WARING'S PROBLEM FOR CUBIC FUNCTIONS*

BY

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1. Introduction

In 1921 it was proved by Kamke that if $f(x)$ is a polynomial with rational coefficients whose value is an integer $\geqq 0$ for every integer $x \geqq 0$, then every integer $\geqq 0$ is a sum of a limited number u of 1's and a limited number v of values of $f(x)$ for integers $x \geqq 0$. This existence theorem was later proved by the method of Hardy and Littlewood by Winogradow and Landau.

For the case of any quadratic function, the writer (and later Dr. Pall) evaluated the limits u and v.

We shall here treat cubic functions (1). The case in which a term x^2 occurs is under investigation by my students. The main result is Theorem 2. For special cubic functions, Theorems 4 and 5 give universal Waring theorems.

2. Determination of all functions (1) with certain properties

We restrict attention to cubic functions of the form

$$f(x) = \frac{\alpha x^3 + \beta x}{d} \qquad (\alpha \neq 0,\ d > 0), \tag{1}$$

$$\alpha,\ \beta,\ d \text{ integers without a common factor } > 1. \tag{2}$$

We assume that $f(x)$ is an integer for every integer $x \geqq 0$. By the values 1 and 2 of x, we see that $\alpha + \beta$ and $8\alpha + 2\beta$ are divisible by d, whence

$$6\alpha \text{ and } 6\beta \text{ are divisible by } d. \tag{3}$$

If d has a prime factor $p > 3$, then α and β are divisible by p, contrary to (2). Hence 2 and 3 are the only possible prime factors of d.

To discuss only a pure Waring problem, we assume that $f(x) \geqq 0$ if x is any integer $x \geqq 0$, and that 1 is a value of $f(\xi)$ for a certain integer $\xi \geqq 0$ (otherwise, sums of values of f would never give the number 1).

I. *Case d a multiple of* 6. Write $d = 6t$. By (3), α and β are divisible by t, whence $t = 1$, by (2), and $d = 6$. By $f(\xi) = 1$,

$$\alpha\xi^3 + \beta\xi = 6, \tag{4}$$

whence ξ is a positive divisor of 6.

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I_1. Let $\xi=3w$, $w=1$ or 2. By (4), $9\alpha w^3+\beta w=2$. Elimination of β gives

$$f(x) = \frac{1}{6}\left[\alpha x^3 + \frac{2-9\alpha w^3}{w}x\right],$$

$$f(w) = \tfrac{1}{3}(1-4\alpha w^3) \geqq 0,\ \alpha \leqq 0; \quad f(6w) = 2 + 27\alpha w^3 \geqq 0,\ \alpha \geqq 0,$$

whence $\alpha=0$, contrary to hypothesis. Thus Case I_1 is excluded.

I_2. Let $\xi=2$. By (4), $4\alpha+\beta=3$. Then

$$f(1) = \tfrac{1}{2}(1-\alpha) \geqq 0,\ \alpha \leqq 1; \quad f(3) = \tfrac{1}{6}(15\alpha+9) \geqq 0,\ \alpha \geqq 0,$$

whence $\alpha=1$, and $f(x)$ is the pyramidal number

$$P(x) = \tfrac{1}{6}(x^3 - x). \tag{5}$$

This function satisfies all of our preceding assumptions.

I_3. There remains only the case $\xi=1$. Then $\alpha+\beta=6$,

$$f(x) = x + \frac{\alpha}{6}(x^3 - x) \equiv x + \alpha P(x),\ \alpha > 0, \tag{6}$$

which is an integer $\geqq 0$ for every integer $x \geqq 0$, since the same is true of $P(x)$.

II. *Case d a multiple of* 3, *but not of* 6. Then $d=3t$, where t is a positive odd integer. By (3), α and β are divisible by t, whence $t=1$ by (2). Thus

$$f(x) = \tfrac{1}{3}(\alpha x^3 + \beta x), \quad \alpha\xi^3 + \beta\xi = 3,$$

whence ξ is a positive divisor of 3.

II_1. Let $\xi=3$. Then $\beta=1-9\alpha$,

$$f(1) = \tfrac{1}{3}(1-8\alpha) \geqq 0,\ \alpha \leqq 0;$$
$$f(4) = \tfrac{1}{3}(4+28\alpha) \geqq 0,\ \alpha \geqq 0.$$

Since $\alpha \neq 0$, Case II_1 is excluded.

II_2. Hence $\xi=1$, $\beta=3-\alpha$,

$$f(x) = x + \frac{\alpha}{3}(x^3 - x) \equiv x + 2\alpha P(x),\ \alpha > 0. \tag{7}$$

III. *Case d a multiple of* 2, *but not of* 3. Similarly as in II, we find that $d=2$, $\xi \neq 2$, $\xi=1$, $\alpha+\beta=2$,

$$f(x) = x + \frac{\alpha}{2}(x^3 - x) \equiv x + 3\alpha P(x),\ \alpha > 0. \tag{8}$$

IV. $d=1$. Then $\xi=1$, $\alpha+\beta=1$, $f(x)=x+6\alpha P(x)$.

THEOREM 1. *If $f(x)=(\alpha x^3+\beta x)/d$ $(\alpha\neq 0)$ is an integer $\geqq 0$ for every integer $x\geqq 0$, and if $f(x)=1$ for some integer $x>0$, then $f(x)$ is either a pyramidal number $P(x)=\frac{1}{6}(x^3-x)$, or is $x+\epsilon P(x)$, where ϵ is a positive integer. Conversely, each of the resulting functions is an integer $\geqq 0$ for every integer $x\geqq 0$, while $f(x)=1$ has a positive integral solution.*

3. THE MAIN THEOREM AND THREE LEMMAS

THEOREM 2. *To each positive integer ϵ prime to 3 there correspond positive integers C and $\nu\geqq 8$ such that every integer $\geqq C\cdot 3^{3\nu}$ is a sum of nine values of*

$$f(x) = x + \tfrac{1}{6}\epsilon(x^3 - x) \tag{9}$$

for integral values $\geqq 0$ of x.

We shall first give the parts of the proof which hold both for $\epsilon=1$ and $\epsilon>1$, and then establish the few simple additional facts required in the case $\epsilon=1$, and hence prove Theorem 2 for $\epsilon=1$ with $C=168$, $\nu=8$. Then we shall present the more elaborate theory for $\epsilon>1$ (which does not hold for $\epsilon=1$). That theory gives a reconstructed proof which provides an explicit program actually to express any sufficiently large integer as a sum of nine values of $f(x)$.

LEMMA 1. *There exists an integer m' such that any given integer is congruent to $f(3m')$ modulo 3^n.*

The difference $f(z+3r)-f(z)$ has the value

$$\Delta = \tfrac{1}{2}\epsilon(3rz^2 + 9r^2z + 9r^3) + 3r - \tfrac{1}{2}\epsilon r. \tag{10}$$

Since 3 is a factor of all terms except the last, while ϵ is prime to 3, $\Delta\not\equiv 0$ if $r\not\equiv 0 \pmod{3^n}$. Take $r=m'-k$, $z=3k$, $0<r<3^n$. Then

$$f(3m') - f(3k) = f[3k + 3(m' - k)] - f(3k) \not\equiv 0 \pmod{3^n}.$$

Hence for $j=0, 1, \cdots, 3^n-1$, the 3^n integers $f(3j)$ are incongruent modulo 3^n, so that any integer is congruent to one of them.

LEMMA 2. *If η is an odd constant integer, $v(\eta-v)$ is even and can be made congruent to any assigned even integer modulo 2^k by choice of an integer v.*

Let $V(\eta-V)\equiv v(\eta-v) \pmod{2^k}$. Then the product of $V-v$ by $V+v-\eta$ is divisible by 2^k, while the factors are of unlike parity. Hence one factor is odd and the other is divisible by 2^k. Thus $V\equiv v$ or $\eta-v \pmod{2^k}$. Hence when v ranges over the 2^k values $0, 1, \cdots, 2^k-1$, we obtain at most (and hence exactly) $\frac{1}{2}\cdot 2^k$ values of $v(\eta-v)$ incongruent modulo 2^k, and the latter values are all even. This proves Lemma 2.

LEMMA 3. *If* $n>1$ *and* $m<\epsilon\cdot 3^n$, *then* $f(3m)<\gamma\cdot 3^{3n}$, *where*

$$\gamma = \tfrac{1}{2}(9\epsilon^4 + 1). \tag{11}$$

Since $9m^3-m$ increases with m,

$$f(3m) = 3m + \tfrac{1}{2}\epsilon(9m^3 - m) < 3\epsilon 3^n + \tfrac{1}{2}\epsilon(9\epsilon^3 3^{3n} - \epsilon 3^n),$$

which will be $<\gamma 3^{3n}$ if $3\epsilon-\frac{1}{2}\epsilon^2<\frac{1}{2}3^{2n}$. The latter holds for every n if $\epsilon\geqq 6$ and holds for $n>1$ if $\epsilon\leqq 5$, since the maximum of $6\epsilon-\epsilon^2$ is its value 9 for $\epsilon=3$.

4. PLAN OF PROOF WITH THE NECESSARY FORMULAS

If s and C are given positive numbers, we can evidently choose a positive integer n so that

$$C\cdot 27^n \leqq s < C\cdot 27^{n+1}.$$

In Theorem 2, $s\geqq C\cdot 27^\nu$. Hence we may take $n\geqq\nu\geqq 8$.

Any such s is one of the integers s_i falling in the following three sub-intervals:

$$3^{i-1}C3^{3n} \leqq s_i < 3^iC3^{3n} \qquad (i = 1, 2, 3).$$

By Lemma 1 we can choose an integer m_i so that

$$s_i = f(3m_i) + 3^nM_i, \quad 0 \leqq m_i < 3^n,$$

where M_i is an integer. Since $f(m_i)\geqq 0$, $3^nM_i\leqq s_i<3^iC3^{3n}$. Using also Lemma 3, we get

$$(3^{i-1}C - \gamma)3^{2n} < M_i < 3^iC3^{2n}.$$

Write $M_i=\epsilon 3^{2n}+N_i$. Then

$$(3^{i-1}C - \gamma - \epsilon)3^{2n} < N_i < (3^iC - \epsilon)3^{2n} \qquad (i = 1, 2, 3). \tag{12}$$

Take $l=3^n$ in the identity

$$f(l - x) + f(l + x) \equiv 2l + \tfrac{1}{3}\epsilon(l^3 - l + 3lx^2)$$

and sum for three values x_j of x. Thus

$$\sum_{j=1}^{3} [f(3^n - x_j) + f(3^n + x_j)] = T,$$

$$T = \epsilon 3^{3n} + 3^n(\epsilon Q - \epsilon + 6), \quad Q = x_1^2 + x_2^2 + x_3^2.$$

Write ϕ_i for $f(v_i)+f(w_i)$ and Q_i for Q. Then will

$$s_i = f(3m_i) + 3^n(\epsilon 3^{2n} + N_i) = f(3m_i) + \phi_i + T$$

(and hence s_i will be a sum of nine values of f) if

$$3^n(N_i + \epsilon - 6) = \phi_i + \epsilon 3^n Q_i \qquad (i = 1, 2, 3). \tag{13}$$

We impose on the unknowns v_i, w_i the restrictions

$$v_i + w_i = 3b_i 3^n \qquad (i = 1, 2, 3; b_i \text{ a positive odd integer}). \tag{14}$$

The identity

$$\phi_i \equiv (v_i + w_i)\left[1 + \frac{\epsilon}{6}\{(v_i + w_i)^2 - 3v_i w_i - 1\}\right]$$

gives $\phi_i = 3^n B_i$, where

$$B_i = 3b_i\left[1 + \frac{\epsilon}{6}\{9b_i^2 3^{2n} - 1 - 3v_i(3b_i 3^n - v_i)\}\right]. \tag{15}$$

Inserting $\phi_i = 3^n B_i$ into (13) and cancelling 3^n, we get

$$\epsilon Q_i = N_i + \epsilon - 6 - B_i. \tag{16}$$

We shall later choose the v_i so that

$$0 \leqq v_i \leqq 3b_i 3^n, \quad 0 \leqq N_i + \epsilon - 6 - B_i \leqq \epsilon 3^{2n}. \tag{17}$$

These and (14) and (16) imply

$$0 \leqq w_i, \quad 0 \leqq Q_i \leqq 3^{2n}. \tag{18}$$

We shall later prove that we may take Q_i to be an integer which is a sum of three squares of integers $x_j \geqq 0$. Thus $x_j \leqq 3^n$ by (18). It will then follow that s_i is the sum of the values of $f(x)$ for the nine values $3m_i$, v_i, w_i, $3^n - x_j$, $3^n + x_j$ of x, each an integer $\geqq 0$.

Employ the abbreviation

$$V_i = v_i - \frac{3}{2} b_i 3^n. \tag{19}$$

In the right member of (16), we insert the value of B_i and get

$$S_i = N_i + \epsilon - 6 - 3b_i\left[1 + \frac{\epsilon}{6}\left(3V_i^2 + \frac{9}{4} b_i^2 3^{2n} - 1\right)\right].$$

The final condition (17) is $0 \leqq S_i \leqq \epsilon 3^{2n}$. Now $S_i \geqq 0$ if $\frac{1}{3}A_i \geqq V_i^2$, where

$$A_i = \frac{6}{\epsilon}\left[\frac{N_i + \epsilon - 6}{3b_i} - 1\right] - \frac{9}{4} b_i^2 3^{2n} + 1. \tag{20}$$

Hence $S_i \geqq 0$ if

$$A_i \geqq 0, \quad V_i \geqq 0, \quad (\tfrac{1}{3}A_i)^{1/2} \geqq V_i. \tag{21}$$

Next, $S_i \leqq \epsilon 3^{2n}$ if $\frac{1}{3}G_i \leqq V_i^2$, where

$$G_i = A_i - 2\cdot 3^{2n}/b_i, \tag{22}$$

and hence if

$$G_i \geqq 0, \quad V_i \geqq 0, \quad (\tfrac{1}{3}G_i)^{1/2} \leqq V_i. \tag{23}$$

If we assume that $G_i \geqq 0$ (whence $A_i \geqq 0$), as well as

$$\frac{3}{2} b_i 3^n + (\tfrac{1}{3}G_i)^{1/2} \leqq v_i \leqq \frac{3}{2} b_i 3^n + (\tfrac{1}{3}A_i)^{1/2}, \quad (\tfrac{1}{3}A_i)^{1/2} \leqq \frac{3}{2} b_i 3^n, \tag{24}$$

we see that (21) and (23) follow and that $v_i \leqq 3b_i 3^n$, whence (17) hold.

By using the values (20) and (22) of A_i and G_i, we see that condition $G_i \geqq 0$ and the final inequality (24) are equivalent to

$$l_i \leqq N_i \leqq L_i, \quad l_i = \epsilon 3^{2n} + \frac{9}{8} \epsilon b_i^3 3^{2n} + \beta_i,$$

where

$$L_i = \frac{9}{2} \epsilon b_i^3 3^{2n} + \beta_i, \quad \beta_i \equiv b_i\left(3 - \frac{\epsilon}{2}\right) + 6 - \epsilon = (1 + \tfrac{1}{2}b_i)(6 - \epsilon). \tag{25}$$

This inequality will evidently follow if l_i is $\leqq$ the lower limit in (12) and if L_i is $\geqq$ the upper limit in (12), and hence if

$$2\epsilon + \gamma + \frac{9}{8} \epsilon b_i^3 + \frac{\beta_i}{3^{2n}} \leqq 3^{i-1} C \leqq \frac{3}{2} \epsilon b_i^3 + \frac{\epsilon}{3} + \frac{\beta_i}{3^{2n+1}} \quad (i = 1, 2, 3). \tag{26}$$

When $\epsilon = 1$, $n \geqq 8$, inequalities* (26) all hold if

$$b_1 = 5, \quad b_2 = 7, \quad b_3 = 11, \quad C = 168. \tag{27}$$

Since we shall need to assign to v_i a prescribed residue modulo 8, we desire that at least 8 consecutive integral values of v_i satisfy the first inequality (24) for every $i = 1, 2, 3$. The difference between its limits is

$$D_i = (\tfrac{1}{3}A_i)^{1/2} - (\tfrac{1}{3}G_i)^{1/2}.$$

Write $\mu_i = 2\cdot 3^{2n}/(b_i A_i)$. By (22) and (23_1), $0 < \mu_i \leqq 1$. Thus D_i is the product of $(\frac{1}{3}A_i)^{1/2}$ by

$$1 - (1 - \mu_i)^{1/2} = \frac{\mu_i}{1 + (1 - \mu_i)^{1/2}} > \frac{\mu_i}{2}.$$

Hence

* The limits for C are approximately 147, 188 if $i = 1$; 130.8, 172.6 if $i = 2$; 167.1, 221.9 if $i = 3$. Since we desire a minimum C independent of i, we take $C = 168$.

$$D_i > \frac{3^{2n}}{b_i(3A_i)^{1/2}}\cdot \tag{28}$$

By (12) and (20),

$$3A_i \leqq \frac{18}{\epsilon}\left[\frac{(3^iC-\epsilon)3^{2n}+\epsilon-6}{3b_i}-1\right]-\frac{27}{4}b_i^2 3^{2n}+3. \tag{29}$$

5. Proof of Theorem 2 when $\epsilon=1$

When $\epsilon=1$, $n\geqq 8$, (27)–(29) give

$$3A_1 < 435\cdot 3^{2n} < 21^2\cdot 3^{2n},$$
$$D_1 > 3^n/105 > 62,$$

and $D_2>29$, $D_3>14$, whence each $D_i>8$. By (15),

$$2B_i - 6b_i = b_i\epsilon F, \tag{30}$$

where F denotes the quantity in $\{\quad\}$ in (15). By Lemma 2, we can choose v_i (mod 8) so that F is congruent modulo 8 to any assigned even integer. Thus in (16) we can choose v_i (mod 8) so that $2\epsilon Q_i\equiv 2z$ (mod 8), where z is an arbitrary integer. Take $z=\epsilon=1$. Then $Q_i\equiv 1$ (mod 4). But $Q_i>0$. Hence Q_i is a sum of three integral squares. This proves Theorem 2 when $\epsilon=1$, with $C=168$, $\nu=8$.

6. Proof of Theorem 2 when ϵ is prime to 6 and $\epsilon>1$

We shall first determine b_1, b_2, b_3, C, ν so that all three inequalities (26) hold when $n\geqq\nu$, viz.,

$$I_i \leqq 3^{i-1}C \leqq S_i \qquad (i = 1, 2, 3). \tag{31}$$

Minimum values of C and b_i may be found by the following scheme. Take b_1 to be the least positive odd integer for which $I_1\leqq S_1$. Take C to be the least integer $\geqq I_1$. Take b_2 and b_3 to be the least positive odd integers for which $S_2\geqq 3C$, $S_3\geqq 9C$. We find that

ϵ	b_1	b_2	b_3	C
5	13	19	27	15182
7	17	25	35	49510
11	27	39	57	309485
13	31	45	65	564244

For these values we find that $I_2\leqq 3C$, $I_3\leqq 9C$, whence (31) hold.

For a general ϵ, we shall choose b_i to be a linear function of ϵ which has the value in the tablette when $\epsilon=5, \cdots, 13$, and is such that (31) hold as regards the coefficients of the highest power of ϵ.

A. For $\epsilon=6e+1$, we take* $b_1=14e+3$ and get

$$C = 24354e^4 + 18882e^3 + 5508e^2 + 728e + 38. \tag{32}$$

Then $I_1<C<S_1$.

A₁. If e is odd, take $b_2=21e+4$. We find that I_2 is termwise (as to coefficients of powers of e) less than $3C$, and that S_2 is termwise $>3C$. Taking $b_3=29e+6$, we find that $I_3<9C<S_3$.

A₂. If e is even, $e>0$, take $b_2=21e+3$, $b_3=29e+7$. Since b_3 exceeds b_3 in Case A₁, evidently $S_3>9C$. Similarly, $I_2<3C$. Computation gives $S_2>3C$, $I_3<9C$, since $e\geqq 2$.

B. For $\epsilon=6e-1$, take $b_1=14e-1$. Then

$$C = 24354e^4 - 10944e^3 + 1917e^2 - 150e + 5, \tag{33}$$

and $I_1<C<S_1$.

B₁. If e is odd, take $b_2=21e-2$, $b_3=29e-2$. For every $e\geqq 1$ computation gives $I_2\leqq 3C$, $S_3\geqq 9C$. See B₁₂.

B₂. If e is even, we accent the letters b, I, S. Take $b_2'=21e-3$, $b_3'=29e-1$. Computation gives $S_2'\geqq 3C$ if $e\geqq 2$, $I_3'\leqq 9C$ for $e\geqq 1$. See B₁₂.

B₁₂. Since $b_2'<b_2$, $b_3<b_3'$, we have $I_2'<I_2$, $S_2'<S_2$, $I_3<I_3'$, $S_3<S_3'$. The results proved in Cases B₁ and B₂ therefore imply $I_2'<3C$, $I_3<9C$, $S_3'>9C$, and also $S_2>3C$ if $e\geqq 2$ (while $S_2\geqq 3C$ by the remark above our tablette). These inequalities together with those in B₁ and B₂ give all the inequalities (31).

By (29) and the square of (28) we see that $D_i>8$ if 3^{2n} exceeds a certain function of e and i, and hence if n is sufficiently large.

If s_i is any given integer, Lemma 1 shows the existence of an integer m_i' such that

$$s_i = f(3m_i') + 3^n M_i', \; 0 \leqq m_i' < 3^n,$$

where M_i' is an integer. In (10) take $z=3m_i'$, $r=3^n y_i$. Since $\Delta\equiv 3r \pmod{\epsilon}$ and since Δ has the factor r and hence 3^n, we have $\Delta=3^n E$. Since ϵ is prime to 3^n, we get $E\equiv 3y_i \pmod{\epsilon}$. Write

$$m_i = m_i' + 3^n y_i, \; M_i = M_i' - E.$$

Then

$$f(3m_i) - f(3m_i') = \Delta = 3^n E, \; s_i = f(3m_i) + 3^n M_i.$$

* Actually the least odd b_1 when $e=1, 2, 3$, or 4.

Since ϵ is prime to 3, we can choose integers y_i so that

$$M_i' - 3y_i - 6 - 3b_i \equiv 0 \pmod{\epsilon}, \quad 0 \leqq y_i < \epsilon.$$

The last inequality shows that the maximum m_i is $3^n - 1 + 3^n(\epsilon - 1) = \epsilon \cdot 3^n - 1$. Hence $0 \leqq m_i < \epsilon \cdot 3^n$, as desired in Lemma 3.

As before, we write $M_i = \epsilon 3^{2n} + N_i$. Thus

$$N_i \equiv M_i' - 3y_i,$$

$$N_i - 6 - 3b_i \equiv 0 \pmod{\epsilon}.$$

It has been noted that the quantity in $\{\quad\}$ in (15) is even, whence $B_i \equiv 3b_i \pmod{\epsilon}$. Hence

$$N_i + \epsilon - 6 - B_i \equiv 0 \pmod{\epsilon}.$$

Hence (16) yields an integral value of Q_i.

We proved that there exist more than 8 consecutive integral values of v_i which satisfy inequalities (24). By (15),

$$2B_i \equiv 6b_i - 3b_i \epsilon v_i(\eta - v_i) \pmod{8}, \quad \eta = 3b_i 3^n.$$

By Lemma 2, we can choose v_i (mod 8) so that $v_i(\eta - v_i) \equiv 2\zeta_i$ (mod 8), where ζ_i is any assigned integer. Then

$$N_i + \epsilon - 6 - B_i \equiv N_i + \epsilon - 6 - 3b_i + 3b_i \epsilon \zeta_i \pmod{4}.$$

We choose ζ_i so that the second member is $\equiv \epsilon$ (mod 4). Then (16) gives $Q_i \equiv 1$ (mod 4).

Since inequalities (26) were satisfied at the outset, we know that $G_i \geqq 0$ and that the final inequality (24) holds. Also (24_1) was shown to hold. Hence (17) hold. By (17_2), $0 \leqq \epsilon Q_i \leqq \epsilon 3^{2n}$. Since Q_i is an integer $\geqq 0$ such that $Q_i \equiv 1$ (mod 4), it is well known that $Q_i = \sum_{j=1}^{3} x_j^2$, where the x_j are integers $\geqq 0$. But $Q_i \leqq 3^{2n}$, whence each $x_j \leqq 3^n$.

In view of (17_1), the integer w_i defined by (14) is $\geqq 0$. Then $f(v_i) + f(w_i) \equiv \phi_i$ has the value $3^n B_i$. Hence (16) yields (13), which is the condition that s_i be the sum of the values of $f(x)$ for the nine values $3m_i$, v_i, w_i, $3^n - x_j$, $3^n + x_j$ of x, each an integer $\geqq 0$.

7. Proof of Theorem 2 when $\epsilon = 2\delta$, δ prime to 3

We shall determine the b_i and C to satisfy (26), viz., (31). For the following four values of δ, our later results do not apply:

δ	b_1	b_2	b_3	C
1	7	11	15	1025
2	11	17	25	7943
5	23	31	45	145900
8	37	55	77	1206699

In the preceding tablette, the b_i and C have their minimum values and all inequalities (31) hold.

J. When $\delta=3d+1$, $d\geqq 1$, we take $b_1=14d+5$ and get

$$C = 24354d^4 + 33795d^3 + 17591d^2 + 4082d + 358, \tag{34}$$

and find that $I_1<C<S_1$.

J_1. If d is odd, take $b_2=21d+8$, $b_3=29d+12$. Then $S_2\geqq 3C$, $I_3\leqq 9C$.

J_2. If d is even, take $b_2=21d+7$, $b_3=29d+11$. Then $I_2<3C$, $S_3>9C$.

J_{12}. The remaining inequalities (31) follow from J_1 and J_2 as in B_{12}.

K. When $\delta=3d-1$, $d\geqq 4$, take $b_1=14d-5$. Then

$$C = 24354d^4 - 33795d^3 + 17591d^2 - 4058d + 350. \tag{35}$$

K_1. If d is even, take $b_2=21d-9$, $b_3=29d-9$. Then $I_2\leqq 3C$, $S_3\geqq 9C$ for $d\geqq 1$.

K_2. If d is odd, take $b_2=21d-10$, $b_3=29d-8$. Then $S_2\geqq 3C$, $I_3\leqq 9C$ for $d\geqq 4$.

K_{12}. Exactly as in B_{12}, the remaining inequalities (31) follow from K_1, K_2.

Let $\epsilon=2^eE$, where E is odd and $e\geqq 1$. By Lemma 2, we can choose v_i (mod 8) so that the number in $\{\quad\}$ in (15) is $\equiv 2z_i$ (mod 8), where z_i is arbitrary, whence $B_i=3b_i+b_i\epsilon(z_i+4u_i)$, where u_i is an integer. Write $\zeta_i=b_iEz_i$, so that ζ_i has a preassigned residue modulo 4, and $B_i=3b_i+2^e(\zeta_i+4b_iEu_i)$.

By choice of y_i, we made $N_i+\epsilon-6-3b_i$ a multiple of ϵ, say $2^e\cdot q_i$. Then

$$N_i + \epsilon - 6 - B_i = 2^eF_i,\ F_i = q_i - \zeta_i - 4b_iEu_i.$$

We may choose ζ_i so that $F_i\equiv E$ (mod 4). Then by (16), $\epsilon Q_i\equiv 2^eE$ (mod $4\cdot 2^e$), whence $Q_i\equiv 1$ (mod 4). But Q_i is an integer $\geqq 0$. Hence Q_i is a sum of three squares.

8. Reduction of D_i from 8 to 6

The lower limit ν of n obtained from $D_i\geqq 8$ may be reduced in certain cases by using $D_i\geqq 6$.

First, it suffices to have $D_i\geqq 7$. Then (24) holds for 7 consecutive integral values of v_i. If f is the first of them, then the seven together with $f-1$ form a complete set of residues modulo 8. Two values of v whose sum is $\eta = 3b_i 3^n$ yield the same value of $P = v(\eta - v)$. The value of P which is apparently lacking in view of the missing value $f-1$ of v is actually obtained from the value $u = \eta - f + 1$ of v, and $u \not\equiv f-1$ since $\eta \not\equiv 2f-2 \pmod 8$, η being odd.

Second, it suffices to have $D_i \geqq 6$. Let f be the first of six consecutive integral values of v. Then the six together with $f-1$ and $f-2$ form a complete set of residues modulo 8. We saw that the two values $f-1$ and $u = \eta - f + 1$ of v give the same value of P; likewise $f-2$ and $w = \eta - f + 2$. Each of $u \equiv f-2$ and $w \equiv f-1 \pmod 8$ reduces to

$$2f - 3 \equiv \eta \pmod{8}. \tag{36}$$

Hence if (36) does not hold, we may employ u and w instead of the missing values $f-1$ and $f-2$ of v, as well as the four residues which together with these four make a complete set of residues modulo 8, and obtain all four even residues of 8 as values of P.

When (36) holds, we modify our proof as follows: We no longer secure the value $P' \equiv (f-1)u \equiv (f-2)w \equiv (f-1)(f-2) \pmod 8$ of P. But when v ranges over six incongruent residues, no one congruent to $f-1$ or $f-2$ modulo 8, we obtain three incongruent even values of P (viz., all except P'). In the notations at the end of §6, we may assign to ζ_i any one of three values incongruent modulo 4 (viz., any except $\frac{1}{2}P'$), and hence choose ζ_i so that $\epsilon Q_i \equiv \epsilon$ or $2\epsilon \pmod 4$. Then $Q_i \equiv 1$ or $2 \pmod 4$ and Q_i is a sum of three squares. Similarly, in the notations at the end of §7, we secure $F_i \equiv E$ or $2E \pmod 4$, whence $Q_i \equiv 1$ or $2 \pmod 4$.

9. Universal theorem $\epsilon = 2$

When $\epsilon = 2$, $\delta = 1$, and we saw that (26) are satisfied if $b_1 = 7$, $b_2 = 11$, $b_3 = 15$, $C = 1025$. The older condition $D_3 \geqq 8$ requires $n \geqq 9$. The condition $D_3 \geqq 7$ barely fails to reduce n from 9 to 8. The best condition $D_3 \geqq 6$ holds if $n \geqq 8$. Then $D_2 > 14$, $D_1 > 40$. Hence Theorem 2 shows that every integer $\geqq 1025 \cdot 3^{24}$ is a sum of nine values of

$$f(x) = x + \tfrac{1}{3}(x^3 - x) = \tfrac{1}{3}(x^3 + 2x)$$

for integral values $\geqq 0$ of x. We employ

THEOREM 3. *Let a polynomial $f(x)$ take integral values $\geqq 0$ for all integers $x \geqq 0$; let $f(x+1) - f(x)$ increase with x. Suppose that every integer n for which $l < n \leqq g + f(0)$ is a sum of $k-1$ values of $f(x)$ for integers $x \geqq 0$. Let m be the maximum integer for which $f(m+1) - f(m) < g - l$. Then every integer N for which $l + f(0) < N \leqq g + f(m+1)$ is a sum of k values of $f(x)$ for integers $x \geqq 0$.*

Tables were made showing all integers 1–3000 which are sums of 2, 3, 4, or 5 values of $f(x)$. In particular, all integers 1–3000 except only 42 and 66 are sums of 5 values. This result was proved to hold also for 3000–4000 as follows. A list was made of the integers 2076–3000 which are missing from the table of sums by 4. To this list we added $924=f(14)$ and subtracted $1135 =f(15)$ from the sums (actually we subtracted $1135-924=211$ from our list), and noted whether or not each difference is in the table of sums by 4. If not, we subtracted other values of $f(x)$ from the sum.

We may apply Theorem 3 to $f(x)=\frac{1}{3}(x^3+2x)$ since $f(x+1)-f(x) =x^2+x+1$ increases with x. As just proved by tables, every integer n for which $66<n\leqq 4000$ is a sum of 5 values of $f(x)$. The maximum integer m for which

$$(2m+1)^2 < 4(4000-66)-3 = 15733$$

has $2m+1=125$, $m=62$. Since $f(63)=83391$, Theorem 3 shows that all integers from 67 to 87391 are sums of 6 values of $f(x)$. The same is true of 42 and 66. Hence we may apply Theorem 3 with $l=0$, $g=87391$, $k=7$; the maximum m is 295; hence all integers $\leqq g+f(296)=8732367$ are sums of 7 values of $f(x)$. The next maximum m is 2954, and all integers $\leqq 860\dot{9}\dot{7}\dot{7}\dot{7}000$ are sums of 8 values of $f(x)$. The next maximum m is 97788, and all integers $\leqq 311716 \times 10^9$ are sums of 9 values of $f(x)$. This product exceeds $1025\cdot 3^{24}=2894902749 \times 10^5$. This proves

THEOREM 4. *Every integer* $\geqq 0$ *is a sum of* 9 *values of* $\frac{1}{3}(x^3+2x)$ *for integers* $x\geqq 0$.

Besides the functions treated, there arose in Theorem 1 the special pyramidal function $P(x)$. For this case, K. C. Yang proved in his Chicago doctoral dissertation of 1928

THEOREM 5. *Every integer* $\geqq 0$ *is a sum of nine pyramidal numbers* $\frac{1}{6}(x^3-x)$ *for integers* $x\geqq 0$.

He verified that every integer <7240 is a sum of five values.

I have not completed the proof that every integer is a sum of nine values of function (9) for $\epsilon=1$.

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