INVARIANTS OF PFAFFIAN SYSTEMS*

BY

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1. **Introduction.** The developments in this paper are based on a series of invariant sets of forms associated with a given pfaffian system. These forms are obtained by exterior multiplication of the given pfaffians and their derived forms. The first set Ω^i of r forms, obtained in turn by multiplying the product of all the given pfaffians by each of the derived forms, has been employed by Cartan. The vanishing of this set is a necessary and sufficient condition that the system be passive. The present paper interprets the vanishing of the second set in the light of the notion of a *primitive system*, i.e., one whose derived system is the given set of equations.

Associated with any pfaffian system are invariant pfaffian systems, formed by equating to zero the linear factors common to any set of forms, $\Omega^{ij\cdots k}$. Any arithmetical invariant of one of the latter systems is invariant for the former; such invariants are the number of equations in the system, the class, the species, etc.

Another arithmetical invariant arises from the fact that every system possesses a primitive system.

The importance of arithmetical invariants for pfaffian systems lies in their usefulness in determining the non-equivalence of pfaffian systems. Riquier† has given methods which may be applied to determine whether or not two pfaffian systems are equivalent, but the algebraic operations involved in their application are in general too complicated to carry out, whereas the comparison of arithmetical invariants will often settle the question.

The Ω's are used (§8) to give a criterion for reducibility to a canonical system of a particular type, designated as *completely separable*, and to state necessary and sufficient conditions for the equivalence of such systems.

The reader is assumed to be familiar with the contents of Goursat's treatise.‡

2. **The fundamental invariant forms.** Consider the pfaffian system

$$S:\ \omega^1 = 0, \omega^2 = 0, \cdots, \omega^r = 0. \tag{2.1}$$

* Presented to the Society, December 27, 1932; received by the editors February 18, 1933. The results in this paper are taken from a doctoral dissertation in mathematics presented at Duke University, June, 1933.

† C. Riquier, *Les Systèmes d'Equations aux Dérivées Partielles*, Paris, 1910.

‡ E. Goursat, *Leçons sur le Problème de Pfaff*, Paris, 1922.

Let the symbol $\Omega^{i_1 i_2 \cdots i_k}$ be defined as follows:

$$\Omega^{i_1 i_2 \cdots i_k} = \omega^1 \omega^2 \cdots \omega^r \omega'^{i_1} \omega'^{i_2} \cdots \omega'^{i_k}, \tag{2.2}$$

where $i_1 i_2 \cdots i_k$ represents any combination of numbers selected from $1, 2, \cdots, r$ and ω' is the derived form of ω. The forms $\Omega^{i_1 i_2 \cdots i_k}$ are invariants of the system S. Moreover they are symmetric in every pair of superscripts.

Assuming the non-singular transformation

$$\bar{\omega}^i = a_\alpha^i \omega^\alpha, \quad a = | a_j^i | \neq 0, \tag{2.3}$$

we have

$$\bar{\omega}'^i = a_\alpha^i \omega'^\alpha + d a_\alpha^i \omega^\alpha,$$

where the repeated indices on the right indicate summation over the range $1, 2, \cdots, r$. This transformation induces on the Ω's the linear homogeneous transformation

$$\bar{\Omega}^{i_1 i_2 \cdots i_k} = a a_{\alpha_1}^{i_1} a_{\alpha_2}^{i_2} \cdots a_{\alpha_k}^{i_k} \Omega^{\alpha_1 \alpha_2 \cdots \alpha_k} \tag{2.4}$$

Conversely, there exists a transformation (2.3) which induces a given transformation

$$\bar{\Omega}^{i_1 i_2 \cdots i_k} = B_{\alpha_1 \alpha_2 \cdots \alpha_k}^{i_1 i_2 \cdots i_k} \Omega^{\alpha_1 \alpha_2 \cdots \alpha_k} \tag{2.5}$$

on the Ω's provided the a's satisfy

$$k! B_{\alpha_1 \alpha_2 \cdots \alpha_k}^{i_1 i_2 \cdots i_k} = a P(a_{\alpha_1}^{i_1} a_{\alpha_2}^{i_2} \cdots a_{\alpha_k}^{i_k}), \tag{2.6}$$

where P indicates the summation of all terms obtained by permuting $\alpha_1 \alpha_2 \cdots \alpha_k$. The conditions for compatibility of (2.6) are

$$k! B_{\alpha_1 \alpha_2 \cdots \alpha_k}^{i_1 i_2 \cdots i_k} = P((B_{\alpha_1 \alpha_1 \cdots \alpha_1}^{i_1 i_1 \cdots i_1} B_{\alpha_2 \alpha_2 \cdots \alpha_2}^{i_2 i_2 \cdots i_2} \cdots B_{\alpha_k \alpha_k \cdots \alpha_k}^{i_k i_k \cdots i_k})^{1/k}). \tag{2.7}$$

If $k = 1$, conditions (2.7) are identically satisfied.

3. **Linear dependence of the Ω's.** Expressing that the relations

$$C_{\alpha_1 \alpha_2 \cdots \alpha_k} \Omega^{\alpha_1 \alpha_2 \cdots \alpha_k} = 0 \tag{3.1}$$

are identically satisfied in the differentials gives a set of linear homogeneous equations on the C's, whose rank can be proved invariant under transformations (2.3). We shall employ Sylvester's term "nullity" as a name for the invariant p_k, which is the number of linearly independent solutions of (3.1).

In the case of the Ω's with a single index, by a transformation (2.3) an Ω

can be made zero corresponding to each relation (3.1). This gives the theory of the derived system,* which is an invariant system of the original.

Suppose it is possible to make the following p Ω's of order $k>1$ vanish:

$$\Omega^{11\cdots 1}=0,\ \Omega^{22\cdots 2}=0,\ \cdots,\ \Omega^{pp\cdots p}=0, \tag{3.2}$$

where p is the corresponding nullity. It can be proved by methods similar to those developed in greater detail in §8 that any transformation (2.3) which leaves (3.2) invariant permutes the first p equations of S among themselves. Those equations therefore form an invariant system of S.

4. **Further invariant systems associated with a pfaffian system.** All the Ω's of sufficiently high order for a given system are zero since the degree of $\Omega^{i_1 i_2\cdots i_k}$ finally exceeds the class of the system. Suppose every Ω with $\rho+1$ indices is identically zero, whereas some Ω with ρ indices does not vanish. Then 2ρ is an arithmetical invariant of the system. It is, in fact, easily identified with the invariant defined in a different manner by Engel† and called by him the rank of the system. For $q\leqq\rho$ let S_q be defined as the system composed of the equations formed by setting all common factors of $\Omega^{i_1 i_2\cdots i_q}$ equal to zero. The system S_q is always contained in S_{q+1}. We have then a sequence of invariant pfaffian systems all of whose arithmetical invariants are also invariants for S.

THEOREM 1. *A system S is of species one if and only if the corresponding S_1 is passive and does not coincide with S.*‡

THEOREM 2. *If S is of rank two, it can be put in a form satisfying*

$$\begin{aligned} &\omega'^1\equiv 0,\ \cdots,\ \omega'^{r-3}\equiv 0,\ \omega'^{r-2}\equiv\phi^2\phi^3,\\ &\omega'^{r-1}\equiv\phi^3\phi^1,\ \omega'^r\equiv\phi^1\phi^2,\ \mathrm{mod}\ \omega^1,\omega^2,\cdots,\omega^r, \end{aligned} \tag{4.1}$$

or

$$\omega'^1\equiv 0,\ \cdots,\ \omega'^{r'}\equiv 0,\ \omega'^{r'+1}\equiv\phi\psi^{r'+1},\ \cdots,\ \omega'^r\equiv\phi\psi^r,\quad \mathrm{mod}\ \omega^1,\omega^2,\cdots,\omega^r. \tag{4.2}$$

In the first case, $S_1=S$; in the second, $S_1>S$. Conversely, if $S_1>S$, then S is of rank two and satisfies (4.2).

The proof of Theorem 2 follows. Since S is of rank two, every Ω of order two must vanish. The vanishing of Ω^{ij} when $i=j$ indicates that every ω'^i must vanish or be of rank two mod $\omega^1,\cdots,\omega^r$; thus the derived form of every ω

* Goursat, p. 294.

† F. Engel, Leipziger Berichte, vol. 52 (1890). This invariant is zero if and only if the system is passive.

‡ For the definition of species and for material which facilitates the proof of the above theorem, see J. M. Thomas, *Pfaffian systems of species one*, these Transactions, vol. 35 (1933).

not contained in the derived system is of rank two mod $\omega^1, \cdots, \omega^r$. Since this is true, the vanishing of Ω^{ij} when $i \neq j$ implies that every pair of non-vanishing derived forms possesses a common factor. This is possible in only two ways: S must satisfy (4.1) or (4.2). When (4.1) is satisfied, the derived system contains $r-3$ equations.

5. **Primitive systems.** If Σ has S for its derived system, Σ will be called a *primitive system* of S.

THEOREM 3. *Every pfaffian system has a primitive system.*

Let us assume that the derived system of

$$S\colon\ \omega^1 = 0, \cdots, \omega^{r'} = 0, \cdots, \omega^r = 0 \tag{5.1}$$

is

$$S^1\colon\ \omega^1 = 0, \cdots, \omega^{r'} = 0. \tag{5.2}$$

The first r' derived forms of S then vanish by virtue of the system. When reduced by (5.1), the non-vanishing forms are quadratic in $\phi^1, \cdots, \phi^{n-r}$, which with the ω's constitute an independent set of forms. Consequently, they vanish by virtue of

$$\omega^{r+1} = \phi^2 - \lambda^2\phi^1 = 0, \cdots, \omega^{n-1} = \phi^{n-r} - \lambda^{n-r}\phi^1 = 0. \tag{5.3}$$

If in addition the λ's form with the original variables an independent set, the derived form of no left member of this set of equations can vanish by virtue of S and (5.3). Hence the system $\Sigma = S + (5.3)$ is a primitive system of S.

THEOREM 4. *The minimum number of equations which adjoined to S yield a primitive system is an invariant of S.*

Let primitive systems for two equivalent systems S, $\bar{S}$ be

$$\Sigma\colon\ \omega^1 = 0, \cdots, \omega^r = 0, \phi^1 = 0, \cdots, \phi^k = 0, \tag{5.4}$$

$$T\colon\ \bar{\omega}^1 = 0, \cdots, \bar{\omega}^r = 0, \psi^1 = 0, \cdots, \psi^l = 0, \tag{5.5}$$

respectively, where k and l are least. The derived forms of S, being linear homogeneous combinations of those of $\bar{S}$, vanish whenever those of $\bar{S}$ do, and vice versa. This shows that $l \leqq k$ and $k \leqq l$, whence $k = l$.

Every passive system (species zero) of r equations is the derived system of a system of $r+1$ equations. When the adjunction of a single equation to a system of species one furnishes a primitive system is answered by the following:

THEOREM 5. *Every system of r equations whose species is one is the derived system of $2r-r'$ equations, but of no smaller number.*

A transformation (2.3) and a change of variables will put any system of species one in the form

$$(5.6)\qquad \omega^1 = \cdots = \omega^{r'} = 0,\ dx^{r'+1} - A^{r'+1}dx^{r+1} = \cdots = dx^r - A^r dx^{r+1} = 0,$$

where the first r' equations constitute the derived system. A system has (5.6) in its derived system if and only if it implies the vanishing of the forms $dA^\alpha dx^{r+1}$, that is, if and only if it implies

$$(5.7)\qquad dA^\alpha - \lambda^\alpha dx^{r+1} = 0.$$

If the forms dA^α were linearly dependent by virtue of (5.6), there would be more than r' equations in the derived system of (5.6). Therefore equations (5.7) form with (5.6) an independent set of $2r-r'$ equations, and no primitive system contains fewer equations. If x, A, λ constitute a set of independent variables, the system composed of (5.6) and (5.7) is a primitive system of (5.6).

THEOREM 6. *When the species exceeds one, the adjunction of a single equation gives a primitive system if and only if the class is $2r-r'+1$ and the rank two.*

If we put

$$\omega'^i \equiv G^i \mod \omega^1, \cdots, \omega^r \qquad (i = r'+1, \cdots, r),$$

the conditions of the theorem can be restated: the class of the set G^i is $r-r'+1$, and

$$(5.8)\qquad G^iG^j = 0 \qquad (i = r'+1, \cdots r).$$

If the adjunction of

$$(5.9)\qquad \phi = 0$$

gives a primitive system,

$$G^i = \phi\psi^i.$$

Hence (5.8) are satisfied. If the forms ω, ϕ, ψ were not independent, the G's would be linearly dependent and the derived system would contain more than r' equations. Hence ω, ϕ, ψ are independent and the class of the system G^i is $r-r'+1$. Conversely, when (5.8) are satisfied, Theorem 2 shows that S can be displayed as (4.1) or (4.2). The class of (4.1) is $r+3$, however, whereas the expression $2r-r'+1$ reduces to $r+4$ when $r'=r-3$. Hence S is in the form (4.2), and the ϕ of those formulas furnishes a primitive system.

6. A theorem on matrices. A square matrix is *monomial* if it contains one and only one non-zero element on each row and on each column. The totality of monomial matrices of a given order has the group property under multiplication. It will be called the *monomial group.*

LEMMA. *If a non-singular square matrix is multiplied by a properly chosen monomial matrix, every element in the main diagonal of the resulting matrix is different from zero.*

Let the given matrix be M. Consider any non-vanishing term of the expansion of the determinant $|M|$. In the matrix M replace each element of this term by unity and every other element of the matrix by zero; call the transpose of the resulting matrix N. Then MN has the chosen non-zero elements on its main diagonal.

THEOREM 7. *If the elements a_j^i of a non-singular square matrix M satisfy the conditions*

$$P(a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}) = 0, \tag{6.1}$$

where the $i_1, \cdots, i_r$ are any set of values from the range $1, 2, \cdots, r$ such that at least two of them are equal and P indicates the summation of all the terms obtained by permuting the subscripts, then M is monomial.

Consider the product MN, where N is constructed as in the preceding lemma. This matrix MN also satisfies conditions (6.1) because multiplication on the right by N simply permutes the columns, thus permuting the subscripts in (6.1). Hence MN is a matrix satisfying (6.1) and also

$$a_1^1a_2^2\cdots a_r^r \neq 0. \tag{6.2}$$

If MN is monomial, then M is too. Therefore it suffices to show that any matrix satisfying conditions (6.1) and (6.2) is monomial.

The theorem holds for $r=2$ because conditions (6.1) and (6.2) are

$$a_1^1a_2^1 = 0, \quad a_1^2a_2^2 = 0; \quad a_1^1a_2^2 \neq 0.$$

Assume the theorem true for every matrix of order less than r and suppose (6.1), (6.2) satisfied by a matrix of order r. From (6.1) we have $a_1^1a_2^1\cdots a_r^1 = 0$, whence $a_2^1a_3^1\cdots a_r^1 = 0$. To prove

$$a_2^1 = a_3^1 = \cdots = a_r^1 = 0 \tag{6.3}$$

we employ induction. Suppose $k-1$ elements on the first row are zero. By interchanging, if necessary, certain rows and the corresponding columns, we make those elements $a_2^1, a_3^1, \cdots, a_k^1$ and preserve the condition (6.2). From (6.1),

$$P(a_1^1a_2^{i_2}\cdots a_k^{i_k}a_{k+1}^1\cdots a_r^1) = 0,$$

where $i_2, \cdots, i_k$ are chosen from the range $2, \cdots, k$ in every possible way. Since $a_2^1 = a_3^1 = \cdots = a_k^1 = 0$ and $a_1^1 \neq 0$, this reduces to

$$P(\overset{i_2}{a_2} \cdots \overset{i_k}{a_k}) \overset{1}{a_{k+1}} \cdots \overset{1}{a_r} = 0.$$

We wish to show that

$$\overset{1}{a_{k+1}} \cdots \overset{1}{a_r} = 0 \tag{6.4}$$

holds. Assuming the contrary, we have

$$P(\overset{i_2}{a_2} \cdots \overset{i_k}{a_k}) = 0. \tag{6.5}$$

But the conditions of the theorem are then satisfied for the range $2, \cdots, k$ and by assumption they imply

$$a_j^i = 0 \qquad (i, j = 2, 3, \cdots, k;\ i \neq j). \tag{6.6}$$

The condition $P(a_2^2 \cdots a_k^k) = 0$, where the upper indices are permuted, is also implied by (6.5). Because of (6.6) it reduces to $a_2^2 \cdots a_k^k = 0$. This contradiction forces us to conclude that (6.4) is true. Hence we may assume $\overset{1}{a_{k+1}} = 0$, and by induction we reach the result that all elements except a_1^1 in the first row are zero. Since any row can be made the first by a transformation that preserves (6.2), the same argument can subsequently be applied to each of the other rows to show that

$$a_j^i = 0 \qquad (i, j = 1, \cdots, r;\ i \neq j),$$

and the theorem therefore is true for the matrix of order r.

7. **Partition of pfaffian systems.** Suppose that for a pfaffian system certain relations

$$F_1 = 0, F_2 \neq 0, \cdots \tag{7.1}$$

are satisfied. Let $\mathcal{G}$ be the subgroup of transformations (2.3) leaving (7.1) invariant. If $\mathcal{G}$ is intransitive or imprimitive,* the transformation (2.3) which displays its ultimate sets of intransitivity or imprimitivity will exhibit S as the sum† of a number of pfaffian systems

$$S = S_1 + S_2 + \cdots + S_q.$$

This will be called a *partition* of S.

If $\mathcal{G}$ is intransitive, each S_i is an invariant system of S. If $\mathcal{G}$ is imprimitive,

* These terms are employed in the sense defined by H. F. Blichfeldt, *Finite Collineation Groups*, Chicago, 1917, p. 17 and p. 76.

† We apply the symbol + only to aggregates having no elements in common.

the systems S_i all contain the same number of equations and are permuted when S is subjected to the general transformation (2.3).*

If each of the systems S_i contains only one equation, the partition will be called *complete*; this occurs only when the group $\mathcal{G}$ is monomial.

THEOREM 8. *The conditions*

$$\Omega^{12\cdots r} \neq 0,\ \Omega^{i_1 i_2 \cdots i_r} = 0 \quad (i_1 i_2 \cdots i_r \neq 12 \cdots r) \tag{7.2}$$

define a complete partition of the pfaffian system.

By (2.4) the conditions that the second of (7.2) be preserved under (2.3) are

$$a^{i_1}_{\alpha_1} a^{i_2}_{\alpha_2} \cdots a^{i_r}_{\alpha_r} \Omega^{\alpha_1 \alpha_2 \cdots \alpha_r} = 0 \quad (i_1 i_2 \cdots i_r \neq 12 \cdots r)$$

and reduce precisely to (6.1). The result follows from Theorem 7.

A system admitting partition is

$$dx^1 + x^2 dx^3 = 0,\ dx^4 + x^7 dx^6 = 0,\ dx^5 + x^8 dx^6 = 0. \tag{7.3}$$

For it the defining relations (7.1) are

$$\Omega^{11} = \Omega^{22} = \Omega^{33} = \Omega^{23} = 0,\ p_2 = 4.$$

8. **Separable systems.** A system will be called *separable* if it is equivalent to a system S expressed in terms of a set of independent variables X such that

$$S = S_1 + S_2,\ X = X_1 + X_2,$$

where S_1 is expressed in terms of the variables X_1, S_2 in terms of X_2, and neither S_1 nor S_2 is vacuous.

The system is *completely separable* if it is equivalent to

$$S = S_1 + S_2 + \cdots + S_r,$$

expressed in terms of variables

$$X = X_1 + X_2 + \cdots + X_r,$$

each S_i containing a single equation and being expressed in terms of the variables in the corresponding X_i. The following is readily proved:

THEOREM 9. *A completely separable system can be written in a canonical form each equation of which is in the canonical form for a single equation.*

Generalization of a known method† proves

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* Invariant systems like those of §3 arise when $\mathcal{G}$ is merely reducible.

† Cf. Goursat, p. 308, where the method is employed in reducing a system of two equations in four variables to canonical form.

THEOREM 10. *A pfaffian system of r equations having $r-1$ independent integrals is completely separable.*

We shall now derive a necessary and sufficient condition that a system be completely separable. Suppose a completely separable system written in canonical form. Let $\omega'^i \equiv G^i$, mod $\omega^1, \cdots, \omega^r$. The number of equations in the derived system is the number of independent solutions of

$$\lambda_\alpha G^\alpha = 0. \tag{8.1}$$

Since $G^1, \cdots, G^r$ have no differentials in common,

$$\lambda_1 G^1 = 0, \cdots, \lambda_r G^r = 0. \tag{8.2}$$

Suppose exactly q of the G's are zero. They can be made $G^1, \cdots, G^q$, and the first q equations of S are

$$dx^1 = 0, \cdots, dx^q = 0. \tag{8.3}$$

The remaining λ's are zero and the number of independent solutions of (8.1) is q. Hence (8.3) is the derived system S^1 of S. The derived system of a completely separable system is therefore passive. We have

$$\Omega^1 = 0, \cdots, \Omega^q = 0, \tag{8.4}$$

whereas the remaining Ω's of the first order are linearly independent.

Let the class of ω^i be $2m_i+1$. It is easily verified that every Ω of order $m_1+m_2+\cdots+m_r$ is zero except

$$\Omega^{(q+1)(q+1)\cdots(q+1)\cdots rr\cdots r} \neq 0 \tag{8.5}$$

where the superscript i occurs m_i times. We denote the conditions that the Ω's other than (8.5) vanish by

$$\Omega_{m_1+m_2+\cdots+m_r} = 0. \tag{8.6}$$

Consider a transformation (2.3) which leaves the two sets of conditions above invariant. The preservation of (8.4) gives

$$a_j^i = 0 \qquad (i = 1, 2, \cdots, q;\ j = q+1, \cdots, r). \tag{8.7}$$

The preservation of the other conditions (8.6) gives

$$P\left(a_{q+1}^{\alpha_1} a_{q+1}^{\alpha_2} \cdots a_{q+1}^{\alpha_{m_{q+1}}} a_{q+2}^{\beta_1} a_{q+2}^{\beta_2} \cdots a_{q+2}^{\beta_{m_{q+2}}} \cdots a_r^{\gamma_1} a_r^{\gamma_2} \cdots a_r^{\gamma_{m_r}}\right) = 0, \tag{8.8}$$

where $\alpha, \beta, \cdots, \gamma$ have any values from the range $q+1, \cdots, r$ except the values occurring in (8.5). In particular, when all the α's are equal, all the β's are equal, etc., we have

$$P\{(a^{\alpha}_{q+1})^{m_{q+1}}(a^{\beta}_{q+2})^{m_{q+2}}\cdots(a^{\gamma}_{r})^{m_r}\} = 0, \tag{8.9}$$

where $\alpha, \beta, \cdots, \gamma$ is any set from $q+1, \cdots, r$ which is not a permutation of $q+1, \cdots, r$ and the m's denote powers.

The substitution

$$(a^{\alpha}_{q+1})^{m_{q+1}} = b^{\alpha}_{q+1}, (a^{\beta}_{q+2})^{m_{q+2}} = b^{\beta}_{q+2}, \cdots, (a^{\gamma}_{r})^{m_r} = b^{\gamma}_{r} \tag{8.10}$$

gives

$$P(b^{\alpha}_{q+1}b^{\beta}_{q+2}\cdots b^{\gamma}_{r}) = 0.$$

Hence the theorem of §6 can be applied to show that the matrix

$$\|b^{i}_{j}\| \qquad (i, j = q+1, \cdots, r)$$

and consequently the matrix

$$\|a^{i}_{j}\| \qquad (i, j = q+1, \cdots, r),$$

is monomial.

If we write

$$S = S^1 + S_{q+1} + S_{q+2} + \cdots + S_r, \tag{8.11}$$

the system S^1 is left invariant by any transformation (2.3) preserving (8.4), (8.5), (8.6); and the systems

$$S^1 + S_{q+1}, S^1 + S_{q+2}, \cdots, S^1 + S_r, \tag{8.12}$$

each of which contains $q+1$ equations, are permuted among themselves by such a transformation. The class values for the canonical form of systems (8.12) are

$$q + 2m_{q+1} + 1, q + 2m_{q+2} + 1, \cdots, q + 2m_r + 1, \tag{8.13}$$

and consequently the class values must be these whenever conditions (8.4), (8.5), (8.6) are satisfied. Thus we have an additional necessary condition on a completely separable system.

The conditions given above are also sufficient. By Theorem 10 systems (8.12) can be written in canonical form. From the result already established concerning the derived system of a completely separable system in canonical form, the first q equations in each of the systems (8.12) must be equations (8.2). Since the class values are the set (8.13), the system is expressed in terms of $x^1, \cdots, x^q$ and $2(m_{q+1}+\cdots+m_r)+r-q$ other variables. Since the total number of these variables is the same as the degree of the Ω in (8.5), that Ω is the product of their differentials, and its non-vanishing declares the set of variables independent. Thus we have

THEOREM 11. *A pfaffian system is completely separable into r equations of class* $1, \cdots, 1, 2m_{q+1}+1, \cdots, 2m_r+1$, *where no m is zero, if and only if the following conditions are satisfied. It must be possible to determine a transformation* (2.3) *which realizes* (8.4), (8.5), (8.6). *When such a transformation* (2.3) *has been found and applied to the system, the numbers* (8.13) *must be the class values of* (8.12).

The determination of the transformation (2.3) involves finding a particular solution of a system of homogeneous algebraic equations in the a's whose degree is $m_{q+1}+ \cdots +m_r$. It is important to note that the second condition in Theorem 11 is either satisfied for all solutions of this system of algebraic equations or for none.

A completely separable system having no integrals admits a complete partition. That the converse is not true is evident from (7.3).

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