (implying $|r-s| > 1$).

In (a) we consider two consecutive odd digits in h, and affirm that the two corresponding digits in x (and $x+h$) cannot be both 0 or both 2. In fact $b_{n_2} = 2$ would imply that we had in the formal addition $x+h$ to carry 1 from the n_2th place right back to the n_1th, where it would compound to 2 with a_{n_1}, and imply $b_{n_1} = 0$ if c_{n_1} is to be even. And similarly $b_{n_2} = 0$ implies $b_{n_1} = 2$.

In (b) we consider two places in each of which h and x have the same digits, but different in the two places, and we conclude that between those two places h must have at least one digit $=1$. We may assume that between those two places no further coincidences occur, and we see at once that if the formal addition $x+h$ is to yield only even digits between those two places, and h had none $=1$ there, these digits in $x+h$ would all be 0 with 1 to carry or all 2, with nothing to carry, and a 1 would appear in the sum in the earlier of the two places r and s.

From Lemmas 3 and 4, we deduce that, *for any function* $g(x)$ *of period* 1, *constant in the complementary intervals of Cantor's set* H_0 *and possessing the above property* (A), *we must have, whenever* h *is a non-terminating ternary fraction,*

$$\int_{H_h} |dg| = \int_{H_0 \cdot H_h} |dg| = 0.$$

From this and Lemma 1, we deduce (as in the proof of our principal theorem), that *for all such functions* $g(x)$,

$$\int_0^1 |d[g(x+h) - g(x)]| = 2\int_0^1 |dg|$$

whenever h *is a non-terminating ternary fraction.*

The exceptional values of h thus belong to the countable set of terminating ternary fractions.

5. In the case of Cantor's function (7, 7′), property (A) holds with $l=1$ and $\delta=\frac{1}{2}$, and equality sign. This provides therefore an instance of a function $g(x)$ for which the exceptional values of h are at most countable. It may be seen moreover that in this case every terminating ternary fraction is an exceptional value of h. For this it suffices to remark that (5) can certainly not hold when $g(x+h)-g(x)$ is constant in an interval in which neither $g(x)$ nor $g(x+h)$ is constant. And if, when

$$h = \cdot a_1 a_2 \cdots a_i \cdots a_n,$$

we choose (as we can)

$$x_0 = \cdot b_1 b_2 \cdots b_i \cdots b_n, \ \cdot b_i = 0 \text{ or } 2,$$

so that x_0+h has also only digits 0 or 2, then the ternary interval of order n and left-hand end point x_0 satisfies this condition.* If we go into the question more nearly, in order to investigate the lower limit of

$$\int_0^1 |d[g(x+h) - g(x)]| \text{ as } h \to 0,$$

* Cf. also Hille and Tamarkin, loc. cit., p. 261.

we find that for our present function, whereas

$$\int_0^1 | d[g(x + h) - g(x)] | = 2$$

for h non-terminating, we have for h terminating of exactly n digits,

$$1 \leqq \int_0^1 | d[g(x + h) - g(x)] | \leqq 2(1 - 2^{-n}), \tag{9}$$

with actual equality on either side for suitable values of h and each n. For let

$$h = \cdot a_1 a_2 \cdots a_n, \; a_i = 0, 1 \text{ or } 2, \; a_n \neq 0;$$

and suppose to fix the ideas that $a_n = 2$. Then every ternary interval of order n and left-hand end point

$$\cdot b_1 b_2 \cdots b_n, \; b_i = 0 \text{ or } 2,$$

with $b_n = 2$, translates through h into an interval in which g is constant, that is to say is itself an interval in which $g(x+h)$ is constant. Similarly if $a_n = 1$ and we take $b_n = 0$. So over half the ternary intervals of order n, the relation (5) certainly holds. Similarly it holds for all the intervals of length 2^{-n} and left-hand end points $\cdot b_1 b_2 \cdots b_k$ with $b_k = 1, k \leqq n$, over which $g(x)$ is constant. There only remain the other half of the ternary intervals of order n, over which $g(x)$ has variation 1/2, and $g(x+h)$ at most 1/2. So in this case the right-hand side of (5′) cannot exceed the left by more than 1.

The other inferences follow also very simply. For instance

$$a_n = a_{n-k_1} = \cdots = a_{n-k_r} = 1, \; a_i = 0 \text{ or } 2 \text{ for } n - k_j < i < n - k_{j-1},$$
$$j = 1, 2, \cdots, r + 1, \; k_0 = 0, \; k_{r+1} = n,$$

according as j is odd or even, give some of the (everywhere dense set of) values of h for which equality occurs on the right of (9), while

$$a_i = 2 \text{ for } i < n, \; a_n = 1 \text{ or } a_i = 0 \text{ for } i < n, \; a_n = 2$$

give the only values for which equality occurs on the left.

6. The classical monotone function just discussed expresses the total mass in $(0, x)$ due to mass unity, evenly distributed among the ternary intervals of order n, for each n, and its constancy in each complementary interval of the ternary set H_0 expresses the fact that there is no mass in these complementary intervals.

We now modify this construction so that the function is constant not only in the complementary intervals of H_0, but also in a set (everywhere dense on H_0) of the ternary intervals themselves, and show that we can choose these additional intervals of constancy (in a more than countable number of ways) so that the resulting functions satisfy (5) for every terminating ternary fraction h. The functions will all possess the property (A), and therefore satisfy (5) for non-terminating values of h, and so we shall have our example of functions for which h has no exceptional values, (5) being always true.

The definition is obtained inductively by supposing the mass distribution effected in the ternary intervals of order $2n$ and now dividing the mass of each such interval equally among *three* (instead of all) of the four ternary intervals of order $2(n+1)$ which it contains. The choice of these three intervals may be effected in four different ways, and as there will be exactly 3^n ternary intervals of order $2n$ actually containing mass, this means that we have $4\cdot 3^n$ different modes of distribution to choose from at each stage. In any case, each ternary interval of order $n+2$ will contain at most $1/3$ of the mass of the interval of order n containing it (viz. either 0 or exactly $1/3$ of it).

As the mass of any interval is the increment of the monotone function $g(x)$ representing the ultimate mass in $(0, x)$, we see that each function so obtained possesses the property (A) of page 332 with $l=2$, $\delta=1/3$.

The construction provides moreover a more than countable set of functions of this kind, for all of which (5) therefore holds whenever h is a non-terminating ternary fraction. By eliminating a certain minority of these functions, we may arrange so that (5) holds also for each of the remaining countable set of values of h. The simplest way of doing this is to associate with each of these values of h, i.e., terminating of say k digits, an infinite sequence of indices $>k$:

$$N_i = N_i^{(h)} \to \infty$$

such that

$$N_i^{(h)} \neq N_j^{(h')} (\text{all } i, j)$$

whenever $h' \neq h$ (this can always be done in any number of ways*), and to stipulate that for each $n = N_i^{(h)}$, two ternary intervals of order $2n$ which are h-translations one of the other (for that h) should never both be intervals devoid of mass. For instance if

$$h = \cdot a_1 a_2 \cdots a_k, \qquad a_i = 0, 1 \text{ or } 2, a_k \neq 0,$$

and

* For instance if $\{h_i\}$ be the considered set of values of h in any countable order, and $\{p_i\}$ the sequence of primes in increasing order, we can associate with each h_i the sequence of integers $p_i p_{i+n}$, $n = n_0, n_0+1, \cdots$.

$$x = \cdot b_1 b_2 \cdots b_k b_{k+1} \cdots b_{2n}, \quad b_i = 0 \text{ or } 2,$$

is the left-hand end point of a ternary interval of order $2n$ devoid of mass, and if $x+h$ is still a left-hand end point of a ternary interval of order $2n$, i.e.

$$x + h = \cdot c_1 c_2 \cdots c_k b_{k+1} \cdots b_{2n}, \; c_i = 0 \text{ or } 2,$$

then this latter interval must not be devoid of mass. Since $2n > k+1$, the two ternary intervals considered are necessarily in two different intervals of order $2(n-1)$, so that at the stage $n-1$ in our construction we can certainly make our choice (in at least 3^{3^n} different ways) so that the condition be fulfilled for that n. By the fact that to each index n that we have particularly to take into consideration corresponds only one h (with reference to which the restriction is made), a question of incompatibility for the different values of h cannot arise, and we are certainly left with a majority of the functions considered. For this residue, we see that for each h, (5) holds outside at most two of the ternary intervals of order $2n$ in each ternary interval of order $2(n-1)$ for each $n = N_i^{(h)}$. Thus if σ_i represent the intervals of this order $2n$ for which (5) may possibly not hold (more precisely in which neither $g(x)$ nor $g(x+h)$ is constant), we have as in the proof of Lemma 2 (inequality (8)),

$$\int_{\sigma_i} |\, dg \,| \leqq \delta \int_{\sigma_{i-1}} |\, dg \,|, \; \delta = \tfrac{2}{3},$$

and hence this variation tends to 0 with $1/i$. Thus (5) holds in the complementary intervals of a set over which $g(x)$ has total variation 0, that is, holds absolutely, for the arbitrary considered terminating h.

7. The object of rarifying the exceptional values of h as much as possible was achieved by an increased concentration of the unit mass distributed over our interval (0, 1), introducing additional intervals of constancy for $g(x)$. Conversely, we can multiply the exceptional set of values of h by breaking up the intervals of constancy, and diffusing the total mass more over the whole interval.

By this means we shall obtain a function with a more than countable set of exceptional h, and at the same time we shall find our example of a case in which the lower limit corresponding to (3) is 0.*

Let $n_1, n_2, \cdots$ be a sequence of integers increasing so rapidly that $\sum_1^\infty 1/n_k$ converges. Let x be any number in (0, 1). It is a theorem due to Cantor that x may always be expressed in the form

$$x = \frac{m_1}{n_1} + \frac{m_2}{n_1 n_2} + \frac{m_3}{n_1 n_2 n_3} + \cdots, \tag{10}$$

* The actual construction is adapted from a suggestion of Mr. A. S. Besicovitch.

where $m_1<n_1$, $m_2<n_2$, $\cdots$, and that this representation is unique, save for the ambiguity

$$\frac{1}{n_1n_2\cdots n_k}=\frac{n_{k+1}-1}{n_1\cdots n_{k+1}}+\frac{n_{k+2}-1}{n_1\cdots n_{k+2}}+\cdots,$$

which gives a non-terminating alternative to any terminating expression of a number. Let us assume that every n_k is even, equaling $2\nu_k$. If x is expressed as in (10), let

$$f(x)=\frac{m_1/2}{\nu_1}+\frac{m_2/2}{\nu_1\nu_2}+\frac{m_3/2}{\nu_1\nu_2\nu_3}+\cdots,$$

if every m_k is even, and

$$f(x)=\frac{m_1/2}{\nu_1}+\frac{m_2/2}{\nu_1\nu_2}+\cdots+\frac{m_{n-1}/2}{\nu_1\nu_2\cdots\nu_{n-1}}+\frac{[m_n/2]+1}{\nu_1\nu_2\cdots\nu_n},$$

if m_n is the first odd m_k. Clearly $f(x)$ is a monotone function, constant over every interval

$$\left(\frac{2\mu_k+1}{n_1n_2\cdots n_k},\ \frac{2\mu_k+2}{n_1n_2\cdots n_k}\right)\text{ where }\mu_k<\nu_k.$$

These intervals for any given value of k fill up half the line, and for $k=1, 2, 3, \cdots, K$ fill up a set of intervals of measure $1-2^{-K}$.

Let h be defined as

$$\frac{a_1}{n_1}+\frac{a_2}{n_1n_2}+\cdots+\frac{a_k}{n_1n_2\cdots n_k}+\cdots,$$

where every a_k is 0 or 2. We have

$$f(x+h)-f(x)=\frac{a_1}{2\nu_1}+\frac{a_2}{2\nu_1\nu_2}+\frac{a_3}{2\nu_1\nu_2\nu_3}+\cdots$$

whenever

$$2\leqq m_r\leqq n_r-2,\ 2\leqq m_{r+1}\leqq n_{r+1}-2,\cdots,$$

where r is the first index for which $a_k\neq 0$. The total variation of $f(x)$ over the set of points for which $m_k<2$ does not however exceed $1/\nu_k$, and the same is true of the total variation of $f(x+h)$ over those intervals. A similar result applies to the range $n_k-m_k<2$. Thus the total variation of $f(x+h)-f(x)$ over these intervals does not exceed

$$4(1/\nu_r+1/\nu_{r+1}+\cdots+1/\nu_{r+k}+\cdots) \tag{11}$$

which is hence an upper bound for the total variation of $f(x+h)-f(x)$ over $(0, 1)$. The total variation of $f(x)$ over the same interval is 1. Thus we have here a set of values of h of the power of the continuum including arbitrarily small values, for which

$$\int_0^1 |\, d[f(x+h) - f(x)]\,| \neq 2 \int_0^1 |\, df(x)\,|,$$

and the total variation on the left-hand side tends to zero (since (11) does so) when h tends to zero through this set.

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