WARING'S PROBLEM FOR CUBIC FUNCTIONS*

BY

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1. Introduction

In 1921 it was proved by Kamke that if \( f(x) \) is a polynomial with rational coefficients whose value is an integer \( \geq 0 \) for every integer \( x \geq 0 \), then every integer \( \geq 0 \) is a sum of a limited number \( u \) of 1's and a limited number \( v \) of values of \( f(x) \) for integers \( x \geq 0 \). This existence theorem was later proved by the method of Hardy and Littlewood by Winogradow and Landau.

For the case of any quadratic function, the writer (and later Dr. Pall) evaluated the limits \( u \) and \( v \).

We shall here treat cubic functions (1). The case in which a term \( x^2 \) occurs is under investigation by my students. The main result is Theorem 2. For special cubic functions, Theorems 4 and 5 give universal Waring theorems.

2. Determination of all functions (1) with certain properties

We restrict attention to cubic functions of the form

\[
(1) \quad f(x) = \frac{\alpha x^3 + \beta x}{d} \quad (\alpha \neq 0, \ d > 0),
\]

\[
(2) \quad \alpha, \beta, \ d \text{ integers without a common factor } > 1.
\]

We assume that \( f(x) \) is an integer for every integer \( x \geq 0 \). By the values 1 and 2 of \( x \), we see that \( \alpha + \beta \) and \( 8\alpha + 2\beta \) are divisible by \( d \), whence

\[
(3) \quad 6\alpha \text{ and } 6\beta \text{ are divisible by } d.
\]

If \( d \) has a prime factor \( p > 3 \), then \( \alpha \) and \( \beta \) are divisible by \( p \), contrary to (2). Hence 2 and 3 are the only possible prime factors of \( d \).

To discuss only a pure Waring problem, we assume that \( f(x) \geq 0 \) if \( x \) is any integer \( x \geq 0 \), and that 1 is a value of \( f(\xi) \) for a certain integer \( \xi \geq 0 \) (otherwise, sums of values of \( f \) would never give the number 1).

I. Case \( d \) a multiple of 6. Write \( d = 6t \). By (3), \( \alpha \) and \( \beta \) are divisible by \( t \), whence \( t = 1 \), by (2), and \( d = 6 \). By \( f(\xi) = 1 \),

\[
(4) \quad \alpha \xi^3 + \beta \xi = 6,
\]

whence \( \xi \) is a positive divisor of 6.

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I. Let $\xi = 3w$, $w = 1$ or 2. By (4), $9\alpha w^3 + \beta w = 2$. Elimination of $\beta$ gives

$$f(x) = \frac{1}{6} \left[ \alpha x^3 + \frac{2 - 9\alpha w^3}{w} x \right],$$

$$f(w) = \frac{1}{3}(1 - 4\alpha w^3) \geq 0, \alpha \leq 0; \quad f(6w) = 2 + 27\alpha w^3 \geq 0, \alpha \geq 0,$$

whence $\alpha = 0$, contrary to hypothesis. Thus Case I₁ is excluded.

I₂. Let $\xi = 2$. By (4), $4\alpha + \beta = 3$. Then

$$f(1) = \frac{1}{4}(1 - \alpha) \geq 0, \alpha \leq 1; \quad f(3) = \frac{1}{4}(15\alpha + 9) \geq 0, \alpha \geq 0,$$

whence $\alpha = 1$, and $f(x)$ is the pyramidal number

$$P(x) = \frac{1}{6}(x^3 - x).$$

This function satisfies all of our preceding assumptions.

I₃. There remains only the case $\xi = 1$. Then $\alpha + \beta = 6$, 

$$f(x) = x + \frac{\alpha}{6} (x^3 - x) \equiv x + \alpha P(x), \alpha > 0,$$

which is an integer $\geq 0$ for every integer $x \geq 0$, since the same is true of $P(x)$.

II. Case $d$ a multiple of 3, but not of 6. Then $d = 3t$, where $t$ is a positive odd integer. By (3), $\alpha$ and $\beta$ are divisible by $t$, whence $t = 1$ by (2). Thus

$$f(x) = \frac{1}{3}(\alpha x^3 + \beta x), \quad \alpha \xi^3 + \beta \xi = 3,$$

whence $\xi$ is a positive divisor of 3.

II₁. Let $\xi = 3$. Then $\beta = 1 - 9\alpha$,

$$f(1) = \frac{1}{3}(1 - 8\alpha) \geq 0, \alpha \leq 0; \quad f(4) = \frac{1}{3}(4 + 28\alpha) \geq 0, \alpha \geq 0.$$

Since $\alpha \neq 0$, Case II₁ is excluded.

II₂. Hence $\xi = 1, \beta = 3 - \alpha$,

$$f(x) = x + \frac{\alpha}{3} (x^3 - x) \equiv x + 2\alpha P(x), \alpha > 0.$$

III. Case $d$ a multiple of 2, but not of 3. Similarly as in II, we find that $d = 2, \xi \neq 2, \xi = 1, \alpha + \beta = 2$,

$$f(x) = x + \frac{\alpha}{2} (x^3 - x) \equiv x + 3\alpha P(x), \alpha > 0.$$

IV. $d = 1$. Then $\xi = 1, \alpha + \beta = 1, f(x) = x + 6\alpha P(x)$. 
Theorem 1. If \( f(x) = (ax^3 + bx)/d \) \((a \neq 0)\) is an integer \( \geq 0 \) for every integer \( x \geq 0 \), and if \( f(x) = 1 \) for some integer \( x > 0 \), then \( f(x) \) is either a pyramidal number \( P(x) = \frac{1}{6}(x^3 - x) \), or is \( x + eP(x) \), where \( e \) is a positive integer. Conversely, each of the resulting functions is an integer \( \geq 0 \) for every integer \( x \geq 0 \), while \( f(x) = 1 \) has a positive integral solution.

3. The main theorem and three lemmas

Theorem 2. To each positive integer \( e \) prime to 3 there correspond positive integers \( C \) and \( \nu \geq 8 \) such that every integer \( \geq C \cdot 3^\nu \) is a sum of nine values of

\[
f(x) = x + \frac{1}{3}e(x^3 - x)
\]

for integral values \( \geq 0 \) of \( x \).

We shall first give the parts of the proof which hold both for \( e = 1 \) and \( e > 1 \), and then establish the few simple additional facts required in the case \( e = 1 \), and hence prove Theorem 2 for \( e = 1 \) with \( C = 168, \nu = 8 \). Then we shall present the more elaborate theory for \( e > 1 \) (which does not hold for \( e = 1 \)). That theory gives a reconstructed proof which provides an explicit program actually to express any sufficiently large integer as a sum of nine values of \( f(x) \).

Lemma 1. There exists an integer \( m' \) such that any given integer is congruent to \( f(3m') \mod 3^n \).

The difference \( f(z+3r) - f(z) \) has the value

\[
\Delta = \frac{1}{3}e(3rz^2 + 9r^2z + 9r^3) + 3r - \frac{1}{3}er.
\]

Since 3 is a factor of all terms except the last, while \( e \) is prime to 3, \( \Delta \neq 0 \) if \( r \neq 0 \mod 3^n \). Take \( r = m' - k, z = 3k, 0 < r < 3^n \). Then

\[
f(3m') - f(3k) = f[3k + 3(m' - k)] - f(3k) \equiv 0 \mod 3^n.
\]

Hence for \( j = 0, 1, \ldots, 3^n - 1 \), the \( 3^n \) integers \( f(3j) \) are incongruent modulo \( 3^n \), so that any integer is congruent to one of them.

Lemma 2. If \( \eta \) is an odd constant integer, \( v(\eta - v) \) is even and can be made congruent to any assigned even integer modulo \( 2^k \) by choice of an integer \( v \).

Let \( V(\eta - V) \equiv v(\eta - v) \mod 2^k \). Then the product of \( V - v \) by \( V + v - \eta \) is divisible by \( 2^k \), while the factors are of unlike parity. Hence one factor is odd and the other is divisible by \( 2^k \). Thus \( V \equiv v \) or \( \eta - v \mod 2^k \). Hence when \( v \) ranges over the \( 2^k \) values 0, 1, \ldots, \( 2^k - 1 \), we obtain at most (and hence exactly) \( \frac{1}{2} \cdot 2^k \) values of \( v(\eta - v) \) incongruent modulo \( 2^k \), and the latter values are all even. This proves Lemma 2.
Lemma 3. If \( n > 1 \) and \( m < e \cdot 3^n \), then \( f(3m) < \gamma \cdot 3^n \), where

\[
\gamma = \frac{1}{2}(9e^4 + 1).
\]

Since \( 9m^3 - m \) increases with \( m \),
\[
f(3m) = 3m + \frac{1}{2}e(9m^3 - m) < 3e3^n + \frac{1}{2}e(9e^33^{2n} - e3^n),
\]
which will be \( < \gamma 3^n \) if \( 3e - \frac{1}{2}e^2 < \frac{1}{2}3^{2n} \). The latter holds for every \( n \) if \( e \geq 6 \) and holds for \( n > 1 \) if \( e \leq 5 \), since the maximum of \( 6e - e^2 \) is its value 9 for \( e = 3 \).

4. Plan of Proof with the Necessary Formulas

If \( s \) and \( C \) are given positive numbers, we can evidently choose a positive integer \( n \) so that
\[
C \cdot 27^n \leq s < C \cdot 27^{n+1}.
\]
In Theorem 2, \( s \geq C \cdot 27^n \). Hence we may take \( n \geq \nu \geq 8 \).

Any such \( s \) is one of the integers \( s_i \) falling in the following three sub-intervals:
\[
3^{i-1}C3^n \leq s_i < 3^iC3^n \quad (i = 1, 2, 3).
\]

By Lemma 1 we can choose an integer \( m_i \) so that
\[
s_i = f(3m_i) + 3^nM_i, \quad 0 \leq m_i < 3^n,
\]
where \( M_i \) is an integer. Since \( f(m_i) \geq 0, 3^nM_i \leq s_i < 3^iC3^n \). Using also Lemma 3, we get
\[
(3^{i-1}C - \gamma)3^{2n} < M_i < 3^iC3^{2n}.
\]
Write \( M_i = \varepsilon 3^{2n} + N_i \). Then
\[
(3^{i-1}C - \gamma - \varepsilon)3^{2n} < N_i < (3^iC - \varepsilon)3^{2n} \quad (i = 1, 2, 3).
\]

Take \( l = 3^n \) in the identity
\[
f(l - x) + f(l + x) = 2l + \frac{1}{3}e(l^3 - l + 3lx^2)
\]
and sum for three values \( x_j \) of \( x \). Thus
\[
\sum_{i=1}^{3} [f(3^n - x_i) + f(3^n + x_i)] = T,
\]
\[
T = \varepsilon 3^{3n} + 3^n(\varepsilon Q - \varepsilon + 6), \quad Q = x_1^2 + x_2^2 + x_3^2.
\]
Write \( \phi_i \) for \( f(v_i) + f(w_i) \) and \( Q_i \) for \( Q \). Then will
\[
s_i = f(3m_i) + 3^n(\varepsilon 3^{2n} + N_i) = f(3m_i) + \phi_i + T
\]
(and hence \( s_i \) will be a sum of nine values of \( f \)).
(13) \[ 3^n(N_i + \epsilon - 6) = \phi_i + \epsilon 3^n Q_i \quad (i = 1, 2, 3). \]

We impose on the unknowns \( v_i, w_i \) the restrictions

(14) \[ v_i + w_i = 3b_i 3^n \quad (i = 1, 2, 3; b_i \text{ a positive odd integer}). \]

The identity

\[ \phi_i = (v_i + w_i) \left[ 1 + \frac{\epsilon}{6} \left\{ (v_i + w_i)^2 - 3v_i w_i - 1 \right\} \right] \]

gives \( \phi_i = 3^n B_i \), where

(15) \[ B_i = 3b_i \left[ 1 + \frac{\epsilon}{6} \left\{ 9b_i^2 3^{2n} - 1 - 3v_i (3b_i 3^n - v_i) \right\} \right]. \]

Inserting \( \phi_i = 3^n B_i \) into (13) and cancelling \( 3^n \), we get

(16) \[ \epsilon Q_i = N_i + \epsilon - 6 - B_i. \]

We shall later choose the \( v_i \) so that

(17) \[ 0 \leq v_i \leq 3b_i 3^n, \quad 0 \leq N_i + \epsilon - 6 - B_i \leq \epsilon 3^n. \]

These and (14) and (16) imply

(18) \[ 0 \leq w_i, \quad 0 \leq Q_i \leq 3^n. \]

We shall later prove that we may take \( Q_i \) to be an integer which is a sum of three squares of integers \( x_j \geq 0 \). Thus \( x_j \leq 3^n \) by (18). It will then follow that \( s_i \) is the sum of the values of \( f(x) \) for the nine values \( 3m_i, v_i, w_i, 3^n - x_i, 3^n + x_i \) of \( x \), each an integer \( \geq 0 \).

Employ the abbreviation

(19) \[ V_i = v_i - \frac{3}{2} b_i 3^n. \]

In the right member of (16), we insert the value of \( B_i \) and get

\[ S_i = N_i + \epsilon - 6 - 3b_i \left[ 1 + \frac{\epsilon}{6} \left( 3V_i^2 + \frac{9}{4} b_i^2 3^{2n} - 1 \right) \right]. \]

The final condition (17) is \( 0 \leq S_i \leq \epsilon 3^n \). Now \( S_i \geq 0 \) if \( \frac{6}{\epsilon} A_i \geq V_i^2 \), where

(20) \[ A_i = \frac{6}{\epsilon} \left[ \frac{N_i + \epsilon - 6}{3b_i} - 1 \right] - \frac{9}{4} b_i^2 3^{2n} + 1. \]

Hence \( S_i \geq 0 \) if

(21) \[ A_i \geq 0, \quad V_i \geq 0, \quad (\frac{6}{\epsilon} A_i)^{1/2} \geq V_i. \]
Next, $S_i \leq \varepsilon 3^{2n}$ if $\frac{1}{3} G_i \leq V_i^2$, where

$$G_i = A_i - 2 \cdot 3^{2n}/b_i,$$

and hence if

$$G_i \geq 0, \ V_i \geq 0, \ (\frac{1}{3} G_i)^{1/2} \leq V_i.$$

If we assume that $G_i \geq 0$ (whence $A_i \geq 0$), as well as

$$\frac{3}{2} b_i 3^n + (\frac{1}{3} G_i)^{1/2} \leq v_i \leq \frac{3}{2} b_i 3^n + (\frac{1}{3} A_i)^{1/2}, \quad (\frac{1}{3} A_i)^{1/2} \leq \frac{3}{2} b_i 3^n,$$

we see that (21) and (23) follow and that $v_i \leq 3b_i 3^n$, whence (17) hold.

By using the values (20) and (22) of $A_i$ and $G_i$, we see that condition $G_i \geq 0$ and the final inequality (24) are equivalent to

$$l_i \leq N_i \leq L_i, \quad l_i = \varepsilon 3^{2n} + \frac{9}{8} \varepsilon b_i^2 3^{2n} + \beta_i,$$

where

$$L_i = \frac{9}{2} \varepsilon b_i^2 3^{2n} + \beta_i, \quad \beta_i = b_i \left(3 - \frac{\varepsilon}{2}\right) + 6 - \varepsilon = (1 + \frac{1}{2} b_i)(6 - \varepsilon).$$

This inequality will evidently follow if $l_i$ is the lower limit in (12) and if $L_i$ is the upper limit in (12), and hence if

$$2\varepsilon + \gamma + \frac{9}{8} \varepsilon b_i^2 + \frac{\beta_i}{3^{2n}} \leq 3^{i-1} C \leq \frac{3}{2} \varepsilon b_i^3 + \frac{\varepsilon}{3} + \frac{\beta_i}{3^{2n+1}} \quad (i = 1, 2, 3).$$

When $\varepsilon = 1$, $n \geq 8$, inequalities* (26) all hold if

$$b_1 = 5, \ b_2 = 7, \ b_3 = 11, \ C = 168.$$

Since we shall need to assign to $v_i$ a prescribed residue modulo 8, we desire that at least $8$ consecutive integral values of $v_i$ satisfy the first inequality (24) for every $i = 1, 2, 3$. The difference between its limits is

$$D_i = (\frac{1}{3} A_i)^{1/2} - (\frac{1}{3} G_i)^{1/2}.$$

Write $\mu_i = 2 \cdot 3^{2n}/(b_i A_i)$. By (22) and (23), $0 < \mu_i \leq 1$. Thus $D_i$ is the product of $(\frac{1}{3} A_i)^{1/2}$ by

$$1 - (1 - \mu_i)^{1/2} = \frac{\mu_i}{1 + (1 - \mu_i)^{1/2}} > \frac{\mu_i}{2}.$$

Hence

* The limits for $C$ are approximately 147, 188 if $i = 1$; 130.8, 172.6 if $i = 2$; 167.1, 221.9 if $i = 3$. Since we desire a minimum $C$ independent of $i$, we take $C = 168$. 
(28) \[ D_i > \frac{3^{2n}}{b_i(3A_i)^{1/2}}. \]

By (12) and (20),

(29) \[ 3A_i \leq \frac{18}{\epsilon} \left[ \frac{(3^iC - \epsilon)3^{2n} + \epsilon - 6}{3b_i} - 1 \right] - \frac{27}{4} b_i^2 3^{2n} + 3. \]

5. **Proof of Theorem 2 when \( \epsilon = 1 \)**

When \( \epsilon = 1, n \geq 8 \), (27)-(29) give

\[ 3A_1 < 435 \cdot 3^{2n} < 21^2 \cdot 3^{2n}, \]

\[ D_1 > 3^n/105 > 62, \]

and \( D_2 > 29, D_3 > 14, \) whence each \( D_i > 8 \). By (15),

(30) \[ 2B_i - 6b_i = b_i \epsilon F, \]

where \( F \) denotes the quantity in \( \{ \} \) in (15). By Lemma 2, we can choose \( v_i (\text{mod} 8) \) so that \( F \) is congruent modulo 8 to any assigned even integer. Thus in (16) we can choose \( v_i (\text{mod} 8) \) so that \( 2 \epsilon Q_i = 2z (\text{mod} 8) \), where \( z \) is an arbitrary integer. Take \( z = \epsilon = 1 \). Then \( Q_i = 1 (\text{mod} 4) \). But \( Q_i > 0 \). Hence \( Q_i \) is a sum of three integral squares. This proves Theorem 2 when \( \epsilon = 1 \), with \( C = 168, \nu = 8 \).  

6. **Proof of Theorem 2 when \( \epsilon \) is prime to 6 and \( \epsilon > 1 \)**

We shall first determine \( b_1, b_2, b_3, C, \nu \) so that all three inequalities (26) hold when \( n \geq \nu \), viz.,

(31) \[ I_i \leq 3^{i-1}C \leq S_i \quad (i = 1, 2, 3). \]

Minimum values of \( C \) and \( b_i \) may be found by the following scheme. Take \( b_1 \) to be the least positive odd integer for which \( I_1 \leq S_1 \). Take \( C \) to be the least integer \( \geq I_1 \). Take \( b_2 \) and \( b_3 \) to be the least positive odd integers for which \( S_2 \geq 3C, S_3 \geq 9C \). We find that

<table>
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<th>( C )</th>
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</table>

For these values we find that \( I_2 \leq 3C, I_3 \leq 9C \), whence (31) hold.
For a general $e$, we shall choose $b_i$ to be a linear function of $e$ which has the value in the tablette when $e = 5, \cdots, 13$, and is such that (31) hold as regards the coefficients of the highest power of $e$.

A. For $e = 6e + 1$, we take* $b_1 = 14e + 3$ and get

$$C = 24354e^4 + 18882e^3 + 5508e^2 + 728e + 38.$$  

Then $I_1 < C < S_1$.

A. If $e$ is odd, take $b_2 = 21e + 4$. We find that $I_2$ is termwise (as to coefficients of powers of $e$) less than $3C$, and that $S_2$ is termwise $> 3C$. Taking $b_3 = 29e + 6$, we find that $I_3 < 9C < S_3$.

A. If $e$ is even, $e > 0$, take $b_2 = 21e + 3$, $b_3 = 29e + 7$. Since $b_3$ exceeds $b_2$ in Case A.1, evidently $S_3 > 9C$. Similarly, $I_2 < 3C$. Computation gives $S_2 > 3C$, $I_3 < 9C$, since $e \geq 2$.

B. For $e = 6e - 1$, take $b_1 = 14e - 1$. Then

$$C = 24354e^4 - 10944e^3 + 1917e^2 - 150e + 5,$$

and $I_1 < C < S_1$.

B. If $e$ is odd, take $b_2 = 21e - 2$, $b_3 = 29e - 2$. For every $e \geq 1$ computation gives $I_3 \leq 3C$, $S_3 \geq 9C$. See B.2.

B. If $e$ is even, we accent the letters $b, I, S$. Take $b'_2 = 21e - 3$, $b'_3 = 29e - 1$. Computation gives $S'_2 \geq 3C$ if $e \geq 2$, $I'_2 \leq 9C$ for $e \geq 1$. See B.2.

B. Since $b'_2 < b_2$, $b_3 < b'_3$, we have $I'_2 < I_2$, $S'_2 < S_2$, $I_3 < I'_3$, $S_3 < S'_3$. The results proved in Cases B.1 and B.2 therefore imply $I'_2 < 3C$, $I_3 < 9C$, $S'_3 > 9C$, and also $S_2 > 3C$ if $e \geq 2$ (while $S_2 \geq 3C$ by the remark above our tablette). These inequalities together with those in B.1 and B.2 give all the inequalities (31).

By (29) and the square of (28) we see that $D_i > 8$ if $3^{2n}$ exceeds a certain function of $e$ and $i$, and hence if $n$ is sufficiently large.

If $s_i$ is any given integer, Lemma 1 shows the existence of an integer $m_i'$ such that

$$s_i = f(3m_i') + 3^n M_i', 0 \leq m_i' < 3^n,$$

where $M_i'$ is an integer. In (10) take $z = 3m_i'$, $r = 3^n y_i$. Since $\Delta \equiv 3r$ (mod $e$) and since $\Delta$ has the factor $r$ and hence $3^n$, we have $\Delta = 3^n E$. Since $e$ is prime to $3^n$, we get $E \equiv 3y_i$ (mod $e$). Write

$$m_i = m_i' + 3^n y_i, M_i = M_i' - E.$$  

Then

$$f(3m_i) - f(3m_i') = \Delta = 3^n E, s_i = f(3m_i) + 3^n M_i.$$  

* Actually the least odd $b_i$ when $e = 1, 2, 3, or 4.$
Since \( e \) is prime to 3, we can choose integers \( y_i \) so that

\[
M_i' - 3y_i - 6 - 3b_i \equiv 0 \pmod{e}, \quad 0 \leq y_i < e.
\]

The last inequality shows that the maximum \( m_i \) is \( 3^n - 1 + 3^n(e - 1) = e \cdot 3^n - 1 \). Hence \( 0 \leq m_i < e \cdot 3^n \), as desired in Lemma 3.

As before, we write \( M_i = e3^n + N_i \). Thus

\[
N_i = M_i' - 3y_i,
\]

\[
N_i - 6 - 3b_i \equiv 0 \pmod{e}.
\]

It has been noted that the quantity in \{ \} in (15) is even, whence \( B_i = 3b_i \pmod{e} \). Hence

\[
N_i + e - 6 - B_i \equiv 0 \pmod{e}.
\]

Hence (16) yields an integral value of \( Q_i \).

We proved that there exist more than 8 consecutive integral values of \( v_i \) which satisfy inequalities (24). By (15),

\[
2B_i = 6b_i - 3b_i \varphi v_i (\eta - v_i) \pmod{8}, \quad \eta = 3b_i 3^n.
\]

By Lemma 2, we can choose \( v_i \pmod{8} \) so that \( v_i (\eta - v_i) \equiv 2\xi_i \pmod{8} \), where \( \xi_i \) is any assigned integer. Then

\[
N_i + e - 6 - B_i \equiv N_i + e - 6 - 3b_i + 3b_i \xi_i \pmod{4}.
\]

We choose \( \xi_i \) so that the second member is \equiv e \pmod{4} \). Then (16) gives \( Q_i = 1 \pmod{4} \).

Since inequalities (26) were satisfied at the outset, we know that \( G_i \geq 0 \) and that the final inequality (24) holds. Also (24) was shown to hold. Hence (17) hold. By (17a), \( 0 \leq Q_i \leq e3^n \). Since \( Q_i \) is an integer \( \geq 0 \) such that \( Q_i = 1 \pmod{4} \), it is well known that \( Q_i = \sum_{j=1}^{3} x_j^2 \), where the \( x_i \) are integers \( \geq 0 \). But \( Q_i \leq 3^n \), whence each \( x_i \leq 3^n \).

In view of (17a), the integer \( w_i \) defined by (14) is \( \geq 0 \). Then \( f(v_i) + f(w_i) = \phi_i \) has the value \( 3^n B_i \). Hence (16) yields (13), which is the condition that \( s_i \) be the sum of the values of \( f(x) \) for the nine values \( 3m_i, v_i, w_i, 3^n - x_j, 3^n + x_j \) of \( x_i \), each an integer \( \geq 0 \).

7. Proof of Theorem 2 when \( e = 2\delta \), \( \delta \) prime to 3

We shall determine the \( b_i \) and \( C \) to satisfy (26), viz., (31). For the following four values of \( \delta \), our later results do not apply:
In the preceding tablette, the $b_i$ and $C$ have their minimum values and all inequalities (31) hold.

J. When $\delta = 3d+1$, $d \geq 1$, we take $b_1 = 14d+5$ and get

$$C = 24354d^4 + 33795d^3 + 17591d^2 + 4082d + 358,$$

and find that $I_1 < C < S_1$.

J1. If $d$ is odd, take $b_2 = 21d+8$, $b_3 = 29d+12$. Then $S_2 \geq 3C$, $S_3 \geq 9C$.

J2. If $d$ is even, take $b_2 = 21d+7$, $b_3 = 29d+11$. Then $I_2 < 3C$, $S_3 > 9C$.

J12. The remaining inequalities (31) follow from J1 and J2 as in B12.

K. When $\delta = 3d-1$, $d \geq 4$, take $b_1 = 14d-5$. Then

$$C = 24354d^4 - 33795d^3 + 17591d^2 - 4058d + 350.$$

K1. If $d$ is even, take $b_2 = 21d-9$, $b_3 = 29d-9$. Then $I_2 \leq 3C$, $S_3 \geq 9C$ for $d \geq 1$.

K2. If $d$ is odd, take $b_2 = 21d-10$, $b_3 = 29d-8$. Then $S_2 \geq 3C$, $I_3 \leq 9C$ for $d \geq 4$.

K12. Exactly as in B12, the remaining inequalities (31) follow from K1, K2.

Let $\epsilon = 2^s \epsilon$, where $\epsilon$ is odd and $\epsilon \geq 1$. By Lemma 2, we can choose $v_i (\mod 8)$ so that the number in \{ \} in (15) is $\equiv 2z_i (\mod 8)$, where $z_i$ is arbitrary, whence $B_i = 3b_i + b_i \epsilon(z_i + 4u_i)$, where $u_i$ is an integer. Write $\xi_i = b_i \epsilon z_i$, so that $\xi_i$ has a preassigned residue modulo 4, and $B_i = 3b_i + 2^s(\xi_i + 4b_i \epsilon u_i)$.

By choice of $y_i$, we made $N_i + \epsilon - 6 - 3b_i$ a multiple of $\epsilon$, say $2^s \cdot q_i$. Then

$$N_i + \epsilon - 6 - B_i = 2^s F_i, \ F_i = q_i - \xi_i - 4b_i \epsilon u_i.$$

We may choose $\xi_i$ so that $F_i \equiv E (\mod 4)$. Then by (16), $\epsilon Q_i \equiv 2^s E (\mod 4 \cdot 2^s)$, whence $Q_i \equiv 1 (\mod 4)$. But $Q_i$ is an integer $\geq 0$. Hence $Q_i$ is a sum of three squares.

8. Reduction of $D_i$ from 8 to 6

The lower limit $\nu$ of $n$ obtained from $D_i \geq 8$ may be reduced in certain cases by using $D_i \geq 6$.

First, it suffices to have $D_i \geq 7$. Then (24) holds for 7 consecutive integral values of $v_i$. If $f$ is the first of them, then the seven together with $f - 1$ form
a complete set of residues modulo 8. Two values of \( v \) whose sum is \( \eta = 3b_33^a \) yield the same value of \( P = v(\eta - v) \). The value of \( P \) which is apparently lacking in view of the missing value \( f - 1 \) of \( v \) is actually obtained from the value \( u = \eta - f + 1 \) of \( v \), and \( u \neq f - 1 \) since \( \eta \neq 2f - 2 \) (mod 8), \( \eta \) being odd.

Second, it suffices to have \( D_t \geq 6 \). Let \( f \) be the first of six consecutive integral values of \( v \). Then the six together with \( f - 1 \) and \( f - 2 \) form a complete set of residues modulo 8. We saw that the two values \( f - 1 \) and \( u = \eta - f + 1 \) of \( v \) give the same value of \( P \); likewise \( f - 2 \) and \( w = \eta - f + 2 \). Each of \( u = f - 2 \) and \( w = f - 1 \) (mod 8) reduces to

\[
2f - 3 \equiv \eta \quad \text{(mod 8)}.
\]

Hence if (36) does not hold, we may employ \( u \) and \( w \) instead of the missing values \( f - 1 \) and \( f - 2 \) of \( v \), as well as the four residues which together with these four make a complete set of residues modulo 8, and obtain all four even residues of 8 as values of \( P \).

When (36) holds, we modify our proof as follows: We no longer secure the value \( P' = (f - 1)u = (f - 2)w = (f - 1)(f - 2) \) (mod 8) of \( P \). But when \( v \) ranges over six incongruent residues, no one congruent to \( f - 1 \) or \( f - 2 \) modulo 8, we obtain three incongruent even values of \( P \) (viz., all except \( P' \)). In the notations at the end of §6, we may assign to \( \xi \), any one of three values incongruent modulo 4 (viz., any except \( \frac{1}{2}P' \)), and hence choose \( \xi \), so that \( \epsilon Q_i \equiv \epsilon \) or \( 2\epsilon \) (mod 4). Then \( Q_i \equiv 1 \) or \( 2 \) (mod 4) and \( Q_i \) is a sum of three squares. Similarly, in the notations at the end of §7, we secure \( F_i \equiv E \) or \( 2E \) (mod 4), whence \( Q_i \equiv 1 \) or \( 2 \) (mod 4).

9. Universal theorem \( \epsilon = 2 \)

When \( \epsilon = 2, \delta = 1 \), and we saw that (26) are satisfied if \( b_1 = 7, b_2 = 11, b_3 = 15, C = 1025 \). The older condition \( D_t \geq 8 \) requires \( n \geq 9 \). The condition \( D_t \geq 7 \) barely fails to reduce \( n \) from 9 to 8. The best condition \( D_t \geq 6 \) holds if \( n \geq 8 \). Then \( D_t > 14, D_t > 40 \). Hence Theorem 2 shows that every integer \( \geq 1025 \cdot 3^{24} \) is a sum of nine values of

\[
f(x) = x + \frac{2}{3}(x^3 - x) = \frac{2}{3}(x^3 + 2x)
\]

for integral values \( \geq 0 \) of \( x \). We employ

**Theorem 3.** Let a polynomial \( f(x) \) take integral values \( \geq 0 \) for all integers \( x \geq 0 \); let \( f(x + 1) - f(x) \) increase with \( x \). Suppose that every integer \( n \) for which \( l < n \leq g + f(0) \) is a sum of \( k - 1 \) values of \( f(x) \) for integers \( x \geq 0 \). Let \( m \) be the maximum integer for which \( f(m + 1) - f(m) < g - l \). Then every integer \( N \) for which \( l + f(0) < N \leq g + f(m + 1) \) is a sum of \( k \) values of \( f(x) \) for integers \( x \geq 0 \).
Tables were made showing all integers 1–3000 which are sums of 2, 3, 4, or 5 values of \( f(x) \). In particular, all integers 1–3000 except only 42 and 66 are sums of 5 values. This result was proved to hold also for 3000–4000 as follows. A list was made of the integers 2076–3000 which are missing from the table of sums by 4. To this list we added \( 924 = f(14) \) and subtracted \( 1135 = f(15) \) from the sums (actually we subtracted \( 1135 - 924 = 211 \) from our list), and noted whether or not each difference is in the table of sums by 4. If not, we subtracted other values of \( f(x) \) from the sum.

We may apply Theorem 3 to \( f(x) = \frac{1}{3}(x^3 + 2x) \) since \( f(x+1) - f(x) = x^2 + x + 1 \) increases with \( x \). As just proved by tables, every integer \( n \) for which \( 66 < n \leq 4000 \) is a sum of 5 values of \( f(x) \). The maximum integer \( m \) for which

\[
(2m + 1)^2 < 4(4000 - 66) - 3 = 15733
\]

has \( 2m + 1 = 125, \ m = 62 \). Since \( f(63) = 83391 \), Theorem 3 shows that all integers from 67 to 87391 are sums of 6 values of \( f(x) \). The same is true of 42 and 66. Hence we may apply Theorem 3 with \( l = 0, g = 87391, k = 7 \); the maximum \( m \) is 295; hence all integers \( \leq g + f(296) = 8732367 \) are sums of 7 values of \( f(x) \). The next maximum \( m \) is 2954, and all integers \( \leq 8609777000 \) are sums of 8 values of \( f(x) \). The next maximum \( m \) is 97788, and all integers \( \leq 311716 \times 10^8 \) are sums of 9 values of \( f(x) \). This product exceeds \( 1025 \times 3^{34} = 2894902749 \times 10^8 \). This proves

**Theorem 4.** Every integer \( \geq 0 \) is a sum of 9 values of \( \frac{1}{3}(x^3 + 2x) \) for integers \( x \geq 0 \).

Besides the functions treated, there arose in Theorem 1 the special pyramidal function \( P(x) \). For this case, K. C. Yang proved in his Chicago doctoral dissertation of 1928

**Theorem 5.** Every integer \( \geq 0 \) is a sum of nine pyramidal numbers \( \frac{1}{3}(x^3 - x) \) for integers \( x \geq 0 \).

He verified that every integer \( < 7240 \) is a sum of five values.

I have not completed the proof that every integer is a sum of nine values of function (9) for \( \epsilon = 1 \).

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