ON RIESZ AND CESÀRO METHODS OF SUMMABILITY*

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1. Introduction. Marcel Riesz‡ formulated the following method of summability: Let \( r \) be any complex constant and, given a series \( u_0 + u_1 + u_2 + \cdots \), let

\[
A_r: \quad \alpha_n = \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n} \right)^r u_k \quad (n = 1, 2, 3, \cdots);
\]

if \( \lim_{n \to \infty} \alpha_n = L \), then \( \sum u_n \) is said to be summable \( A_r \) to \( L \).

In a second note, Riesz§ gave a method which, when \( r > 0 \), equivalent|| (vide Theorem 4.4) to the following: Let \( r \) be a complex constant and let

\[
B_r: \quad \beta(t) = \sum_{k=0}^{[t]-1} \left( 1 - \frac{k}{t} \right)^r u_k, \quad 1 \leq t < \infty;
\]

if \( \beta(t) \) approaches a limit \( L \) as \( t \) becomes infinite over the real set \( t \geq 1 \), then \( \sum u_n \) is summable \( B_r \) to \( L \). The second method of Riesz is the following: Let \( \Re(r) > 0 \), and let

\[
D_r: \quad \delta(t) = \sum_{k=0}^{[t^{-1}]} \left( 1 - \frac{k}{t} \right)^r u_k, \quad 1 \leq t < \infty;
\]

if \( \delta(t) \) converges to \( L \) as \( t \) becomes infinite continuously, then \( \sum u_n \) is summable \( D_r \) to \( L \). The method \( D_r \) is known as the Riesz method of order \( r \) and type \( \lambda_n = n^** \), and has proved to be one of the most useful of all methods of summability.

In his second note, Riesz outlined a proof that \( D_r \) is equivalent to \( C_r \),

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‡ Comptes Rendus, vol. 149 (1909), pp. 18–22. In this note Riesz considered only real positive orders \( r \).
§ Comptes Rendus, vol. 152 (1911), pp. 1651–1654. Here again Riesz considered only the case \( r > 0 \).
|| The terminology used in this paper is that given by W. A. Hurwitz, Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 17–36.
¶ We use the symbols \([t]\) and \([r^{-}]\) to denote respectively the greatest integer \( \leq t \) and the greatest integer \( < t \).
the Cesàro method of order \( r \), when \( r > 0 \). Chapman* has stated that Riesz's proof of equivalence of \( D_r \) and \( C_r \) holds when \( r > -1 \); but this statement is incorrect as Theorem 2.1 shows. Hobson† has given a more detailed proof of equivalence of \( D_r \) and \( C_r \) when \( r > 0 \).

In a third note, Riesz‡ outlined a proof that \( A_r \) and \( C_r \) are equivalent when \(-1 < r < 1 \), and showed that this equivalence does not hold for certain values of \( r > 1 \).

It is the object of this paper to discuss \( A_r \), \( B_r \), \( D_r \), \( C_r \), and closely related methods of summability. We shall be especially interested in orders \( r \) with real part \( \Re(r) < 0 \).

In §2 we show that \( D_r \) does not constitute a useful method of summability when \( \Re(r) < 0 \); and in §§2–3 we discuss modifications of \( D_r \) which may be expected to be useful when \( \Re(r) < 0 \). For each complex \( r \), these modifications are found to be equivalent to \( B_r \). In §4 we show that \( B_r \) and \( D_r \) are equivalent when \( r \geq 0 \). In §§5–7 we obtain auxiliary results from which it follows in §8 that \( B_r \) and \( C_r \) are equivalent when \(-1 < \Re(r) < 0 \). The theorems of §8 give a complete solution of the problem which furnished the point of departure of this investigation. In §9 we give relations between methods \( B_r \) of different orders. We show in §§10–11 that \( A_r \) does not possess certain properties of \( B_r \) when \( \Re(r) < -1 \); in particular when \( \Re(r) \) is less than a certain constant between \(-2 \) and \(-1 \), \( A_r \) is not consistent with convergence. Finally, in §12 we point out that when \( \Re(r) < 0 \), the methods \( A_r \), \( B_r \), and \( C_r \) are equivalent over a certain class of series.

2. Ineffectiveness of \( D_r \) when \( \Re(r) < 0 \). It is well known that \( D_r \) is regular when \( r \) is real and \( \geq 0 \). It can be shown further that \( D_r \) is regular when \( \Re(r) > 0 \); and that \( D_r \) is not regular when \( \Re(r) = 0 \) but \( r \neq 0 \). To show that \( D_r \) does not constitute a useful method of summability when \( \Re(r) < 0 \), we will prove the following Theorem.

**Theorem 2.1.** In order that \( \sum u_n \) may be summable \( D_r \) when \( \Re(r) < 0 \), it is necessary and sufficient that \( \sum u_n \) have at most a finite number of terms different from zero.

Sufficiency is easily established. To prove necessity, let us suppose that \( u_p \neq 0 \) for a certain index \( p > 0 \); then \( \lim_{k \to 0^+} |\delta(p + h)| = +\infty \). Hence if \( u_k \neq 0 \) for an infinite set of values of \( k \), then \( \delta(t) \) is unbounded over every interval

Theorem 2.1 and its proof make it clear that if a useful generalization to orders with real part \( R(r) < 0 \) of the Riesz method \( D_r \) is to be obtained, the upper index of summation with respect to \( k \) must be a function of \( t \) which is definitely less than \( [t^-] \). The two methods defined by the transformations

\[
\pi(t) = \sum_{k=0}^{[t-\theta]} \left( 1 - \frac{k}{t} \right)^r u_k, \quad \theta < t < \infty,
\]

\[
\rho(t) = \sum_{k=0}^{[t-\theta]} \left( 1 - \frac{k}{t} \right)^r w_k, \quad \theta < t < \infty,
\]

where \( \theta \) is a positive constant, suggest themselves at once as modifications of \( D_r \), which may be useful for every complex order.

Let \( r \) be any complex number. Then, corresponding to any given series \( \sum u_n \), the functions \( \pi(t) \) and \( \rho(t) \) are equal except when \( t \) is of the form \( t = n + \theta \) and \( u_n \neq 0 \), in which case \( \pi(n + \theta) = \rho(n + \theta) \). Furthermore the transforms \( \pi(t) \) and \( \rho(t) \) are continuous except when \( t = n + \theta \) and \( u_n \neq 0 \); here \( \pi(t) \) and \( \rho(t) \) have finite jumps, \( \pi(t) \) having right-hand continuity and \( \rho(t) \) having left-hand continuity. It follows that if either \( \pi(t) \) or \( \rho(t) \) converges as \( t \to \infty \), then the other must also converge to the same value as \( t \to \infty \). Hence the methods (2.2) and (2.3) are equivalent. We elect to consider the first rather than the second of these.

3. Consideration of (2.2) for different values of \( \theta \). In this section we will establish a theorem which will be of fundamental importance in the sequel; and will show that, for any fixed complex \( r \), the methods (2.2) obtained by selecting different positive values of \( \theta \) are equivalent to \( B_r \).

**Theorem 3.1.** If \( \sum u_n \) is summable (2.2) with \( r \) a fixed complex constant and \( \theta \) a fixed positive constant, then

\[
\lim_{n \to \infty} u_n/n^r = 0.
\]

Suppose \( \sum u_n \) is summable (2.2) to \( L \). Then

\[
\lim_{t \to \infty} \sum_{k=0}^{[t-\theta]} \left( 1 - \frac{k}{t} \right)^r v_k = 0
\]

* There is an apparent inconsistency between Theorem 2.1 and Chapman's statement (loc. cit., p. 401) that the Dirichlet series \( 1 - 2^r + 3^r - 4^r + \cdots, s > 0 \), is summable (\( R \), \( n, -r \)), i.e. \( D_{-r} \), when \( r < s \). The last equation of p. 399 shows that Chapman has used the transformation \( B_r \) rather than \( D_r \); and furthermore the second equation of p. 400 is correct only when \( [n] = n \). Therefore, as a matter of fact, Chapman has not shown that \( \sum (-1)^n (n+1)^{-r} \) is summable \( D_{-r} \) when \( r < s \); what he has shown is that the series is summable \( A_{-r} \) when \( r < s \). However, it follows from this result and Theorem 12.1 that the series \( \sum (-1)^n (n+1)^{-r} \) is summable \( B_{-r} \) and \( C_{-r} \) as well as \( A_{-r} \) when \( r < s \).
where \( v_0 = u_0 - L \) and \( v_n = u_n \) when \( n > 0 \). Given \( \epsilon > 0 \), choose \( T > \theta \) such that

\[
\left| \sum_{k=0}^{[t-\theta]} \left( 1 - \frac{k}{t} \right)^r v_k \right| < \frac{\epsilon}{2}, \quad t > T.
\]

Let \( n > T + 1 \), let \( 0 < h < 1 \), and set \( t = n + \theta - h \) in (3.12) to obtain

\[
\left| \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n + \theta - h} \right)^r v_k \right| < \frac{\epsilon}{2}, \quad 0 < h < 1.
\]

The left member of (3.13) is a continuous function of \( h \) over the closed interval \( 0 \leq h \leq 1 \); hence we may take the limit as \( h \to 0 \) to obtain

\[
\sum_{k=0}^{n-1} \left( 1 - \frac{k}{n + \theta} \right)^r v_k \leq \frac{\epsilon}{2}, \quad n > T + 1.
\]

Again we may set \( t = n + \theta \) in (3.12) and write the last term of the sum as a separate term to obtain

\[
\left| \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n + \theta} \right)^r v_k + \left( \frac{\theta}{n + \theta} \right)^r v_n \right| < \frac{\epsilon}{2}, \quad n > T + 1.
\]

Combining (3.14) and (3.15), we find that \( \theta^r v_n / (n + \theta)^r \) \( < \epsilon \) when \( n > T + 1 \). Hence \( \lim_{n \to \infty} \theta^r v_n / (n + \theta)^r = 0 \) so that \( \lim_{n \to \infty} v_n / n^r = 0 \) and, since \( v_n = u_n \) when \( n > 0 \), (3.11) follows. Thus Theorem 3.1 is proved.

A slight modification of the preceding argument shows that if \( \sum u_n \) is bounded (2.2), then \( u_n / n^r \) is bounded for all \( n > 0 \).

**Theorem 3.2.** If \( r \) is a complex constant and

\[
\pi^{(\theta)}(t) = \sum_{k=0}^{[t-\theta]} \left( 1 - \frac{k}{t} \right)^r u_k, \quad \pi^{(\theta')}(t) = \sum_{k=0}^{[t-\theta']} \left( 1 - \frac{k}{t} \right)^r u_k
\]

represent two different methods of the form (2.2) with \( \theta > 0, \theta' > 0 \), and if furthermore \( \lim_{n \to \infty} u_n / n^r = 0 \), then

\[
\lim_{t \to \infty} \left\{ \pi^{(\theta)}(t) - \pi^{(\theta')}(t) \right\} = 0.
\]

To establish this result, we may assume that \( \theta > \theta' \) and show that the difference in the left member of (3.21) consists of a finite number of terms each of which approaches zero as \( t \to \infty \).

From the two preceding theorems we obtain at once

**Theorem 3.3.** When \( r \) is any complex constant, the methods obtained by assigning different positive values to \( \theta \) in (2.2) are equivalent.
For if \( \sum u_n \) is summable (2.2) for a positive value of \( \theta \), then \( \lim u_n/n^r = 0 \) by Theorem 3.1; hence the hypotheses of Theorem 3.2 are satisfied and the conclusion (3.21) completes the proof of Theorem 3.3.

The only representative of the set of methods (2.2) which we will consider in the sequel is that for which \( \theta = 1 \); in this case (2.2) becomes \( B_r \).

4. Relations between \( B_r \) and \( D_r \) when \( R(r) \geq 0 \). Before passing to a study of \( B_r \) when \( R(r) < 0 \), we wish to point out that \( B_r \) is closely related to the familiar Riesz method \( D_r \) when \( R(r) \geq 0 \).

Theorem 4.1. If \( R(r) \geq 0 \) and \( \lim u_n/n^r = 0 \), then

\[
\lim_{t \to \infty} (\delta(t) - \beta(t)) = 0.
\]

We find from (1.2) and (1.3) that \( |\delta(t) - \beta(t)| \leq \frac{1}{u_{\infty}} |\lfloor t \rfloor| \) when \( R(r) \geq 0 \) and \( t > 1 \), and Theorem 4.1 follows.

Theorem 4.2. If \( R(r) \geq 0 \), then \( D_r \) includes \( B_r \).

If \( \sum u_n \) is summable \( B_r \) to \( L \) so that \( \lim \beta(t) = L \), then \( \lim u_n/n^r = 0 \) by Theorem 3.1 with \( \theta = 1 \); hence the hypotheses of Theorem 4.1 are satisfied, the conclusion (4.11) shows that \( \lim \delta(t) = L \), and Theorem 4.2 is proved.

Theorem 4.3. If \( r \geq 0 \), then \( B_r \) includes \( D_r \).

The proposition being evident when \( r = 0 \), we suppose \( r > 0 \). Let \( \sum u_n \) be summable \( D_r \) to \( L \) so that \( \lim \delta(t) = L \). Then, using the fact (§1) that \( C_r \) includes \( D_r \) when \( r > 0 \), we see that \( \sum u_n \) must be summable \( C_r \), and hence, as is well known, that \( \lim u_n/n^r = 0 \). Hence Theorem 4.1 shows that \( \lim \beta(t) = L \) and Theorem 4.3 is proved.

Combining Theorems 4.2 and 4.3 with the fact that \( D_r \) and \( C_r \) are equivalent when \( r = 0 \), we obtain

Theorem 4.4. If \( r \geq 0 \), then \( B_r \), \( D_r \), and \( C_r \) are equivalent.

5. A relation between the \( A_r \) and \( B_r \) transforms when \( R(r) < 0 \). We proceed to establish some preliminary propositions, interesting in themselves, which will enable us to obtain relations between \( B_r \) and \( C_r \).

Theorem 5.2. When \( R(r) < 0 \), the assumption that

\[
\lim_{t \to \infty} \beta(t) = \lim_{t \to \infty} \sum_{k=0}^{[t]-1} \left( 1 - \frac{k}{t} \right)^r u_k = L
\]

is equivalent to the two assumptions

\[
\lim_{n \to \infty} u_n/n^r = 0
\]
and
\begin{align*}
\lim_{n \to \infty} \alpha_n &= \lim_{n \to \infty} \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n} \right)^r u_k = L. *
\end{align*}

That (5.11) implies (5.12) follows from Theorem 3.1 with \( \theta = 1 \); and that (5.11) implies (5.13) follows from the fact that \( \alpha_n = \beta(n) \). Hence our problem here is to show that (5.12) and (5.13) together imply (5.11).

A consideration of the sequence \( \nu_n \) defined by \( \nu_0 = u_0 - L \) and \( \nu_n = u_n \), \( n > 0 \), shows that it is sufficient to prove (5.12) and (5.13) imply (5.11) when \( L = 0 \). We suppose therefore that \( R(r) < 0 \), that (5.12) holds, and that (5.13) holds with \( L = 0 \); we will show that (5.11) holds with \( L = 0 \).

Given \( \epsilon > 0 \), choose an index \( N > 0 \) so great that
\begin{align*}
\left| \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n} \right)^r u_k \right| < \frac{\epsilon}{2},
\end{align*}
and
\begin{align*}
\left| u_n / n^r \right| < \frac{\epsilon}{(4B \left| r \right|)},
\end{align*}
where \( B = 2^{-r'}(1 + \sum_{r=1}^\infty \beta r^{-1}) \) and \( r' = R(r) \). Next, choose an index \( P > N \) so great that
\begin{align*}
\left| \sum_{k=0}^{N-1} \frac{k}{t} \left( 1 - \frac{k}{t} \right)^{r-1} u_k \right| < \frac{\epsilon}{4}, \quad t > P.
\end{align*}

Let \( n > P \) and consider the function
\begin{align*}
\beta(t) &= \sum_{k=0}^{n-1} \left( 1 - \frac{k}{t} \right)^r u_k, \quad n \leq t < n + 1.
\end{align*}
Using (5.14), we see that
\begin{align*}
\left| \beta(n) \right| < \frac{\epsilon}{2}.
\end{align*}

Differentiating (5.17) we find
\begin{align*}
\beta'(t) = \frac{r}{t^2} \sum_{k=0}^{n-1} \frac{k}{t} \left( 1 - \frac{k}{t} \right)^{r-1} u_k, \quad n \leq t < n + 1,
\end{align*}
where the derivative for \( t = n \) is a right-hand derivative. Hence
\begin{align*}
\beta'(t) \leq \frac{|r|}{t^2} \sum_{k=0}^{N-1} \left( 1 - \frac{k}{t} \right)^{r-1} u_k + \frac{|r|}{t^2} \sum_{k=N}^{n-1} k^{r-1} \left( 1 - \frac{k}{t} \right)^{r-1} \left| u_k / k^r \right|,
\end{align*}
and using (5.16) and (5.15) we obtain

\* Consideration of independence of (5.12) and (5.13) is relegated to §10 where we study \( A_r \).
(5.19) \[ |\beta'(t)| < \epsilon/4 + \epsilon \Phi_n(t)/(4B), \]
where

(5.20) \[ \Phi_n(t) = \sum_{k=1}^{n-1} \frac{k^{r'}(t-k)^{r'-1}}{t^{r'}}, \quad n \leq t < n+1. \]

But since \( r' < 0 \),

\[ \sum_{k=1}^{|t/2|} \frac{k^{r'}(t-k)^{r'-1}}{t^{r'}} \leq \frac{1}{t} \sum_{k=1}^{|t/2|} \left(1 - \frac{k}{t}\right)^{r'-1} \leq \frac{1}{t} \sum_{k=1}^{|t/2|} \left(\frac{1}{2}\right)^{r'-1} \leq 2^{-r'}, \]

and

\[ \sum_{k=|t/2|+1}^{n-1} \frac{k^{r'}(t-k)^{r'-1}}{t^{r'}} \leq \left(\frac{|t/2|+1}{t}\right)^{r'} \sum_{k=|t/2|+1}^{n-1} (t-k)^{r'-1} \leq 2^{-r'} \sum_{p=1}^{\infty} p^{r'-1}, \]

so that

(5.21) \[ \Phi_n(t) \leq 2^{-r'}\left(1 + \sum_{p=1}^{\infty} p^{r'-1}\right) = B, \quad n \leq t < n+1. \]

From (5.19) and (5.21) we obtain

(5.22) \[ |\beta'(t)| < \epsilon/2, \quad n \leq t < n+1. \]

Using (5.18), (5.22), and the formula

\[ |\beta(t)| \leq |\beta(n)| + \int_n^t |\beta'(t)| \, dt, \quad n \leq t < n+1, \]

we find that \( |\beta(t)| < \epsilon, n \leq t < n+1 \).

We have shown that if \( n > P \), then \( |\beta(t)| < \epsilon, n \leq t < n+1 \). It follows that if \( t > P + 1 \), then \( |\beta(t)| < \epsilon \). Hence \( \lim \beta(t) = 0 \) and Theorem 5.1 is proved.

6. Lemmas involving \( C_r \). The Cesàro method \( C_r \) of order \( r \) (\( r \) not a negative integer) is defined by the transformation

(6.01) \[ C_r: \quad \gamma_n = \sum_{k=0}^{n} a_{nk} u_k \quad (n = 0, 1, 2, \ldots), \]

where

(6.02) \[ a_{nk} = \frac{\Gamma(n + 1)\Gamma(n - k + 1 + r)}{\Gamma(n + r)\Gamma(n - k + 1)}, \quad 0 \leq k \leq n. \]

The following two lemmas will be used in the next section.
**Lemma 6.1.** Corresponding to each complex constant $r$ (not a negative integer) there is a bounded sequence $C_{nk}$ of constants such that for each positive index $n$ and each index $k < n$

$$a_{nk} = \left(1 - \frac{k}{n}\right)^r \left(1 + \frac{C_{nk}}{n - k}\right).$$

Using the familiar asymptotic expansion of the logarithm of the gamma function of a complex argument, we find

$$\log \left\{ \frac{\Gamma(n + 1 + r)}{\Gamma(n + 1)} \right\} = r \log n + H_n/n, \quad n > 0,$$

where $H_n$ is a bounded sequence of constants. Subtracting (6.12) from the relation obtained by replacing $n$ by $n-k$ in it, we obtain

$$\log a_{nk} = r \log \left\{ \frac{(n-k)/n}{n} \right\} - H_n/n + H_{n-k}/(n-k)$$

when $n > 0$ and $k < n$. The lemma results from (6.13). The following lemma is easily deduced from (6.12).

**Lemma 6.2.** When $r$ is not a negative integer

$$\lim_{n \to \infty} n^r a_{nn} = \Gamma(1 + r).$$

In §8 we shall need

**Lemma 6.3.** When $R(r) \leq -1$, $r$ not a negative integer, the condition $\lim n/n^r = 0$ is not necessary in order that $\sum u_n$ may be summable $C_r$.

The inverse of $C_r$ is, when $r$ is not an integer, given by

$$u_n = \sum_{k=0}^{n} \left(1 \right)^{n-k} \binom{k+r}{r} \binom{r+1}{n-k} \gamma_k$$

or

$$u_n = \sum_{k=0}^{n} \frac{\sin \pi r}{\pi} \frac{\Gamma(2 + r)}{\Gamma(k + 1 + r)} \frac{\Gamma(k + 1 + r)}{\Gamma(1 + r)} \frac{\Gamma(n - k - 1 - r)}{\Gamma(n - k + 1)} \gamma_k.$$

Corresponding to each complex $r$ which is not an integer, let the sequence $\gamma_n^{(r)}$ be defined by $\gamma_n^{(r)} = \pi / \left\{ \sin \pi r \Gamma(2 + r) \right\}$ and $\gamma_n^{(r)} = 0$ when $n > 0$; and let $\sum u_n^{(r)}$ be the series whose $C_r$ transform is $\gamma_n^{(r)}$. Substituting in (6.32) we find

$$u_n^{(r)} = \Gamma(n - 1 - r)/\Gamma(n + 1) = n^{-2-r}(1 + o(1))$$

so that

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The series \( \sum u_n^{(r)} \) is summable \( C_r \) to 0 and, when \( R(r) \leq -1 \), the right member of (6.34) fails to converge to 0 as \( n \to \infty \); thus Lemma 6.3 is established.

7. A relation between the \( A_r \) and \( C_r \) transforms when \( R(r) < 0 \). With the lemmas of §6 at our disposal, we are in a position to prove the following theorem.

**Theorem 7.1.** If \( R(r) < 0 \) (\( r \) not a negative integer) and the terms of \( \sum u_n \) satisfy the condition

\[
\lim_{n \to \infty} \frac{u_n}{n^r} = 0,
\]

then

\[
\lim_{n \to \infty} (\gamma_n - \alpha_n) = 0,
\]

where \( \gamma_n \) and \( \alpha_n \) represent respectively the \( C_r \) and \( A_r \) transform of \( \sum u_n \).

Letting \( \sum u_n \) be any series for which (7.11) holds, we have for each \( n > 1 \)

\[
\gamma_n - \alpha_n = a_{nn} u_n + \sum_{k=0}^{n-1} \left\{ a_{nk} - \left( 1 - \frac{k}{n} \right) \right\} u_k.
\]

Writing \( a_{nn} u_n \) in the form \( (n^r a_{nn}) (u_n / n^r) \), we see from Lemma 6.2 and (7.11) that it approaches zero as \( n \) becomes infinite. Furthermore the coefficient of \( u_0 \) is zero for each \( n \). Hence it follows from (7.13) that

\[
\gamma_n - \alpha_n = o(1) + \sum_{k=1}^{n-1} \left\{ a_{nk} - \left( 1 - \frac{k}{n} \right)^r \right\} u_k,
\]

and we may use Lemma 6.1 to obtain

\[
\gamma_n - \alpha_n = o(1) + \sum_{k=1}^{n-1} \left( 1 - \frac{k}{n} \right)^r \frac{C_{nk}}{n - k} u_k.
\]

Choosing a constant \( C \) such that \( |C_{nk}| < C \) when \( 0 < k < n \), we obtain

\[
|\gamma_n - \alpha_n| \leq o(1) + C \sum_{k=1}^{n-1} \frac{k^r (n - k)^{r-1}}{n^r} |u_k/k^r|,
\]

where \( r' = R(r) \).

* It should be noted that the hypotheses of this theorem are not sufficient to ensure that either of the sequences \( \gamma_n \) or \( \alpha_n \) is convergent, and hence that this theorem gives an especially important relation between Cesàro and Riesz transforms.
Now (7.16) shows that (7.12) will follow if \( \lim v_n = 0 \) implies \( \lim V_n = 0 \) when \( V_n \) is defined by

\[
V_n = \sum_{k=1}^{n-1} \frac{k^r(n-k)^{r'-1}}{n^{r'}} v_k.
\]

Thus we can establish Theorem 7.1 by proving that the transformation defined by (7.17) is regular over the set of sequences which converge to zero. To prove the latter result, it is necessary as well as sufficient* to prove that

\[
\lim_{n \to \infty} k^r(n-k)^{r'-1}/n^{r'} = 0 \quad (k = 1, 2, 3, \ldots),
\]

and that

\[
W_n = \sum_{k=1}^{n-1} k^r(n-k)^{r'-1}/n^{r'} < M \quad (n = 2, 3, 4, \ldots),
\]

for some constant \( M \) which may depend on \( r \) but must be independent of \( n \). It is clear that (7.18) holds for any value of \( r \). That (7.19) holds when \( R(r) < 0 \) follows from (5.20) and (5.21) since \( W_n = \Phi_n(n) \). Thus Theorem 7.1 is proved.

8. Relations between \( B_r \) and \( C_r \). The preceding results enable us to establish the following two theorems.

**Theorem 8.1.** If \( R(r) < 0 \), \( r \) not a negative integer, then \( C_r \) includes \( B_r \).

Suppose \( \sum u_n \) is summable \( B_r \) to \( L \) so that \( \lim \beta(t) = L \). Then by Theorem 5.1, (5.12) and (5.13) hold and we may use Theorem 7.1 to show that \( \lim \gamma_n = L \). Thus Theorem 8.1 is proved.

**Theorem 8.2.** If \( -1 < R(r) < 0 \), then \( B_r \) includes \( C_r \); if \( R(r) \leq -1 \), \( B \) does not include \( C_r \).

Suppose \( -1 < R(r) < 0 \) and \( \sum u_n \) is summable \( C_r \) to \( L \). Then \( \lim \gamma_n = L \).

Since, as is well known, (7.11) is a necessary condition for summability \( C_r \) when \( R(r) > -1 \), we can apply Theorem 7.1 to obtain \( \lim \alpha_n = L \); an application of Theorem 5.1 completes the proof of the first part of Theorem 8.1. To prove the second part suppose \( R(r) \leq -1 \), and, of course, that \( r \) is not a negative integer. By Lemma 6.3, there is a series \( \sum u_n \) summable \( C_r \), for which (5.12) fails; hence by Theorem (5.1), \( \sum u_n \) is not summable \( B_r \), and the second part of Theorem 8.2 is proved.

Theorems 8.1 and 8.2 yield

Theorem 8.3. If \(-1 < \Re(r) < 0\), then \(B_r\) and \(C_r\) are equivalent; if \(\Re(r) \leq -1\), \(B_r\) and \(C_r\) are not equivalent.

Theorem 8.4. If \(r\) is real and \(> -1\), then \(B_r\) and \(C_r\) are equivalent.

When \(-1 < r < 0\), this is included in Theorem 8.3. When \(r > 0\), the result is included in Theorem 4.4.

Cesàro's method \(C_r\) of summability is, as is well known, not regular when \(\Re(r) < 0\). When \(-1 < \Re(r) < 0\), \(C_r\) will evaluate only a subset of the set of all convergent series, and will evaluate no divergent series; hence, as might be expected, \(C_r\) occupies, for this range of values of \(r\), a prominent place in the theory of series. On the other hand when \(r\) is real and \(< -1\), \(C_r\) can evaluate to zero certain divergent series of positive terms (see, for example, §6). Owing to this fact, and also to the fact that many useful properties which hold when \(\Re(r) > -1\) fail when \(\Re(r) \leq -1\), the method \(C_r\) has received little attention when \(\Re(r) \leq -1\).

It is of interest to note that Theorems 8.1 and 8.2 show that \(B_r\) is equivalent to \(C_r\) over precisely the range of values of \(r\) with negative real parts over which \(C_r\) has been useful, namely the range \(-1 < \Re(r) < 0\).

In the next section, we will show that summability \(B_r\) is significant even when \(\Re(r) < -1\).

9. Relations between methods \(B_r\) of different orders. In this section we prove six theorems on relations between methods \(B_r\) of different orders.

Theorem 9.1. If \(-1 < \Re(r) < -s\) and \(\Re(s) \leq \Re(r)\), then \(B_r\) includes \(B_s\).

Let \(\sum u_n\) be summable \(B_r\) to \(L\) so that \(\lim \beta^{(r)}(t) = L\). Then, by Theorem 5.1, \(\lim u_n/n^r = 0\). We may write

\[
\beta^{(r)}(t) - \beta^{(s)}(t) = \sum_{k=0}^{[t]-1} \left\{ \left(1 - \frac{k}{t}\right)^r - \left(1 - \frac{k}{t}\right)^s \right\} u_k
\]

\[
= \sum_{k=1}^{[t]-1} k^r \left\{ \left(1 - \frac{k}{t}\right)^r - \left(1 - \frac{k}{t}\right)^s \right\} u_k/k^r.
\]

We see that Theorem 9.1 will follow if the transformation

\[
W(t) = \sum_{k=1}^{[t]-1} k^r \left\{ \left(1 - \frac{k}{t}\right)^r - \left(1 - \frac{k}{t}\right)^s \right\} w_k
\]

is regular over the set of all sequences \(w_n\) which converge to zero.

Letting \(d_k(t)\) represent the coefficient of \(w_k\) in (9.11), we have evidently

\[
\lim_{t \to \infty} d_k(t) = 0 \quad (k = 1, 2, 3, \ldots).
\]
Also
\[
\sum_{k=1}^{[t/2]} k^{r'} \left( 1 - \frac{k}{t} \right)^{r'} \leq 2^{-r'} \sum_{k=1}^{[t/2]} k^{r'} < 2^{-r'} \sum_{k=1}^{\infty} k^{r'},
\]
where \( r' = R(r) \) and \( s' = R(s) \). Since \( r' < -1 \),
\[
\sum_{k=1}^{[t/2]} k^{r'} \left( 1 - \frac{k}{t} \right)^{r'} \leq 2^{-r'} \sum_{k=1}^{[t/2]} k^{r'} < 2^{-r'} \sum_{k=1}^{\infty} k^{r'},
\]
and
\[
\sum_{k=[t/2]+1}^{[t]-1} k^{r'} \left( 1 - \frac{k}{t} \right)^{r'} \leq 2^{-r'} \sum_{k=[t/2]+1}^{[t]-1} (t - k)^{r'} < 2^{-r'} \sum_{k=1}^{\infty} k^{r'}.
\]
Hence
\[
(9.13) \quad \sum_{k=1}^{[t]-1} |d_k(t)| < 2^{-r'+2} \sum_{k=1}^{\infty} k^{r'}.
\]

The conditions (9.12) and (9.13) ensure that (9.11) has the desired property and Theorem 9.1 is proved.

From Theorem 9.1, we obtain at once

**Theorem 9.2.** If \( R(r) = R(s) < -1 \), then \( B_r \) and \( B_s \) are equivalent.

From Theorems 9.1 and 5.1 we obtain

**Theorem 9.3.** If \( i(R(r)) < -1 \) and \( \sum u_n \) is summable \( B_r \) to \( L \), then \( \sum u_n \) converges to \( L \), the convergence being absolute.

That \( \sum u_n \) must converge to \( L \) follows from the fact that \( B_0 \), which includes \( B_r \) by Theorem 9.1, represents convergence. Again, by Theorem 5.1, \( \lim u_n/n^r = 0 \); hence \( |u_n| < n^r, r' = R(r) < -1 \), for all sufficiently great \( n \), and absolute convergence of \( \sum u_n \) follows. Thus Theorem 9.3 is proved.

**Theorem 9.4.** If \( -1 < R(r) < R(s) < 0 \), then \( B_r \) includes \( B_s \). If \( -1 < r < s \), then \( B_s \) includes \( B_r \).

The first part of the Theorem follows from the fact (Theorem 8.3) that \( B_r \) and \( C_r \) are equivalent when \( -1 < R(r) < 0 \) and the fact that \( C_s \) includes \( C_r \) when \( -1 < R(r) < R(s) \). The second part follows from a similar application of Theorem 8.4.

To complete Theorems 9.1 and 9.4, it would be desirable to determine whether \( B_r \) includes \( B_s \), when \( -1 = R(r) < R(s) < 0 \). Neither the method of proof of Theorem 9.1 nor that of Theorem 9.4 throws light on this question. A partial answer to this question is given by the following theorem.
Theorem 9.5. If $-1 = r < \mathcal{R}(s) < 0$, then $B_r$ includes $B_s$.

We shall give a proof of Theorem 9.5 after having proved Theorem 10.1 below. After having proved Theorem 9.5, we can use Theorems 9.1, 9.4, and 9.5 to give a relation of inclusion between any two methods $B_r$ of real orders, namely

Theorem 9.6. If $r < s$, then $B_r$ includes $B_s$.

10. Consideration of $A_r$. Since it is sometimes convenient to use transformations involving a continuous parameter, and at other times a discontinuous parameter, it is important to know whether $A_r$ and $B_r$ are equivalent, and whether the results which we have established for $B_r$ hold also for $A_r$.

Using Theorem 8.4 and the result of Riesz that $A_r$ and $B_r$ are equivalent when $-1 < r < 1$, we see that $A_r$ and $B_r$ are equivalent when $-1 < r < 1$. We proceed to show that $A_r$ and $B_r$ have very different properties when $\mathcal{R}(r) < -1$.

A series $\sum u_k$ is said to be summable by the Abel method $P$ to $L$ if $\sum u_k x_k$ converges for $|x| < 1$ and $\lim_{x \to 1^{-}} \sum u_k x_k = L$. We shall say that $\sum u_k$ is summable $P^*$ to $L$ if $\sum u_k x_k$ converges for all sufficiently small $|x|$ and generates an analytic function $u(x)$ such that $\lim_{x \to 1^{-}} u(x) = L$. It is evident that $P^*$ includes $P$ and that $P$ does not include $P^*$.

Theorem 10.1. If $\mathcal{R}(r) \leq -1$ and $r \neq -1$, then $P^*$ does not include $A_r$; if $r$ is real and $\geq -1$, then $P^*$ includes $A_r$.

Let $\sum u_k$ be summable $A_r$ to $L$; then $\alpha_n \to L$ where

\[
(n + 1)^r \alpha_{n+1} = \sum_{k=0}^{n} (n + 1 - k)^r u_k.
\]

From (10.11) we obtain when $|x| < 1$,

\[
\sum_{n=0}^{\infty} (n + 1)^r \alpha_{n+1} x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (n + 1 - k)^r u_k x^n.
\]

Letting $u(x)$ be the analytic function determined by the equation

\[
\sum_{n=0}^{\infty} (n + 1)^r \alpha_{n+1} x^n = u(x) \sum_{n=0}^{\infty} (n + 1)^r x^n,
\]

we see that when $|x|$ is sufficiently small, say $|x| < \delta$, $u(x)$ is a convergent power series in $x$. A comparison of (10.12) and (10.13) suffices to show that

\[
\text{This modification of Abel's method was introduced by Silverman-Tamarkin, Mathematische Zeitschrift, vol. 29 (1928), pp. 161–170; p. 169.}
\]
(10.14) \[ u(x) = \sum_{k=0}^{\infty} u_k x^k, \quad |x| < \delta; \]

hence \( u(x) \) is the analytic function generated by \( \sum u_k x^k \). That \( P^* \) includes \( A_r \) when \( r \) is real and \( \geq -1 \) follows at once from the conditions for regularity\(^*\) of the transformation defined by (10.13); this also follows from a result of Silverman-Tamarkin, loc.cit.

We shall prove the first part of Theorem 10.1 by a method which shows that \( P^* \) and \( A_r \) are inconsistent when \( \Re(r) < -1 \). Corresponding to each complex \( r \), let \( \sum u_n(r) \) be the series having for its \( A_r \) transform the sequence \( \alpha_1 = 1, \alpha_n = 0 \) when \( n > 1 \). Then \( \sum u_n(r) \) is summable \( A_r \) to 0. Using (10.14) and (10.13), we see that the analytic function \( u(r)(x) \) generated by \( \sum u_n(r) x^k \) is given by

(10.15) \[ u(r)(x) \sum_{n=0}^{\infty} (n+1)^r x^n = 1. \]

But when \( r = -1 - ih, h \) real and \( \neq 0 \), we have as \( x \to 1^- \)

\[ \sum_{n=0}^{\infty} (n+1)^{-1-ih} x^n \sim \Gamma(ih) \left\{ \log \left( \frac{1}{x} \right) \right\}^{ih} \]

so that \( \lim_{x \to 1^-} u(r)(x) \) does not exist and \( \sum u_n(r) \) is non-summable \( P^* \). On the other hand, if \( \Re(r) < -1 \), then \( \sum (n+1)^r \) converges to \( \zeta(-r) \) which is finite and different from zero; hence \( \sum u_n(r) \) is summable \( P^* \) to \( 1/\zeta(-r) \) which is finite and different from the \( A_r \) value of \( \sum u_n(r) \). Thus Theorem 10.1 is proved.

We pass now to a proof of Theorem 9.5. Let \( \sum u_k \) be summable \( B_1 \) to \( L \). Then by Theorem 5.1, \( nu_n \to 0 \) and \( \sum u_k \) is summable \( A_{-1} \) to \( L \). Then by Theorem 10.1, \( \sum u_k \) is summable \( P^* \) to \( L \). But \( \sum u_k x^k \) must converge when \( |x| < 1 \) since \( nu_n \to 0 \); hence \( \sum u_k \) is summable \( P \) to \( L \). Therefore, by Tauber's Theorem\(^\S\) \( \sum u_k \) must converge to \( L \). Since \( nu_n \to 0 \) and \( \sum u_k \) converges to \( L \), it follows\(^\|\) that \( \sum u_k \) is summable \( C \), for every \( s \) with \( \Re(s) > -1 \). Finally summability \( B \), for every \( s \) with \( -1 < \Re(s) < 0 \) follows from Theorem 8.3 and Theorem 9.5 is proved.

We have shown in the proof of Theorem 10.1 that when \( \Re(r) < -1 \), the transformation \( A_r \) can evaluate to 0 a series which is not summable \( P \) to 0 and which is therefore not convergent to 0. Using this result and Theorem 9.3, we obtain

\[ \dagger \text{W. A. Hurwitz, loc. cit., p. 20.} \]
\[ \ddagger \text{Lindelöf, Le Calcul des Résidus, p. 139.} \]
\[ \S \text{A. Tauber, Monatshefte für Mathematik und Physik, vol. 8 (1897), pp. 273-277.} \]
\[ \| \text{Hardy and Littlewood, Proceedings of the London Mathematical Society, (2), vol. 11 (1912), p. 462.} \]
Theorem 10.2. If $R(r) < -1$, then $A_r$ and $B_r$ are not equivalent.

Theorem 10.1 also shows that the methods $A_r$ do not, in contrast to the methods $B_r$, form for real values of $r$ a set of consistent methods of summability whose effectiveness increases steadily as $r$ increases.

We can now see that (5.12) is not a consequence of (5.13) when $R(r) < -1$ by proving

Theorem 10.3. When $R(r) < -1$, the condition $u_n/n^{r} \rightarrow 0$ is not necessary in order that $\sum u_n$ may be summable $A_r$.

If the condition were necessary, it would follow from Theorem 5.1 that $A_r$ and $B_r$ would be equivalent and Theorem 10.2 would be contradicted.

In the next section, we give a theorem which is interesting in connection with Theorem 10.3, and give further properties of $A_r$.

11. Consideration of $A_r$ when $R(r) < \xi$. Let $\xi, -2 < \xi < -1$, be the real negative root of the equation.

\[(11.01) 2^r + 3^r + 4^r + \cdots = 1.\]

We shall now prove

Theorem 11.1. If $R(r) < \xi$ and $\sum u_n$ is bounded $A_r$, then $u_n/n^r$ is bounded for all $n > 0$.

Let $\sum u_n$ be bounded $A_r$, $R(r) < \xi$, so that $\alpha_n$, being defined by

\[(11.11) \alpha_n = \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n} \right)^r u_k,\]

is a bounded sequence. Since $r' = R(r) < \xi$, it follows that

\[(11.12) 0 < \theta_r = 2^{r'} + 3^{r'} + 4^{r'} + \cdots < 1.\]

Choose an index $p > 1$ so great that

\[(11.13) 2^{-r'+1} \sum_{k=p}^{\infty} k^{r'} < (1 - \theta_r)/2\]

and let a sequence $v_n$ be defined by the formulas $v_n = 0, n < p; v_p = u_0 + u_1 + \cdots + u_p; \text{ and } v_n = u_n$ for $n > p$. Then

\[\alpha_n = o(1) + \sum_{k=p}^{n-1} \left( 1 - \frac{k}{n} \right)^r v_k = o(1) + \sum_{k=p}^{n-1} k^{r'} \left( 1 - \frac{k}{n} \right)^r (v_k/k^{r'}).\]

Hence we can prove Theorem 10.1 by showing that boundedness of $W_n$...
implies boundedness of \( w_n \) whenever

\[
W_n = \sum_{k=p}^n k^r \left( 1 - \frac{k}{n+1} \right)^r w_k, \quad n > p.
\]

Let \( d_{nk} \) represent the coefficient of \( w_k \) in (11.14). Then when \( n > 2p \),

\[
\sum_{k=p}^{n-p-1} |d_{nk}| \leq 2^{-r+1} \sum_{k=p}^\infty k^{r'} < \frac{1 - \theta r}{2};
\]

the first inequality being obtained by considering separately the sums when \( k \) ranges from \( p \) to \( \left\lfloor \frac{n+1}{2} \right\rfloor \) and from \( \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \) to \( n - p - 1 \). Also

\[
\lim_{n \to \infty} \sum_{k=n-p}^{n-1} |d_{nk}| = \sum_{k=2}^p k^{r'} < \theta r.
\]

Combining (11.15) and (11.16), we obtain

\[
\limsup_{n \to \infty} \left\{ \sum_{k=p}^{n-1} |d_{nk}| \right\} < (1 + \theta r)/2 < 1.
\]

Since

\[
\lim_{n \to \infty} d_{n,n} = 1,
\]

we may use (11.17) and the fact that \( d_{nk} = 0 \) when \( k < p \) to obtain

\[
\liminf_{n \to \infty} \left\{ \left| d_{n,n} \right| - \sum_{k=0}^{n-1} |d_{nk}| \right\} > 0.
\]

Owing to (11.19), the fact that the transformation (11.14) has the desired property results from the following lemma.

**Lemma 11.2.** *If the coefficients in the transformation*

\[
W_n = \sum_{k=0}^n d_{nk} w_k
\]

*satisfy (11.19) and if \( W_n \) is a bounded sequence, then \( w_n \) is a bounded sequence.*

To prove this lemma, let \( w_n \) be an unbounded sequence; we shall show that \( W_n \) is an unbounded sequence. Since \( w_n \) is unbounded, we can choose an increasing sequence \( n_j \) of indices such that \( |w_{n_j}| \geq |w_k| \) when \( 0 \leq k < n_j \). Then

\[
|W_{n_j}| \geq -\sum_{k=0}^{n_j-1} |d_{n_j k}| |w_k| + |d_{n_j n_j}| |w_{n_j}|
\]

\[
\geq \left\{ |d_{n_j n_j}| - \sum_{k=0}^{n_j-1} |d_{n_j k}| \right\} |w_{n_j}|.
\]
But \( \lim |w_{nj}| = +\infty \) and using (10.19) we see that \( \lim |W_{nj}| = +\infty \). Hence \( W_n \) is an unbounded sequence, Lemma 11.2 is proved, and Theorem 11.1 follows.

**Theorem 11.2.** If \( \Re(r) < \xi \), every series summable \( A_r \) is convergent, but not necessarily to the value to which it is summable.

That a series summable \( A_r \) must be convergent follows from Theorem 11.1; in fact boundedness \( A_r \) is sufficient to ensure absolute convergence of \( \sum u_n \). That the \( A_r \) and convergence values need not be equal is shown by the series \( \sum u_n^{(r)} \) used in the proof of Theorem 10.1.

Since \( C_r \) can evaluate certain divergent series when \( \Re(r) < -1, \) \( r \) not a negative integer, and \( A_r \) can evaluate only absolutely convergent series when \( \Re(r) < \xi \), it follows that \( C_r \) and \( A_r \) are not equivalent when \( \Re(r) < \xi \).

The methods \( A_r, \Re(r) < \xi \), may be of use for classification of convergent series; but use of such methods for evaluation of series is open to the objection that they are, by Theorem 11.2, inconsistent with convergence.

12. Conclusion. In conclusion we point out that while \( A_r, B_r \), and \( C_r \) are not mutually equivalent when \( \Re(r) < -1 \), there is an important class of series over which these methods are equivalent. In fact, a combination of Theorems 5.1 and 7.1 yields the following theorem.

**Theorem 12.1.** If \( \Re(r) < 0 \), being \( \neq -1, -2, \ldots \) when \( C_r \) is involved, and \( \lim u_n/n^r = 0 \), and if \( \sum u_n \) is summable by one of the methods \( A_r, B_r \), and \( C_r \), then it is summable to the same value by the other two methods.

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