# ON ANALYTICAL COMPLEXES*

BY

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1. In his Colloquium Lectures† one of us outlined a proof of an important theorem regarding the covering of analytic loci by complexes. A proof for algebraic varieties had previously been given by B. van der Waerden‡ and B. O. Koopman and A. B. Brown§ have recently proved the theorem for analytic loci. The object of this paper is to give a detailed proof along the lines indicated in *Topology*.

2. We begin with certain general observations|| concerning the nature of a configuration $\xi$ (at first complex) represented by an analytic system

$$(2.1) \qquad F_h(x_1, \cdots, x_n) \equiv F_h(x) = 0 \qquad (h = 1, 2, \cdots, r),$$

in the vicinity of a given point $O$ of $\xi$ which we take as the origin throughout for the complex euclidean space $S_n$ containing $\xi$. There is a neighborhood of $O$ relative to $\xi$ consisting of a finite number of algebroid elements, any one of them, say $w_p$, having about its center $O$, in a suitable coordinate system $y_i$, a canonical representation

$$(2.2) \quad \begin{aligned} &\text{(a)} \qquad H(y_1, \cdots, y_p, y_{p+1}) = 0, \\ &\text{(b)} \qquad \frac{\partial H}{\partial y_{p+1}} \cdot y_{p+1+i} + G_i(y_1, \cdots, y_{p+1}) = 0, \end{aligned}$$

where $H$, $G_i$ are *pseudopolynomials* in $y_{p+1}$, i.e. polynomials with coefficients analytic in $y_1, \cdots, y_p$ at $(y) = (0)$, and where moreover $H$ is algebraically irreducible and *special*, i.e. its leading coefficient is unity and its other coefficients are zero at (0). $p$ is the *complex dimension* of $w_p$ (dim $w_p$), and also of $\xi$ at $O$ ($\dim_O \xi$) when dim $w = p$ for some $w$ component of $\xi$ at $O$, and $\leqq p$ for all others. When $O$ is not on $\xi$ we agree to take $\dim_O \xi = -1$.

We have the following basic *irreducibility property*: if $\xi$ does not contain $w_p$, then the intersection $\xi \cdot w_p$ is a $\xi$, whose dimension $< p$ at $O$. For the case

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† S. Lefschetz, *Topology*, Colloquium Publications, vol. XII, New York, 1930, p. 364. Except as introduced here the same notation and terminology will be used as in *Topology*.

‡ Mathematische Annalen, vol. 102, pp. 337–362.

§ These Transactions, vol. 34 (1932), pp. 231–252.

|| Based on Osgood's *Lehrbuch der Funktionentheorie*, vol. II, chapter II.

where $\xi$ is defined by a single relation (2.1) see Osgood's proof (loc. cit., p. 133), and the extension to any $\xi$ is obvious.

We shall now recall a series of properties most of them direct consequences of the preceding.

I. The solution of an infinite system (2.1) about any point $O$ is of the same type as for a finite system.

II. A point of $w_p$ is *singular* if the rank of the Jacobian matrix $J$ of (2.2) is $<n-p$ at the point; it is an *ordinary* point otherwise. The locus $\alpha$ of the singular points is the *singular* locus of $w_p$. Since $J$ contains a minor of order $n-p$ equal to $(\partial H/\partial y_{p+1})^{n-p}\not\equiv 0$ when $H=0$, the conditions that the rank be $<n-p$ define a $\xi$ not containing $w_p$. Hence $\alpha\cdot w_p=\alpha$ is a $\xi$ and $\dim_O \alpha<p$.

The characteristic property of an ordinary point $(a)$ is to have relative to $w_p$ a neighborhood which is a $2p$-cell $E_{2p}$ with a parametric representation

$$x_i - a_i = \phi_i(u_1, \cdots, u_p), \tag{2.3}$$

where at $(u)=(0)$ the $\phi$'s are analytic, vanish and have a Jacobian matrix of rank $p$. Every point of $w_p$ is a limit-point of ordinary points.

III. It is impossible to decompose $w_p$ about $O$ into a sum of $r>1$ sets $w^i_{q_i}$. For otherwise $w^i=w^i\cdot w_p\neq w_p$, hence $q_i<p$. Therefore $w_p$ would have points about which the coordinates depend upon $q_i<p$ parameters, which is untrue. As a noteworthy consequence the resolution of $\xi$ into $w$ components about $O$ is unique and hence $\dim_O \xi$ depends solely upon $O$ and $\xi$.

IV. Given a fixed coordinate system $x_i$ we shall call *vertical* the direction of its $x_n$ axis and denote by $P(\lambda)$ the projection of the locus $\lambda$ on $x_n=0$. If the center $O$ of $w_p$ is an isolated intersection with the vertical through $O$, then $P(w_p)$ is a $w_p$ of center $P(O)$. This does not require that the coordinates $x_i$ be canonical for $w_p$. We may of course assume that $O$ is the origin so that $P(O)=O$. Under the assumption $w_p$ may be represented by a system (2.1) such that no $F_h(0, \cdots, 0, x_n)\equiv 0$, hence we may replace all the $F$'s by pseudopolynomials in $x_n$. The algebraic elimination of $x_n$ yields then a system such as (2.1) without $x_n$, representing $P(w_p)$; hence $P(w_p)$ is a $\xi$. If this $\xi$ had $r>1$ components about $O$ the vertical cylinders erected on them would decompose $w_p$ into a $\xi$ having at least $r$ components $w$ about $O$. Therefore $r=1$ and $P(w_p)$ is a $w$ of center $O$. If a point $Q$ varies on $w_p$, $x_n(Q)$ is a finite-valued function of $P(Q)$, hence $P(Q)$ depends on $p$ parameters and $P(w_p)$ is a $w_p$.

Since $x_n$ is a finite-valued function on $P(w_p)$ we have for $w_p$ a representation (Osgood, loc. cit., p. 114)

$$(2.4)\quad \begin{array}{ll} \text{(a)} & G_i(x_1, \cdots, x_{n-1}) = 0, \\ \text{(b)} & H(x_1, \cdots, x_n) = 0, \end{array}$$

where $H$ is a pseudopolynomial in $x_n$ and (2.4a) represents $P(w_p)$ in $x_n=0$. Since no true subset of $w_p$ is a $w_p$, $H$ is irreducible.

The branch locus $\beta$ of $w_p$ is its intersection with $\partial H/\partial x_n=0$. Just as for the singular locus we have $\dim_O\beta<p$. Hence $w_p$ possesses ordinary points not on $\beta$.

3. We shall now consider a real analytic variety $\eta$. It is a real locus represented by a real system (2.1) or system with $F$'s all real.* The same system represents a $\xi$ to be denoted by $(\eta)$. Let $O$ be a point of $\eta$. On following up Osgood's resolution of $(\eta)$ into $w$ components about $O$ we find that their canonical coordinates $y_i$ may be chosen real. This being assumed done we have for a component $w_p$ three possibilities: (a) The canonical system of $w_p$ is real and $w_p$ possesses real ordinary points. The real subset of $w_p$ (real algebroid element) will be denoted by $v_p$, so that $w_p=(v_p)$; incidentally the form of (2.2) shows that when $p=0$, $O$ is an ordinary point, i.e., it is a $v_0$. (b) This case is the same as the preceding except that the real points of $w_p$ are all singular. (c) The canonical system of $w_p$ cannot be chosen real. When $w_p=w_p$, $H$ and $G_i$ in (2.2) may be replaced by $H+\overline{H}$, $G_i+\overline{G}_i$, both real and of the same form, hence we have cases (a) or (b). Therefore in case (c) necessarily $w_p\neq\widehat{w}_p$.

We shall now show that $\eta$ may be decomposed about $O$ into a finite sum of $v$'s. Let $p=\dim_O(\eta)$. Since the required result holds when $p=0$ we use induction on $p$. The real points of a $w_p$ of type (b) are on the singular locus of $w_p$ which is an $(\eta)$ whose dimension at $O$ is $<p$. As regards the real points of a $w_p$ of type (c), let $f_j=0$ be the canonical equations of $w_p$. Since (2.1) is real, $\bar{f}_j=0$ are the canonical equations of another component of $(\eta)$ which is $\widehat{w}_p$. Hence the real points in question are on $w_p\cdot\widehat{w}_p$ and since $w_p$ does not contain $\widehat{w}_p$, this is a $\xi$ whose dimension at $O$ is $<p$. But this $\xi$, being represented by the real system $f_j+\bar{f}_j=0$, $-i(f_j-\bar{f}_j)=0$, is also an $(\eta)$. The real points of components not of type (a) being thus on varieties $(\eta)$ whose dimensions at $O$ are $<p$, the required result is a consequence of the hypothesis of the induction.

The meaning of $\dim v_p$, $\dim_O\eta$ is as before. As it happens they are precisely the Urysohn-Menger dimensions, but this does not matter for our purpose.

The *irreducibility property* holds for $\eta$: if $\eta$ does not contain $v_p$, $\dim_O\eta\cdot v_p<p$. Its proof is as follows. Under the hypothesis $(\eta)$ does not contain $(v_p)$, hence $p>\dim_O(\eta\cdot v_p)=\dim_O\eta\cdot v_p$.

Properties I, $\cdots$, IV hold with $v$ in place of $w$ and with these modifications: (a) (2.3) represents a real analytic $E_p$; (b) (2.4) still represents $v_p$ in the

* The condition $\overline{F}(\bar{x})=\overline{F(x)}$ defines an analytic function $\overline{F}$, the *conjugate* of $F$, and $F$ is *real* whenever $\overline{F}\equiv F$. The set of the conjugate points of the points of a locus $\lambda$ will be denoted by $\widehat{\lambda}$, the usual "bar" notation being reserved for the closure.

real and $(v_p)$ in the complex domains, but (2.4a) represents a real $v_p'$ which may be $\neq P(v_p)$, since in addition it may contain points which are the projections of pairs of conjugate points of $(v_p)$. Thus we can only assert that $P(v_p)$ is a subset of a $v_p'$. Here again $v_p$ contains an ordinary point $Q$ not on the branch locus $\beta$. $Q$ possesses then relative to $v_p$ a neighborhood which is an analytic $E_p$ homeomorphic with $P(E_p)$. This implies that in (2.3) the Jacobian matrix of $\phi_1, \cdots, \phi_{n-1}$ is of rank $p$ at $(u)=(0)$. Hence $P(Q)$ has a neighborhood relative to $P(v_p)$, and not merely relative to $v_p'$, which is an analytic $E_p$. We may think of $P(Q)$ as an ordinary point of $P(v_p)$.

*Henceforth we shall deal exclusively with the real domain.*

4. **The segments on $v_p$.** Let $\alpha_h$ be direction cosines for $S_n$, so that $(\alpha)$ is a point of the unit-sphere $H_{n-1}$ of $S_n$. A point $(x)$ of $v_p$ $(p<n)$ will not be an isolated intersection of $v_p$ with the line $x_h+s\alpha_h$ ($s$ variable) when and only when the MacLaurin series for $s$ of the functions $f_i(x+s\alpha)$ are $\equiv 0$, where the $f$'s are the left-hand sides of a representation (2.1) for $v_p$. There results a real analytic system

$$\Phi_j(x;\alpha)=0. \tag{4.1}$$

Its solutions for $(x;\alpha)$ in the vicinity of any solution $(x^0;\alpha^0)$ make up a finite number of sets $v_q$. On such a $v_q$ we shall then have a parametric representation

$$\text{(a)}\quad x_i=\phi_i(y_1,\cdots,y_q);\quad \text{(b)}\quad \alpha_i=\psi_i(y_1,\cdots,y_q), \tag{4.2}$$

where $\phi_i$, $\psi_i$ are analytic on $v_q$. The system (4.2b) represents on $H_{n-1}$ the directions near $(\alpha^0)$ corresponding to segments on our given $v_p$ associated with (4.2).

Since (4.2) represents a $v_q$,

$$\left\|\frac{\partial\phi_i}{\partial y_j};\ \frac{\partial\psi_i}{\partial y_j}\right\| \tag{4.3}$$

is of rank $q$ at some points as near as we please to $(y^0)$. On the other hand, for $y_1, \cdots, y_q$ near $y_1^0, \cdots, y_q^0$ and $s$ arbitrary but small, $\phi_i+s\psi_i$ represents a point of our initial $v_p$, and hence among these functions at most $p$ are functionally independent, or

$$\left\|\frac{\partial\phi_i}{\partial y_j}+s\frac{\partial\psi_i}{\partial y_j};\ \psi_i\right\| \tag{4.4}$$

is of rank $\leqq p$, and this must hold for $s$ small but arbitrary. Now any determinant of this matrix containing $s$ is a polynomial in $s$ whose leading coefficient is the corresponding determinant of

$$\left\| \frac{\partial \psi_i}{\partial y_j}; \psi_i \right\| \tag{4.5}$$

which must therefore be of rank $\leqq p$. Owing to the relations

$$\sum \psi_i^2 = 1, \ \sum \psi_i \frac{\partial \psi_i}{\partial y_j} = 0,$$

the new matrix may be bordered with a row $0, \cdots, 0, 1$ without changing its rank. It follows that the rank of

$$\left\| \frac{\partial \psi_i}{\partial y_j} \right\|$$

is at most $p-1 \leqq n-2$. Therefore the directions of segments meeting $v_p$ in an infinite set are represented on $H_{n-1}$ by a variety $\eta$ whose dimension at any point $<n-1$, and hence they are nowhere dense on the sphere.†

5. Analytic complexes. By an *analytic structure* $\zeta$ we shall mean a real point set in a real $S_n$ which constitutes a topological space with varieties $\eta$ as the neighborhoods. Each point $Q$ of $\zeta$ has then a neighborhood relative to $\zeta$ made up of a finite set $v_{q_1}^1, \cdots, v_{q_r}^r$, where the $v$'s have $Q$ as their common center. The largest $q_r$ is the *dimension of* $\zeta$ *at* $Q$ ($\dim_Q \zeta$), and the largest value $p$ of $\dim_Q \zeta$ for $Q$ on $\zeta$ is the *dimension* of $\zeta$ ($\dim \zeta$) which is then designated by $\zeta_p$.

We now define the point $Q$ of $\zeta$, whose neighborhood is $v_{q_1}^1 + \cdots + v_{q_r}^r$, as *singular* when $r>1$, or when $r=1$ and $\dim_Q \zeta < p$, or else $\dim_Q \zeta = p$ and $Q$ is singular for its unique $v$. From property II of §2 for an $\eta$, we have that the set of all singular points or *singular locus* is a $\zeta_r$, $r<p$. A point of $\zeta_p - \zeta_r$ is an *ordinary* point of $\zeta_p$. Its characteristic property is that it possesses relative to $\zeta_p$ a neighborhood which is an analytic $E_p$.

By an *analytic p-element*, or merely $p$-element, $\epsilon_p$, we shall mean a relatively open subset of a structure $\zeta_p$, containing at least one ordinary point, and such that $\bar{\epsilon}_p \subset \zeta_p$. Under these conditions we shall describe $\epsilon_p$ and $\zeta_p$ as *associated* with each other. By an *analytic p-complex*, $\kappa_p$, we shall mean a finite set of non-intersecting elements $\epsilon$, of dimension up to and including $p$, which constitute a closed bounded point set in $S_n$. By convention the empty set is to be a $\zeta_{-1}$, an $\epsilon_{-1}$ or a $\kappa_{-1}$. We shall write $F(\zeta) = \bar{\zeta} - \zeta$. We do not consider here infinite $\kappa$'s, since they may be taken care of as in *Topology*.

The intersection of two or more $\zeta$'s or $\epsilon$'s is respectively a $\zeta$ or an $\epsilon$. If $\kappa = \sum \epsilon$, $\kappa^* = \sum \epsilon^*$ are complexes, so is $\kappa \cdot \kappa^* = \sum \epsilon \cdot \epsilon^*$. Similarly when $\kappa \cdot \zeta$ is closed then $\kappa \cdot \zeta$ is the complex $\sum \epsilon \cdot \zeta$.

† Cf. Koopman and Brown, loc. cit., p. 242.

A complex $\kappa'$ will be called a *subdivision* of $\kappa$ if the two coincide as point sets and if each element of $\kappa'$ is contained in one of $\kappa$. If $\kappa^*$ is any complex on $\kappa$ it is clear from the preceding paragraph that $\kappa$ has a subdivision with one of $\kappa^*$ as a subcomplex. It follows that $\kappa+\kappa^*$ can be covered by a complex having subdivisions of $\kappa$ and $\kappa^*$ as subcomplexes. For $\kappa$ and $\kappa^*$ can each be subdivided to form a complex having a common subcomplex covering $\kappa\cdot\kappa$.*

Whenever throughout $\kappa$ we have $\epsilon_p\cdot\bar{\epsilon}_q=0$ for $q\leqq p$, $\kappa$ is said to be *normal*. When $\kappa$ is normal, it remains closed, and hence a $\kappa$ (moreover a normal $\kappa$) when one or more $p$-elements are removed from it.

*Every complex has a normal subdivision.* Given any $\zeta_p$, we shall denote its singular locus by $\zeta_{p'}$ $(p'<p)$. Let then $\epsilon_p$, $\epsilon_q$ be two elements of $\kappa_p$ and $\zeta_p$, $\zeta_q$ associated structures. We shall first show that there exists a $\zeta_r\supset\epsilon_p\cdot\bar{\epsilon}_q$ such that $r<p$ and that the distance from $\epsilon_p\cdot\bar{\epsilon}_q$ to $F(\zeta_r)>0$. In any case $\epsilon_p\cdot\bar{\epsilon}_q\subset\bar{\epsilon}_p\cdot\bar{\epsilon}_q\subset\zeta_p\cdot\zeta_q=\zeta_s$. Also $F(\zeta_s)\subset F(\zeta_p)+F(\zeta_q)$. Since no $\bar{\epsilon}$ meets its $F(\zeta)$, $\bar{\epsilon}_p\cdot\bar{\epsilon}_q$ does not meet $F(\zeta_s)$, and as $\bar{\epsilon}_p\cdot\bar{\epsilon}_q$ is self-compact the distance of the two sets $>0$. Therefore when $s<p$ we may take $\zeta_r=\zeta_s$.

Let now $s=p$ and let $Q$ be a point of $\epsilon_p\cdot\bar{\epsilon}_q$ not on $\zeta_{s'}$, so that $Q\subset\zeta_s-\zeta_{s'}$ and $\dim_Q\zeta_s=p$. This implies $q=p$ and that the nieghborhoods of $Q$ relative to $\zeta_p$ and $\zeta_q$ have a common $v_p$ which is then wholly on $\epsilon_p$ near $Q$. In that case necessarily $Q\subset\zeta_{q'}$. For otherwise $v_p$ would be a complete neighborhood of $Q$ relative to $\zeta_q$, hence it would contain points of $\epsilon_q$ infinitely near $Q$, and we should have $\epsilon_p\cdot\epsilon_q\neq 0$, which is ruled out. It follows $\epsilon_p\cdot\bar{\epsilon}_q\subset\zeta_{q'}+\zeta_{s'}$.

Since a singular locus is closed relative to its $\zeta$, and since $F(\zeta_s)\subset F(\zeta_p)+F(\zeta_q)$, we find that $\zeta_r=\zeta_{q'}+\zeta_{s'}-F(\zeta_p)-F(\zeta_q)$ satisfies the condition for a structure, with $F(\zeta_r)\subset F(\zeta_p)+F(\zeta_q)$. Since the last two $F$'s do not meet $\bar{\epsilon}_p\cdot\bar{\epsilon}_q$, this is likewise the case as regards $F(\zeta_r)$, which implies also $\zeta_r\supset\bar{\epsilon}_p\cdot\bar{\epsilon}_q\supset\epsilon_p\cdot\bar{\epsilon}_q$, and that the distance condition holds. Since $r=p'$ or $q'$, both $<p$, $\zeta_r$ has all the properties that we require.

We can find a closed polyhedral neighborhood of $\bar{\epsilon}_p\cdot\bar{\epsilon}_q$ not meeting $F(\zeta_r)$, and its intersection with $\zeta_r$ is a $\kappa_r$. The sum of these complexes for all $p$-elements is a $\kappa_t$, $t<p$, and $\epsilon_p'=\epsilon_p-\kappa_t$ is an element. Replacing $\epsilon_p$ by $\epsilon_p'$ together with the sum of the elements $\epsilon_p\cdot\kappa_t$, we obtain a subdivision $\kappa_p'$, such that $\epsilon_p'\cdot\bar{\epsilon}_q'=0$ if $\epsilon_q'\neq\epsilon_p'$. Hence $\kappa_p'-\sum\epsilon_p'$ is a $\kappa$ whose dimension $<p$. The required result follows then by induction on $p$.

6. **The covering theorem.** Let $\kappa$ be any complex and let "vertical" direction or projection have the same meaning as in §2, IV. Every point of $\kappa$ has a neighborhood relative to $\kappa$ made up of a finite number of $v$'s. Since $\kappa$ is self-compact it can be covered with a finite number of $v$'s. It follows then from §4 that the axes may be so chosen that no vertical meets $\kappa$ in an infinite set.

From §§3, 5, we conclude that $\kappa$ has a subdivision (obtained as its intersection with a suitable polyhedral complex) whose elements are each on a $v$ represented by a system (2.4). The subdivision can then be normalized so that at each step in the process the preceding property is preserved. Ultimately we turn the complex into a normal complex, still called $\kappa$, whose elements all have the property just described.

Let $\epsilon_p$ be any element of our new $\kappa$ with its $\zeta_p$ given by (2.4). The branch locus $\zeta^*$ is of dimension $<p$. We verify at once that $\epsilon^* = \epsilon_p \cdot \zeta^*$ and $\epsilon_p - \epsilon^*$ are elements with $\zeta^*$ and $\zeta_p$ as associated structures. Referring to the end of §3 we find also that, when $p<n$, $P(\epsilon_p - \epsilon^*)$ is an $\epsilon_p$. Moreover when $(x_1, \cdots, x_{n-1})$ ranges over $P(\epsilon_p - \epsilon^*)$, to certain real roots $x_n'$ of $H=0$ there will correspond points $(x_1, \cdots, x_{n-1}, x_n')$ which generate elements $\epsilon_p'$ whose sum is $\epsilon_p - \epsilon^*$.

Assuming that our complex is a $\kappa_p$, $p<n$, we decompose every $\epsilon_p$ of $\kappa_p$ in the set of $p$-elements $\epsilon_p'$ plus $\epsilon^*$ (whose dimension $<p$), and repeat the operation for the elements of next lower dimension of the new complex, etc. Ultimately then we have in place of $\kappa_p$ a new normal complex, still to be called $\kappa_p$, such that every $\epsilon$ of $\kappa_p$ has for projection $P(\epsilon)$ an element, and $\epsilon$ is represented by an analytic relation $x_n = f(Q)$, $Q \subset P(\epsilon)$ (analytic homeomorphism).

A final subdivision $\kappa_p' = \sum \epsilon'$ of $\kappa_p$ will now be made, such that $P(\kappa_p')$ can be covered with a $\kappa_p'' = \sum \epsilon''$ having the property that every $P(\epsilon')$ is an exact sum of elements $\epsilon''$. For $p=0$ this is trivial, hence we use induction on $p$. Taking $\kappa_p$ in the reduced form just obtained, $\kappa_q = \kappa_p - \sum \epsilon_p$ is a complex with $q<p$. Under the hypothesis of the induction it has a subdivision $\kappa_q' = \sum \epsilon'$ of the desired type. Let $\delta$ be a positive number such that $-\delta < x_n < \delta$ on $\kappa_p$, and let $C(\kappa_q')$ be the $(q+1)$-complex whose elements are the parts of the vertical cylinders based on the $\epsilon''$s lying between the spaces $x_n = \pm\delta$, together with their intersections with these spaces. Let $\epsilon_r$ be an element of the original $\kappa_p$. Since an $\epsilon_r$ carries no vertical segment, the intersection $\epsilon_r \cdot C(\kappa_q')$ consists of elements of dimension $\leqq q$, some being of dimension $q$ when $r=q$. Therefore $\kappa_p \cdot C(\kappa_q')$ is a $q$-complex, and since $q<p$, it has a subdivision $\kappa_q^{*\prime}$ such that $P(\kappa_q^{*\prime})$ is covered by a $\kappa_q''$ of the required type. Given any $\epsilon_p$ of $\kappa_p$ we form a new element $\epsilon_p' = \epsilon_p - \kappa_q^{*\prime}$. Then $\kappa_p' = \kappa_q^{*\prime} + \sum \epsilon_p'$ is the required subdivision of $\kappa_p$. For let $\kappa_p'$ contain $m$ $p$-elements $\epsilon_p'^{\alpha}$ and let $\eta^\alpha = P(\epsilon_p'^{\alpha})$. When $m=1$, we can take $\kappa_p'' = \kappa_q'' + \eta'$. Therefore we may use induction on $m$. Removing $\epsilon'^m$ from $\kappa_p'$ we have a complex $\kappa_p^{*\prime}$ which, under the hypothesis of the induction, possesses an associated $\kappa_p^{*\prime\prime}$ covering $P(\kappa_p^{*\prime})$. Now $\eta' = \eta^m - \kappa_p^{*\prime\prime}$ is also an element and $\kappa_p'' = \eta' + \kappa_p^{*\prime\prime}$ is a covering of $P(\kappa_p')$ such as we are seeking.

Observe that every $\epsilon'$ is still analytically homeomorphic with its projection $P(\epsilon')$ since this holds as regards the $\epsilon$ of $\kappa_p$ on which it lies. We are now ready for the

THEOREM. *Every analytic complex has a simplicial subdivision.*

We first assume $p<n$, and $\kappa_p$ in its ultimate reduced form, $\kappa_p'$, $\kappa_p''$ having the same meaning as above. The theorem being trivial for $n=0$ we use induction on $n$. Since $\kappa_p''$ is on an $S_{n-1}$ it has then a simplicial subdivision $K_p{}^0=\sum\sigma$. Let $QQ'$ be any vertical with $Q\subset\sigma_q$. It intersects $\kappa_p'$ in points $Q^1, Q^2, \cdots, Q^r$ ($r$ finite) each on a different $\epsilon'$ of $\kappa_p'$, say $Q^i\subset\epsilon'^i$. When $Q$ ranges continuously over $\sigma_q$, by the above $Q^i$ remains on $\epsilon'^i$ and generates a homeomorph of $\sigma_q$, a cell $E_q^i\subset\epsilon'^i$, and no two of these cells intersect. As a consequence $K_p=\sum E_q^i$ is a cellular subdivision of $\kappa_p'$ and hence of $\kappa_p$. We shall now show that the homeomorphism between $E_q$ and $E_q^i$ can be extended to their boundaries. This merely requires that we prove that when $Q\subset F(E_q)$, it has a unique image on $F(E_q^i)$. Suppose that it has $s$ images $Q'^j$. We may choose for each $Q'^j$ a neighborhood relative to $E_q{}^i$ consisting of a cell $E_q{}^{ij}$ whose projection is a simplex $\sigma_q^j$ (in the straightness of $\sigma_q$), no two of the cells $E_q{}^{ij}$ intersecting. As a consequence $\sigma^j\cdot\sigma^h=0$ $(j\neq h)$ and $Q$ has for neighborhood relative to $\sigma_q$ a set of $s$ non-intersecting $q$-simplexes, which can only be if $s=1$, as asserted. Since $\overline{E}_q{}^i$ and $\bar{\sigma}_q$ are homeomorphic, $E^i$ is simplicial and so is $K_p$.

If we have a $\kappa_n$, on removing its $n$-elements we have a $\kappa_p$, $p<n$, which we identify with the $\kappa_p$ just considered. When $Q\subset\sigma_q$, the points of $\kappa_n$ projected on $Q$ may include some of the segments $Q^iQ^{i+1}$ and we observe that, since $r$ is fixed throughout any $\sigma$, if the segment is zero anywhere on a face of $\sigma_q$ it is zero throughout that face. As a consequence we find by an elementary induction that when $Q$ ranges over $\sigma_q$ the segments $\neq 0$ generate $(q+1)$-cells $E_{q+1}^i$ whose structure is that of a truncated simplicial prism. Since these cells are convex, the covering $K_n$ thus obtained for $\kappa_n$ is convex, and its first derived, which is simplicial, answers the question.

COROLLARY. *If $\kappa_q\subset\kappa_p$, $\kappa_p$ has a simplicial subdivision with a subcomplex covering $\kappa_q$.*

For $\kappa_p$ has a subdivision $\kappa_p'$ having a subdivision of $\kappa_q$ as a subcomplex. In particular $\kappa_p$ may be a closed polyhedral region of $S_n$ containing $\kappa_q$. This is substantially the theorem of *Topology*, p. 364.

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