7. We need to recall a few known definitions.

Given an abstract space $E$ (i.e., an arbitrary set of elements), a family $\mathcal{E}$ of sets in $E$ is said to be additive if it satisfies the following conditions:

(i) The empty set $\emptyset$ belongs to $\mathcal{E}$.

(ii) If a set $X$ belongs to $\mathcal{E}$, its complement $C_X$ (with respect to the space $E$) also belongs to $\mathcal{E}$.

(iii) If $\{X_n\}$ is a sequence of sets belonging to $\mathcal{E}$, the set $X=\bigcap X_n$ also belongs to $\mathcal{E}$.

If $F(X)$ is a finite real-valued function of sets, defined for all sets of an additive family $\mathcal{E}$, and if

$$F\left(\bigcap_{n} X_n\right) = \sum_{n} F(X_n)$$

for any finite sequence $\{X_n\}$ of sets of $\mathcal{E}$, of which no two have points in common, then $F(X)$ is called an additive function of sets of $\mathcal{E}$. If (7.1) holds for any finite or infinite sequence $\{X_n\}$ of sets belonging to $\mathcal{E}$, of which no two have points in common, then $F(X)$ is said to be a completely additive function of sets of $\mathcal{E}$.

In this paragraph we assume that $\mathcal{E}^*$ is an additive family in the space $E$, and $\mu(X) \geq 0$ is a completely additive and finite-valued function of sets of $\mathcal{E}^*$. The sets $X$ belonging to $\mathcal{E}^*$ are called measurable, $\mu(X)$ being the measure of $X$. A measurable set $X$ is a singular set if for any measurable subset $Y$ of $X$ either $\mu(Y)=0$ or $\mu(X-Y)=0$.

An additive function $F(X)$ of measurable sets is absolutely continuous if $F(X)=0$ whenever $X$ is of measure zero. This together with the property of being completely additive, is equivalent to the statement that for any $\epsilon > 0$ there exists an $\eta > 0$ such that $\mu(X) < \eta$ implies $|F(X)| < \epsilon$.

The family $\mathcal{E}^*$ of measurable sets may be regarded as a metric complete space with the distance defined by §

\[\text{\marginnote{\textsuperscript{†} This volume, pp. 549-556. In the present addition we extend the results of \S 2 to completely additive functions of sets in an abstract space. The author is indebted to Professor Tamarkin for criticisms.}}\]

\[\text{\marginnote{\textsuperscript{‡} Presented to the Society, April 14, 1933; received by the editors February 16, 1933.}}\]

\[\text{\marginnote{\textsuperscript{§} This definition corresponds to that of distance in the space $R$ of characteristic functions of \S 2.}}\]
If two measurable sets differ by subsets of measure zero they are regarded as the same elements of the space $\mathfrak{M}^\ast$. Any completely additive and absolutely continuous function of measurable sets may be regarded as a continuous functional on the metric space $\mathfrak{M}^\ast$.

**Lemma 1.** If $A$ is a measurable set of positive measure, then, for any positive number $\epsilon$, the set $A$ contains either a singular set of measure $> \epsilon$ or a measurable set of positive measure $\leq \epsilon$.

Suppose that $A$ contains neither a singular set of measure $> \epsilon$, nor a measurable set of positive measure $\leq \epsilon$. Then there will exist a measurable subset $A_1$ of $A$ such that $0 < \mu(A_1) < \mu(A)$. The set $A - A_1$ must be a non-singular set of measure $> \epsilon$, and, by the same argument, $A - A_1$ contains a measurable subset $A_2$ such that $0 < \mu(A_2) < \mu(A - A_1)$. By repeating this process we obtain an infinite sequence of measurable sets $\{A_n\}$ of positive measure, of which no two have points in common. Since the series

$$
\sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\sum_{n=1}^{\infty} A_n\right)
$$

converges, for $n$ sufficiently large, we have $0 < \mu(A_n) < \epsilon$. This, however, contradicts the assumption that $A$ contains no measurable set of measure $\leq \epsilon$.

**Lemma 2.** Given an arbitrary number $\epsilon > 0$, the space $E$ may be expressed as the sum of a finite number of measurable sets $E_1, E_2, \cdots, E_p$ such that $E_iE_j = 0$ for $i \neq j$, while each $E_i$ is either a singular set or a set of measure $\leq \epsilon$.

We observe that for an arbitrary pair of singular sets, either one of them contains the other, with the possible exception of a set of measure zero, or else their common part is of measure zero. Since $\mu(E) < \infty$, on the basis of this remark we can find a finite sequence of singular sets $E_1, E_2, \cdots, E_n$ of measure $> \epsilon$ such that

$$
(7.3) \quad E_iE_j = 0 \text{ for } i \neq j,
$$

while the set

$$
(7.4) \quad A = E - \sum_{i=1}^{m} E_i
$$

contains no singular set of measure $> \epsilon$.

Let $X$ be any measurable set and let $\lambda(X)$ denote the least upper bound of the measures of all measurable subsets $Y$ of $X$ such that $\mu(Y) \leq \epsilon$. It follows from Lemma 1 that $0 < \lambda(X) \leq \epsilon$ for any measurable set $X \subset A$ of positive measure. Hence, by induction, we can determine a sequence $\{X_i\}$ of measurable subsets of $A$ such that
(7.5) \[ X_iX_j = 0 \text{ for } i \neq j, \]

(7.6) \[ \epsilon \leq \mu(X_{n+1}) \leq \frac{1}{2} \lambda \left( A - \sum_{i=1}^{n} X_i \right) \quad (n = 1, 2, \ldots). \]

Upon putting

\[ X_0 = A - \sum_{i=1}^{\infty} X_i \]

from (7.6) we have

(7.7) \[ \lambda(X_0) \leq \lambda \left( A - \sum_{i=1}^{n} X_i \right) \leq 2\mu(X_{n+1}) \quad (n = 1, 2, \ldots). \]

Since, by (7.5),

(7.8) \[ \sum_{i=1}^{\infty} \mu(X_i) \leq \mu(A) < \infty, \]

the series (7.8) converges and \( \lim_{n} \mu(X_n) = 0 \). Thus we infer from (7.7) that \( \lambda(X_0) = 0 \), whence also \( \mu(X_0) = 0 \). Let now \( h \) be a positive integer such that

(7.9) \[ \mu \left( \sum_{n=h+1}^{\infty} X_n \right) = \mu \left( \sum_{n=h+1}^{\infty} X_n \right) \leq \epsilon, \]

and let

\[ E_{m+1} = X_1, \ldots, E_{m+h} = X_h, E_{m+h+1} = X_0 + \sum_{n=h+1}^{\infty} X_n. \]

These sets, by (7.6) and (7.9), are of measure \( \leq \epsilon \), and by (7.5) no two of them have points in common. Hence the sequence \( E_1, E_2, \ldots, E_{m+h+1} \) satisfies the conditions of Lemma 2.

8. We now are able to generalize Theorems 1 and 2 of §2.

THEOREM 5. Let \( \{F_n(X)\} \) be a sequence of completely additive and absolutely continuous functions of measurable sets. If this sequence converges for any set belonging to a class of the second category in the space \( R^n \), then the functions \( F_n(X) \) are equally absolutely continuous† and the sequence \( \{F_n(X)\} \) converges for any measurable set \( X \subset E - (E_1 + E_2 + \cdots + E_m) \) where \( \{E_i\} \) is a finite sequence of singular sets.

Consequently, if \( \{F_n(X)\} \) converges for any measurable set \( X \), the limit function is again a completely additive and absolutely continuous function of measurable sets in \( E \).

† That is, to every \( \epsilon > 0 \) there corresponds an \( \eta > 0 \) which depends only on \( \epsilon \), such that \( |F_n(X)| \leq \epsilon \) for \( n = 1, 2, \ldots \) and for any set \( X \) of measure \( \leq \eta \).
The fact that the functions \( F_n(X) \) are equally absolutely continuous can be established in exactly the same fashion as in Theorem 1, §2, if we interpret the functions \( F_n(X) \) as continuous functionals in the metric complete space \( \mathfrak{R}^* \). Now, since by assumption the sequence \( \{F_n(X)\} \) converges for any \( X \) belonging to a set of the second category in \( \mathfrak{R}^* \), there exists in \( \mathfrak{R}^* \) a sphere, say \( \mathfrak{R}(A_0; r) \), such that \( \{F_n(X)\} \) converges for each \( X \) of a set everywhere dense in \( \mathfrak{R}(A_0; r) \). But the functionals \( F_n(X) \) are equally continuous in \( \mathfrak{R}^* \), hence the sequence \( \{F_n(X)\} \) converges everywhere in the sphere \( \mathfrak{R}(A_0; r) \).

Now let

\[
E = \sum_{i=1}^{p} E_i
\]

be a representation of the space \( E \) mentioned in Lemma 2. We may assume that the sets \( E_1, E_2, \ldots, E_m \) are singular while the sets \( E_{m+1}, \ldots, E_p \) are of measure \( \leq r \).

Let \( X \) be an arbitrary measurable set contained in \( \sum_{i=m+1}^{p} E_i \). Then

\[
X = \sum_{i=m+1}^{p} XE_i.
\]

Each set \( XE_i, i = m+1, \ldots, p \), is of measure \( \leq r \). Consequently the sets
\[
A_0 + XE_i \text{ and } A_0 - A_0 XE_i, \quad i = m+1, \ldots, p
\]

are elements of the sphere \( \mathfrak{R}(A_0; r) \) and both sequences \( \{F_n(A_0 + XE_i)\}, \{F_n(A_0 - A_0 XE_i)\} \) converge. Thus the sequence
\[
F_n(XE_i) = F_n(A_0 + XE_i) - F_n(A_0 - A_0 XE_i)
\]
also converges for \( i = m+1, \ldots, p \). Hence, by (8.1), the sequence \( \{F_n(X)\} \) converges for any measurable set \( X \) contained in \( E - (E_1 + \cdots + E_m) \) where \( E_1, \ldots, E_m \) are singular sets.

**Theorem 6.** If \( \{F_n(X)\} \) is a sequence of completely additive and absolutely continuous functions of measurable sets and if

\[
\lim_n |F_n(X)| < \infty
\]

for any set \( X \) belonging to a class of the second category in the space \( \mathfrak{R}^* \), then there exists a fixed constant \( M \) such that

\[
|F_n(X)| < M
\]

for any measurable set \( X \subset E - (E_1 + \cdots + E_m) \) where \( \{E_i\} \) is a finite sequence of singular sets in \( E \).

Consequently, if the inequality (8.2) holds for every measurable set \( X \), there exists a constant \( M \) such that (8.3) holds for all measurable sets \( X \) in \( E \).
Let $\mathfrak{R}^*_k$ be the aggregate of sets $X$ such that
$$|F_n(X)| \leq k$$
$(n = 1, 2, \cdots)$. By assumption the class $\sum^\infty_1 \mathfrak{R}^*_k$ is of the second category in the space $\mathfrak{R}^*$. By the continuity of the functionals $F_n(X)$, the sets $\mathfrak{R}^*_k$ are closed (in the space $\mathfrak{R}^*$). Hence, for some value $k = k_0$, $\mathfrak{R}^*_k$ contains a sphere, say $\mathfrak{R}(A_0; r)$.

We now introduce the same representation of the space
$$E = \sum^{\rho}_{i=1} E_i$$
as in the proof of Theorem 5. Let $X$ be an arbitrary measurable set. Since, for $i = m + 1, \cdots, \rho$, the sets $XE_i$ are of measure $\leq r$, the sets $A_0 + XE_i$ and $A_0 - A_0XE_i$ belong to the sphere $\mathfrak{R}(A_0; r)$. Thus
$$|F_n(A_0 + XE_i)| \leq k_0, \quad |F_n(A_0 - A_0XE_i)| \leq k_0,$$
and
$$|F_n(XE_i)| = |F_n(A_0 + XE_i) - F_n(A_0 - A_0XE_i)| \leq 2k_0.$$
Hence, for any measurable set $X \subset E - (E_1 + \cdots + E_m)$ we have
$$|F_n(X)| = |F_n\left(\sum^{\rho}_{i=m+1} XE_i\right)| \leq 2(\rho - m)k_0,$$
which completes the proof of Theorem 6.

9. Theorems 5 and 6 contain the corresponding two theorems which have been stated recently by Nikodym.†

I. If $\mathfrak{S}$ is an additive family of sets in an abstract space $E$, and if the sequence $\{F_n(X)\}$ of completely additive functions of sets of $\mathfrak{S}$ converges for every set $X$ of $\mathfrak{S}$, then the limit function is also a completely additive function of sets of $\mathfrak{S}$.

II. If $\mathfrak{S}$ is an additive family of sets in $E$ and if the sequence $\{F_n(X)\}$ of completely additive functions of sets of $\mathfrak{S}$ is bounded for every set $X$ of $\mathfrak{S}$, then there exists a constant $M$ such that $|F_n(X)| \leq M$ for $n = 1, 2, \cdots$ and for all $X \subset \mathfrak{S}$.

In order to reduce these theorems to Theorems 5 and 6 respectively we merely have to introduce a measure $\mu(X)$ for the family $\mathfrak{S}$, with respect to which the functions $F_n(X)$ would be absolutely continuous. This can be achieved by putting, for each set $X \subset \mathfrak{S}$,

\[ \mu(X) = \sum_{n=1}^{\infty} \frac{V_n(X)}{2^n [V_n(E) + 1]}, \]

where \( V_n(X) \) denotes the absolute variation of \( F_n(X) \) on the set \( X \). Since each \( V_n(X) \) is a non-negative and completely additive function of sets of \( \mathcal{E} \) the series (9.1) converges and \( \mu(X) \geq 0 \) is a completely additive and finite-valued function of sets of \( \mathcal{E} \). Hence \( \mu(X) \) may be taken as a measure in \( E \) and, since \( F_n(X) = 0, \ n = 1, 2, \ldots \), for every set \( X \) of \( \mathcal{E} \) such that \( \mu(X) = 0 \), the functions \( F_n(X) \) are absolutely continuous with respect to this measure. Thus the theorems of Nikodym are reduced to our Theorems 5 and 6.

† See for instance H. Hahn, Theorie der reellen Funktionen, 1921, Chapter VI. The absolute variation is called there (p. 400) “absolute Summe.”

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