ALMOST PERIODIC TRANSFORMATIONS*

BY

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1. Introduction

When one studies periodic transformations such as, for example, rotations, he often encounters transformations which are not periodic but which are, in a very real and non-technical sense, *almost* periodic. For instance, repeated rotation through an angle which is an irrational part of a revolution will never bring a point set back point-for-point into itself; yet this object may be as nearly attained as we please by repeating the process an appropriate number of times. Moreover, such "appropriate" numbers are relatively dense† among the integers. This example suggests the definition of an a.p. (almost periodic) transformation; it being only necessary to make precise the meaning of "as nearly as possible" when applied to bringing the points of a set back into themselves.

Consider, for example, a set $\mathfrak{T}$ of uniformly continuous transformations which take each point of a complete metric space $\mathfrak{C}$ into a point of $\mathfrak{C}$. Let $\mathfrak{T}$ contain the identity and the product of any two of its elements. Then if ξ is a variable point of $\mathfrak{C}$ and $\Phi(\xi)$ and $\Psi(\xi)$ are any two elements of $\mathfrak{T}$, let the smaller of the two quantities, unity and the least upper bound for all ξ in $\mathfrak{C}$ of the distance between the points $\Phi(\xi)$ and $\Psi(\xi)$, be called the distance between the transformations $\Phi(\xi)$ and $\Psi(\xi)$, and let it be indicated by $\|\Phi(\xi), \Psi(\xi)\|$. Then $\|\Phi(\xi), \xi\|$ represents one way of telling how nearly the points $\Phi(\xi)$ approximate the points ξ. Moreover a transformation of $\mathfrak{T}$ will be called *a.p.* if to each positive number e there corresponds a positive integer L so great that among each L successive positive integers there is an integer N satisfying $\|\Phi^N(\xi), \xi\| \leq e$. This is merely an example of a definition of an a.p. transformation. A more general definition will be given in the next section. Transformations will be thought of as points in a new space, and a.p. points will be defined. Moreover, for simplicity in wording and notation, most of the theorems on a.p. transformations will be stated in terms of a.p. points. However, the reader may readily re-phrase them in terms of the more natural and significant a.p. transformations.

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† A set of real numbers is called relatively dense if there exists a positive number L so great that every interval of length L contains at least one element of the set.

For the sake of generality the concepts of a.p. functions and sequences in a complete metric space have been introduced. They include ordinary a.p. functions and sequences as special cases; and may also be thought of as including a.p. transformations as a special case. However, this work is not merely a generalization of the standard theory of a.p. functions and sequences, for my most significant theorem—the climax of the whole theory—applies to a.p. transformations (or points) alone, and does not seem to be susceptible of generalization to space functions or sequences. The theorem to which I refer is Theorem V, §8, which shows that every a.p. transformation can be expressed as an infinite product of simpler transformations.

2. Almost periodic points, space functions, and space sequences

Definition. A space $\mathfrak{T}$ will be called a $\mathfrak{G}$-*space* if it satisfies the postulates

a. $\mathfrak{T}$ is metric and complete (Let $\|\phi, \psi\|$ denote the distance from ϕ to ψ.)

b. An operation called *multiplication* is defined so that to each ordered pair of points ϕ and ψ corresponds a unique point $\phi\psi$. The operation is associative, and the space contains an identity point I.

c. $\|\phi\theta, \psi\theta\| \leqq \|\phi, \psi\|$ for any three points ϕ, ψ, θ.

d. The product $\theta\phi$ is a uniformly continuous function of the variable point ϕ for each point θ.

Theorem I. *In any* $\mathfrak{G}$-*space*, $\|\phi\theta, \psi\theta\| = \|\phi, \psi\|$ *if* θ^{-1} *exists.**

Theorem II. *In any* $\mathfrak{G}$-*space the product* $\theta\phi$ *is a continuous function of the points* θ *and* ϕ.

Definition. Let a positive number e and a point ϕ of the $\mathfrak{G}$-space $\mathfrak{T}$ be given. Then an integer N which satisfies $\|\phi^N, I\| \leqq e$ is called an *e-iteration exponent* of ϕ. Moreover a point ψ of $\mathfrak{T}$ is called *a.p.* if to each positive number e there corresponds a positive integer L so great that among every L successive positive integers there is an e-iteration exponent of ψ.

Definition. Let each point of a $\mathfrak{G}$-space $\mathfrak{T}$ be a (1, 1) transformation which takes a set or space $\mathfrak{E}$ into a subset of itself. Then an a.p. point ϕ of $\mathfrak{T}$ will be called an *a.p. transformation*.

It can readily be verified that this definition includes the special case of the definition given above. Moreover a.p. points are no more general than a.p. transformations, for if a $\mathfrak{G}$-space $\mathfrak{T}$ is given, a space $\mathfrak{T}'$ of transformations can always be set up isomorphic with $\mathfrak{T}$. Merely let $\xi' = \phi\xi$ correspond to the point ϕ.

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* The symbol θ^{-1} denotes a point which satisfies the equations $\theta\theta^{-1} = \theta^{-1}\theta = I$.

As an example of a $\mathfrak{G}$-space, consider the set of all complex numbers whose absolute value is unity or less. Take multiplication and distance in their ordinary sense. In this space, all the numbers whose absolute value is unity are a.p. points.

Another example of a $\mathfrak{G}$-space is the set of all complex numbers; the product of two points being the sum of the numbers. In this case the unit point (the number zero) is the only a.p. point.

DEFINITION. Let $\mathfrak{S}$ be a complete metric space, and $\Phi(t)$ a function defined over a set of real numbers and having its set of values in $\mathfrak{S}$. Let s be a real number such that $t+s$ is in the set of definition of $\Phi(t)$ for all values of t in that set of definition. Then s will be called an e-translation number of $\Phi(t)$ if the distance between $\Phi(t)$ and $\Phi(t+s)$ is never greater than the positive number e.

DEFINITION.* A continuous space function $\Phi(t)$ of the real variable t which has its set of values in a complete metric space $\mathfrak{S}$ is called *a.p.* if its e-translation numbers corresponding to each positive e are relatively dense. If each point of $\mathfrak{S}$ is a transformation of the points of a set $\mathfrak{C}$ into a subset of $\mathfrak{C}$, $\Phi(t)$ is called an *a.p. family of transformations.*

DEFINITION.* Let $\{\Gamma_n\}$ be a two-way sequence of points in a complete metric space $\mathfrak{S}$. Then if the e-translation numbers of Γ_n considered as a function of n are relatively dense for each positive e, Γ_n is called an a.p. sequence. If each point of $\mathfrak{S}$ is a transformation, Γ_n is called an *a.p. sequence of transformations.*

THEOREM III. *An a.p. space function is uniformly continuous for all values of its argument.*

THEOREM IV. *An a.p. space function or sequence is bounded.*

These two theorems can be proved in the same way as the special case of numerical a.p. functions or sequences.†

THEOREM V. *In the complete metric space $\mathfrak{S}$ let $\{\Gamma_n\}$ be an a.p. sequence of points such that the distance between Γ_m and Γ_n equals the distance between Γ_{m+1} and Γ_{n+1} for every pair of integers m and n, and let $\mathfrak{T}$ be the subspace consisting of the points Γ_n and their limit points. Then*‡

* The author is indebted to Dr. I. J. Schoenberg for the suggestion which led to this generalization of a.p. transformations.

† H. Bohr, *Zur Theorie der fastperiodischen Funktionen*, Acta Mathematica, vol. 45, pp. 29–127, especially pp. 35 and 36.

‡ The notation
$$\lim_{\phi(\xi)\to\alpha} \theta(\xi)=\beta$$
means that to each $e>0$ there corresponds a $d>0$ such that every point ξ which satisfies $\|\phi(\xi), \alpha\| \leqq d$ also satisfies $\|\theta(\xi), \beta\| \leqq e$.

$$(1) \qquad \phi\psi = \lim_{\Gamma_m \to \phi, \Gamma_n \to \psi} \Gamma_{m+n}$$

exists and is a point of $\mathfrak{T}$ *whenever* ϕ *and* ψ *are both points of* $\mathfrak{T}$. *Moreover if we let* (1) *define multiplication, the space* $\mathfrak{T}$ *will be a* $\mathfrak{G}$*-space having* Γ_1 *as an a.p. point.*

The existence of $\phi\psi$ follows from the inequality

$$\|\Gamma_{m+n}, \Gamma_{m'+n'}\| \leqq \|\Gamma_m, \Gamma_{m'}\| + \|\Gamma_n, \Gamma_{n'}\|$$

and the completeness of $\mathfrak{T}$. It is easy to verify the fact that $\mathfrak{T}$ is a $\mathfrak{G}$-space, and since $\Gamma_1^n = \Gamma_n$, the translation indices of the sequences are iteration exponents of Γ_1.

3. Almost periodic continuation

Definition. Let $\mathfrak{J}$ be the set $a \leqq t < +\infty$ or the set $-\infty < t < +\infty$. Let $\mathfrak{R}$ be an infinite set of real numbers such that the sum and difference of any two numbers in $\mathfrak{R}$ is in $\mathfrak{R}$; then a function $\Phi(t)$ defined on the intersection $\mathfrak{J}\mathfrak{R}$ and having its set of values in a complete metric space $\mathfrak{S}$ will be called *asymptotically periodic* if there exists a sequence of positive numbers $s_1, s_2, \cdots$ in $\mathfrak{R}$ such that $s_n \to \infty$ as $n \to \infty$ and such that uniformly in t on $\mathfrak{J}\mathfrak{R}$,

$$\lim_{n\to\infty} \Phi(t + s_n) = \Phi(t).$$

Lemma 1. *If* $\Phi(t)$ *is asymptotically periodic on* $\mathfrak{J}\mathfrak{R}$, *then there is one and only one asymptotically periodic function* $\Psi(t)$ *which equals* $\Phi(t)$ *on* $\mathfrak{J}\mathfrak{R}$ *but is defined on* $\mathfrak{Y}\mathfrak{R}$, *where* $\mathfrak{Y}$ *is an interval containing* $\mathfrak{J}$.

For if t is any number in $\mathfrak{R}$, for sufficiently great integers m and n the numbers $t+s_m$, $t+s_n$, $t+s_m+s_n$ are all in $\mathfrak{J}\mathfrak{R}$, and

$$\|\Phi(t + s_m), \Phi(t + s_n)\| \leqq \|\Phi(t + s_m + s_n), \Phi(t + s_m)\| + \|\Phi(t + s_m + s_n), \Phi(t + s_n)\|.$$

It follows from the hypothesis concerning Φ that the second member approaches zero as m and n approach infinity, and hence that

$$\Psi(t) = \lim_{n\to\infty} \Phi(t + s_n)$$

is defined for all t on $\mathfrak{R}$; and we have $\Psi(t) = \Phi(t)$ on $\mathfrak{J}\mathfrak{R}$. Now corresponding to an arbitrary $e > 0$ there exists an integer N so great that for all t' on $\mathfrak{J}\mathfrak{R}$ and any $n > N$,

$$\|\Phi(t' + s_n); \Phi(t')\| \leqq e.$$

Thus for any t on $\mathfrak{R}$ and any $n>N$,

$$\|\Psi(t+s_n),\Psi(t)\| = \lim_{p\to\infty} \|\Phi(t+s_n+s_p),\Phi(t+s_p)\| \leqq e,$$

and

$$\lim_{n\to\infty} \Psi(t+s_n) = \Psi(t)$$

uniformly over the whole set $\mathfrak{R}$. Finally suppose $\Theta(t)$ is equal to $\Phi(t)$ on $\mathfrak{J}\mathfrak{R}$ and satisfies uniformly on its set of definition $\mathfrak{Y}\mathfrak{R}$ the equation

$$\lim_{n\to\infty} \Theta(t+s_n') = \Theta(t),$$

where $s_n'\to\infty$. Then on $\mathfrak{Y}\mathfrak{R}$, for sufficiently great m and n,

$$\begin{aligned}\|\Psi(t),\Theta(t)\| \leqq \|\Psi(t),\Psi(t+s_n)\| &+ \|\Theta(t+s_n),\Theta(t+s_n+s_n')\| \\ &+ \|\Psi(t+s_n+s_n'),\Psi(t+s_n')\| + \|\Theta(t+s_n'),\Theta(t)\|,\end{aligned}$$

and since the second member approaches zero as n approaches infinity, $\Theta(t)=\Psi(t)$ on $\mathfrak{Y}\mathfrak{R}$, and the lemma is proved.

THEOREM I. *An a.p. space function or space sequence is completely determined by its values on any half infinite interval.*

THEOREM II. *An a.p. point or transformation ϕ possesses an inverse ϕ^{-1} (in the case of the transformation, a single-valued inverse over its whole set of definition), and its integer powers form an a.p. sequence or sequence of transformations.*

For if N_n is a $(1/n)$-iteration exponent of ϕ and its value is greater than n for each positive n, then uniformly for non-negative k,

$$\lim_{n\to\infty} \phi^{N_n+k} = \phi^k$$

and ϕ^n is an asymptotically periodic function ϕ_n of the non-negative integer n. But by the lemma, ϕ_n is defined for all integers n, and hence

$$\phi\phi_{-1} = \lim_{n\to\infty} \phi\phi_{-1+N_n} = I = \lim_{n\to\infty} \phi_{-1+N_n}\phi = \phi_{-1}\phi;$$

so that $\phi_n=\phi^n$ for all integers n. Moreover negative translation numbers for ϕ_n exist, since $\|\phi^{k-n},\phi^k\|=\|\phi^k,\phi^{k+n}\|$.

4. FOURIER SEQUENCES

In the future it will often be convenient to state two or more theorems or definitions at once; and this will be done by the use of brackets. Where alternative words or sets of words are to be used, both alternatives will be inserted in the brackets and separated by a semi-colon. If no words are needed for one of the alternatives, that will be indicated by a dash. In reading the theorem, read one set of words taken in the same relative position from each pair of brackets. Parenthetical expressions are indicated in the ordinary way and have nothing to do with the brackets.

DEFINITION. A finite or infinite sequence $f_1, f_2, \cdots$ of real numbers will be called an *upper Fourier sequence* of [a continuous space function; a two-way space sequence; a point in a $\mathfrak{T}$-space] Θ if to each $e>0$ correspond a positive integer N and a positive number d such that any [number; integer; integer] t whose multiples $tf_1, tf_2, \cdots, tf_N$ all differ from integers by less than d is an [e-translation number; e-translation index; e-iteration exponent] of Θ.

DEFINITION. A sequence will be called a *lower Fourier sequence* of Θ if to each positive number d and positive integer N (not greater than the number of elements f_i) corresponds a positive number e such that all the multiples $tf_1, tf_2, \cdots, tf_N$ of any [e-translation number; e-translation index; e-iteration exponent] t of Θ differ from integers by less than d.

The relationship between upper and lower Fourier sequences will be given in Theorem V.

DEFINITION. A sequence which is both an upper and a lower Fourier sequence is called a *Fourier sequence.*

THEOREM I. *Every a.p. [function; sequence] in [Bohr's; Walther's] sense has at least one Fourier sequence.*

In the case of the function, two of Bohr's* theorems show that a Fourier sequence can be obtained by dividing each Fourier exponent by 2π and arranging them in countable order. Moreover, Walther† has shown how to construct corresponding to a given a.p. sequence an a.p. function whose set of integer e-translation numbers corresponding to each given $e>0$ will be identical with the set of e-translation indices of the sequence. Thus a Fourier sequence of the function will be a Fourier sequence of the sequence.

DEFINITION. If s and t are variable real [numbers; integers] and $\Theta(t)$ is an a.p. space [function; sequence], the real [function; sequence] $f(t) = \sup\ \Theta(s+t), \Theta(s)$ is called the *Bochner translation [function; sequence] of* $\Theta(t)$.

THEOREM II. *The Bochner translation [function; sequence] of a given a.p. space [function; sequence] is a.p., and its set of e-translation [numbers; indices] for each given $e>0$ is identical with the set of e-translation [numbers; indices] of the given space [function; sequence].*

* *Zur Theorie der fastperiodischen Funktionen,* Acta Mathematica, vol. 46 (1925), pp. 101–214, especially pp. 105 and 110.

† *Fastperiodische Folgen und ihre Fouriersche Analyse,* Atti del Congresso Internazionale dei Matematici 1928 (VII), vol. 2, pp. 289–298, especially p. 290.

From Theorems I and II of this section and Theorem II, §3, we have immediately the following theorem which is fundamental in this work:

THEOREM III. *Every a.p.* [*space function*; *space sequence*; *point*] *has at least one Fourier sequence.*

THEOREM IV. *The necessary and sufficient condition that a* [*continuous space function*; *space sequence*; *point in a* $\mathfrak{G}$*-space*] *be a.p. is that it have an upper Fourier sequence.*

Here sufficiency follows from Wennberg's* theorem on Diophantine approximation.

THEOREM V. *Each element* f_p *of a lower Fourier sequence of an a.p.* [*space function*; *space sequence*; *point*] Θ *is linearly dependent with integer coefficients on a finite number* (*dependent on* p) *of the elements of any upper Fourier sequence* $f_1', f_2', \cdots$ *of* Θ [— —; *and unity*; *and unity*].

For let M be a positive integer greater than unity. Let e_M be a positive number such that tf_p differs from an integer by less than $1/(2M)$ whenever t is an e_M-[translation number; translation index; iteration exponent] of Θ. Let d_M be a positive number less than $1/(2M)$ and N_M a positive integer such that t is an e_M-[translation number; translation index; iteration exponent] of Θ whenever the numbers $tf_1', tf_2', \cdots, tf_{N_M}'$ all differ from integers by less than d_M. Then there exists no number t at all which will make tf_p+1/M, $tf_1', tf_2', \cdots, tf_{MN}'$ all differ from integers by less than d_M. Now according to a theorem of [Bohr†; Giraud‡; Giraud‡], a necessary and sufficient condition that there exist values of the variable [number; integer; integer] t which bring a given set of linear functions a_it+b_i $(i=0, 1, \cdots, Q)$ arbitrarily close to integers is that every set of integer multipliers $g_0, g_1, \cdots, g_Q$ which make the quantity $\sum_{i=0}^{Q} g_ia_i$ become [zero; an integer; an integer] should make the quantity $\sum_{i=0}^{Q} g_ib_i$ an integer. In the present case arbitrarily good approximating values of t do not exist if we put $a_0=f_p$, $a_1=f_1', \cdots, a_{N_M}=f_{N_M}'$ and $b_0=1/M$, $b_1=0, \cdots, b_{N_M}=0$; hence the condition is not satisfied, and there exist integers $g_M, g_{M,1}, g_{M,2}, \cdots, g_{M,N_M}$ such that

$$g_Mf_p + \sum_{i=1}^{N_M} g_{M,i}\, f_i'$$

* *Zur Theorie der Dirichlet'schen Reihen*, Dissertation, Upsala, 1920, p. 19.

† *Neuerer Beweis eines allgemeinen Kronecker'schen Approximationssatzes*, Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser, vol. 6 (1924–25), Article 8.

‡ *Sur la résolution approchée en nombres entiers d'un système d'équations linéaires non homogènes*, Société Mathématique de France, Comptes Rendus des Séances, 1914, pp. 29–32.

is [zero; an integer; an integer] and such that g_M/M is not an integer. Thus each of the quantities g_2f_p, g_3f_p, $\cdots$ can be expressed in a finite linear combination of [— —; unity and; unity and] the quantities f_1', f_2', $\cdots$ with integer coefficients. Now if $k_1, k_2, \cdots, k_n$ are the prime factors of g_2, unity can be expressed as a finite linear combination of g_2, g_{k_1}, g_{k_2}, $\cdots$, g_{k_n}, since g_M/M is not an integer; and hence f_p can be so expressed in terms of g_2f_p, $g_{k_1}f_p$, $\cdots$, $g_{k_n}f_p$.

THEOREM VI. *A sequence whose elements are linearly dependent with integer coefficients on* [— —; *unity and*; *unity and*] *a finite number of the elements of a lower Fourier sequence of an a.p.* [*space function*; *space sequence*; *point*] Θ *is itself a lower Fourier sequence of* Θ. *A sequence on a finite number of whose elements* [— —; *and unity*; *and unity*] *each element of an upper Fourier sequence of* Θ *is linearly dependent with integer coefficients is itself an upper Fourier sequence of* Θ.

For a linear combination of numbers with integer coefficients can be brought as close to an integer as we please by bringing the numbers sufficiently close to integers. The last two theorems lead immediately to the

THEOREM VII. *Let* $f_1, f_2, \cdots$ *be a Fourier sequence of an a.p.* [*space function*; *space sequence*; *point*] Θ. *Then a necessary and sufficient condition that a sequence* $f_1', f_2', \cdots$ *should also be a Fourier sequence of* Θ *is that each* f_p *be linearly dependent with integer coefficients on a finite number of* $f_1', f_2', \cdots$ [— —; *and unity*; *and unity*], *and vice versa.*

DEFINITION. *A number module* is a set of real numbers which forms a group under the operation of addition. It is called *complete* if it contains the number unity; otherwise *incomplete.* A denumerable number module which when arranged as a sequence constitutes a Fourier sequence is called a *Fourier module.*

Obviously any countable arrangement of a Fourier module is a Fourier sequence.

THEOREM VIII. *Each a.p.* [*space function*; *space sequence*; *point*] *has one and only one* [— —; *complete*; *complete*] *Fourier module.*

For the [function; sequence; point] has a Fourier sequence $f_1, f_2, \cdots$. Let ϕ be the set of all numbers which are linearly dependent with integer coefficients on a finite number of the f_i [— —; and unity; and unity]. By Theorem VII, a sequence obtained by ordering ϕ is a Fourier sequence; hence ϕ is a Fourier module. Now if ϕ' is any [— —; complete; complete]

Fourier module, it is linearly dependent on ϕ [— —; and unity; and unity] and vice versa, and must therefore be ϕ.

5. Scalars

Definition. A finite or infinite sequence of real numbers will be called a *scalar*, and the number of elements in it (which may and usually will be the symbol ∞) will be called its *length*. The [*sum*; *product*] *of two scalars* of the same length or *product of one scalar by a number* is the scalar obtained by [adding; multiplying] corresponding elements or multiplying each element by the number. The *scalar identity* ι is the sequence $1, 1, 1, \cdots$; and the *scalar zero* (indicated by an ordinary zero) is the sequence $0, 0, 0, \cdots$; for both ι and 0, the length of the scalar will be indicated by the context. Scalars will be indicated by small Greek letters and their elements by corresponding italics, thus: α; $a_1, a_2, \cdots$.

Definition. The *absolute value* $|\alpha|$ *of an infinite scalar* α; $a_1, a_2, \cdots$ will be the greatest lower bound for all positive integers n and k of

$$\frac{1}{n}+\frac{1}{k}+\max_{0<j\leqq n}\ \min_{-\infty<i<+\infty}\ |a_j+k!i|.$$

The *absolute value* $|\alpha|$ *of a finite scalar* α; $a_1, \cdots, a_p$ will be the greatest lower bound for all positive integers k of

$$\frac{1}{k}+\max_{0<j\leqq p}\ \min_{-\infty<i<+\infty}\ |a_j+k!i|.$$

Using $|\alpha-\beta|$ as the distance between α and β, one can verify that the set of all scalars of the same given length is a metric space.

Definition. A *reduced upper Fourier sequence* of a [space function; space sequence; point in a $\mathfrak{T}$-space] is a sequence on a finite number of whose elements each element of an upper Fourier sequence is rationally linearly dependent. A *base* is a reduced upper Fourier sequence every finite subset of which is rationally linearly independent. A base is called *minimal* if each of its elements is rationally linearly dependent on a finite number of the elements of an incomplete Fourier module, or in case none exists, the complete Fourier module. A base for a space sequence or a point in a $\mathfrak{T}$-space is called *proper* either if it contains a rational element or if unity is not rationally linearly dependent on any finite subset of its elements.

It follows from Theorem IV, §4, that *the statements that a* [*continuous space function*; *space sequence*; *point in a* $\mathfrak{T}$-*space*] *has a reduced upper Fourier sequence, has a base, has a proper minimal base, or is a.p. are all equivalent.*

THEOREM I. *If s and t are variable real [numbers; integers], a necessary and sufficient condition that the scalar γ be a reduced upper Fourier sequence of the [continuous space function $\Theta(t)$; space sequence $\Theta_t = \Theta(t)$] is that uniformly in s*

$$\lim_{t\gamma\to 0} \Theta(s+t) = \Theta(s).$$

To prove sufficiency, let $f_1, f_2, \cdots$ be an upper Fourier sequence of Θ whose elements are rationally linearly dependent on $\gamma: c_1, c_2, \cdots$. By making $|t\gamma|$ sufficiently small, an arbitrarily large number of the quantities $tc_1, tc_2, \cdots$ can be brought arbitrarily close to multiples of an arbitrarily large $k!$. There exist integers p_i such that each $p_i f_i$ is a finite linear combination of the c_j with integer coefficients. Thus an arbitrarily large number of the quantities $tp_i f_i$ can be brought arbitrarily close to multiples of $k!$, and by choosing k large enough so that $k!$ contains all of the corresponding p_i as factors, arbitrarily many of the tf_i may be brought arbitrarily close to integers. Hence t will be an arbitrarily good translation number. To prove necessity we need only notice that the integer sub-multiples of the elements of γ when arranged in countable order form an upper Fourier sequence.

From the above theorem and Theorem II, §3, we obtain

THEOREM II. *If n is an integer, a necessary and sufficient condition that a scalar γ be a reduced upper Fourier sequence of the point Λ in a $\mathfrak{T}$-space having the identity point I is that*

$$\lim_{n\gamma\to 0} \Lambda^n = I.$$

6. Almost periodic properties invariant under multiplication

THEOREM I. *If $[\Phi(t);\ \Gamma_n;\ \Lambda]$ is an a.p. [function; sequence; point] in a $\mathfrak{T}$-space, then $[\Phi(t)\Theta;\ \Gamma_n\Theta;\ \Lambda^n\Theta]$ is a uniformly continuous function of the point Θ uniformly with respect to $[t; n; n]$.*

For any finite set of values of t or n, the theorem is obvious. Since $\Phi(t)$ is uniformly continuous, for an arbitrary $e>0$ we can divide any finite interval $\mathfrak{J}$ up into a sufficiently large number of equal intervals so that on any such interval $\Phi(t)$ varies by less than $e/3$. After choosing a point t_i from each interval, we can bring all the points $\Phi(t_i)\Theta'$ within a distance of $e/3$ from the corresponding points $\Phi(t_i)\Theta''$ by bringing Θ' sufficiently close to Θ''. Thus we can bring $\Phi(t)\Theta'$ within e of $\Phi(t)\Theta''$ for all t on $\mathfrak{J}$ by bringing Θ' sufficiently close to Θ''. Thus the theorem is true for functions, sequences or points on any finite interval. That it is also true for the infinite interval can be seen by choosing a length L corresponding to $e>0$ so great that on any interval of this length there is always an $(e/3)$-translation number or index or iteration exponent. Then $[\Phi(t);\ \Gamma_n;\ \Lambda^n]$ for any value of $[t;\ n;\ n]$ can be replaced by $[\Phi(t_0);\ \Gamma_{n_0};\ \Lambda^{n_0}]$ with an error not greater than $e/3$, where $[t_0; n_0; n_0]$ lies between 0 and L. But by bringing Θ' and Θ'' sufficiently close, $[\Phi(t_0)\Theta';\ \Gamma_{n_0}\Theta';\ \Lambda^{n_0}\Theta']$ can be brought within a distance of $e/3$ from $[\Phi(t_0)\Theta'';\ \Gamma_{n_0}\Theta'';\ \Lambda^{n_0}\Theta'']$ for all $[t_0;\ n_0;\ n_0]$ between zero and L at once. Thus $[\Phi(t)\Theta';\ \Gamma_n\Theta';\ \Lambda^n\Theta']$ would be at a distance not greater than e for all $[t; n; n]$ at once from $[\Phi(t)\Theta'';\ \Gamma_n\Theta'';\ \Lambda^n\Theta'']$.

THEOREM II. *The product of any two* [— —; — —; *permutable*] *a. p.* [*functions*; *sequences*; *points*] *in a* $\mathfrak{G}$*-space is a.p.*

For let s and t be real [numbers; integers; integers] and let $\Theta_1(t)$ and $\Theta_2(t)$ be the [a.p. functions; a.p. sequences; tth powers of the a.p. points] having the reduced upper Fourier sequences γ_1 and γ_2. Let γ^* be a sequence whose elements comprise all the elements of γ_1 and γ_2. Then both $t\gamma_1$ and $t\gamma_2$ can be brought arbitrarily close to zero by bringing $t\gamma^*$ sufficiently close to zero. Thus, uniformly in s,

$$\lim_{t\gamma^*\to 0} \Theta_1(s+t) = \Theta_1(s) \qquad \text{and} \qquad \lim_{t\gamma^*\to 0} \Theta_2(s+t) = \Theta_2(s).$$

Then since $\Theta_1(t)\Gamma$ is uniformly continuous in Γ uniformly with respect to t, it follows that uniformly in s

$$\lim_{t\gamma^*\to 0} \|\Theta_1(s+t)\Theta_2(s+t);\ \Theta_1(s+t)\Theta_2(s)\| = 0$$

and hence that uniformly in s,

$$\lim_{t\gamma^*\to 0} \|\Theta_1(s+t)\Theta_2(s+t);\ \Theta_1(s)\Theta_2(s)\| = 0.$$

Now we note that in case $\Theta_1(t)$ and $\Theta_2(t)$ represent the tth powers of permutable points, $\Theta_1(t)\Theta_2(t)$ represents the tth power of the product of the points; and hence in all three cases our theorem is proved.

COROLLARY. *A sequence whose elements comprise all the elements of* γ_1 *and* γ_2 *which are reduced upper Fourier sequences of two a.p.* [*functions*; *sequences*; *permutable points*] *in a* $\mathfrak{G}$*-space is a reduced upper Fourier sequence for the product of the* [*functions*; *sequences*; *points*].

DEFINITION. A sequence of points $\Lambda_1,\ \Lambda_2,\ \cdots$ will be said to *converge exponentially uniformly* if $\Lambda_1^n,\ \Lambda_2^n,\ \cdots$ converges uniformly with respect to n for all integers n.

THEOREM III. *If a sequence of a.p.* [*space functions*; *space sequences*; *points*] *converges* [— —; — —; *exponentially*] *uniformly, its limit is a.p.*

For, using the natural extension of the notation of the last theorem, it follows from the fact that $\lim_{n\to\infty}\Theta_n(t)$ is uniform in t that

$$\lim_{t\gamma^*\to 0}\lim_{n\to\infty}\Theta_n(s+t)=\lim_{n\to\infty}\lim_{t\gamma^*\to 0}\Theta_n(s+t)=\lim_{n\to\infty}\Theta_n(s),$$

where the limits with respect to t are uniform in s.

COROLLARY. *If $\gamma_1, \gamma_2, \cdots$ are reduced upper Fourier sequences for an [— —; — —; exponentially] uniformly convergent sequence of a.p. [space functions; space sequences; points] and γ^* is a sequence which has each of the γ_i as subsequences, then γ^* is a reduced upper Fourier sequence for the limit [function; sequence; point].*

THEOREM IV. *The [— —; — —; exponentially] uniform limit of an infinite product of [— —; — —; permutable] a.p. [space functions; space sequences; points] in a $\mathfrak{T}$-space is a.p.*

7. Pseudo-arguments, indices, and exponents

DEFINITION. If $[\Theta(t);\ \Gamma_n;\ \Lambda]$ is an a.p. [space function; space sequence; point] having the base γ; and if α is any scalar of the same length as γ, then the symbol $[\Theta(\alpha)_\gamma;\ (\Gamma_\alpha)_\gamma;\ \Lambda^\alpha_\gamma]$ will denote

$$[\lim_{t\gamma\to\alpha\gamma}\Theta(t);\ \lim_{n\gamma\to\alpha\gamma}\Gamma_n;\ \lim_{n\gamma\to\alpha\gamma}\Lambda^n]$$

and will be called the *pseudo- [value of the function; element of the sequence; power of the point] corresponding to the pseudo- [argument; index; exponent] α with respect to the base γ.* The base γ will be omitted from the notation when the context makes clear what base is to be used. If the points of the space are transformations, the same nomenclature will be used except that for a family of transformations the terms *pseudo-value* or *argument* will be replaced by *pseudo-member* or *parameter.* A pseudo- [element; power] of an a.p. [space sequence; point] with respect to a base γ will be called *proper* if γ is a proper base and no non-integer element of the pseudo- [index; exponent] corresponds to a rational element of γ.

THEOREM I. *All [— —; proper; proper] pseudo- [values; elements; powers] of an a.p. [space function; space sequence; point] exist.*

For if t is a variable real [number; integer; integer] and α is any scalar [argument; index; exponent] which satisfies the hypothesis, then $\alpha\gamma$ can be approached arbitrarily closely by the scalar $t\gamma$; and if $\Theta(t)$ is the [function; sequence; tth power of the point], then uniformly in t

$$\lim_{s\gamma,s'\gamma\to\alpha\gamma}\Theta(s-s'+t)=\Theta(t);$$

so that

$$\lim_{s\gamma,s'\gamma\to\alpha\gamma} \|\Theta(s);\Theta(s')\| = 0$$

and the theorem follows.

THEOREM II. *If t is a real [number; integer; integer] and ι is the identity scalar of the same length as the base γ of the a.p. [space function $\Theta(t)$; space sequence Γ_t; point Λ], then $[\Theta(t\iota)_\gamma;\ (\Gamma_{t\iota})_\gamma;\ \Lambda_\gamma^{t\iota}]$ is the same point as $[\Theta(t);\ \Gamma_t;\ \Lambda^t]$.*

THEOREM III. *The [— —; proper; proper] pseudo- [value $\Theta(\alpha)_\gamma$; element $(\Gamma_\alpha)_\gamma$; power Λ_γ^α] of the a.p. [space function $\Theta(t)$; space sequence Γ_n; point Λ] is a uniformly continuous function of the scalar $\alpha\gamma$ for all [— —; admissible; admissible] values of α.*

For in the case of the function having the base γ, to a given $e>0$ corresponds $d>0$ so small that for all t and t' satisfying $|(t-t')\gamma|\leqq d$,

$$\|\Theta(t);\Theta(t')\| \leqq e. \tag{1}$$

Now let α and α' be any scalar satisfying $|(\alpha-\alpha')\gamma|<d$. Then when $t\gamma$ and $t'\gamma$ are sufficiently close to $\alpha\gamma$ and $\alpha'\gamma$ respectively, the equation (1) is satisfied; and hence

$$\|\Theta(\alpha);\Theta(\alpha')\| = \lim_{t\gamma\to\alpha\gamma,\ t'\gamma\to\alpha'\gamma} \|\Theta(t);\Theta(t')\| \leqq e.$$

Similar arguments show that the theorem holds for sequences and points also.

THEOREM IV. *If Λ^α and Λ^β are proper pseudo-powers of the a.p. point Λ taken with respect to the same base, then*

$$\Lambda^\alpha\Lambda^\beta = \Lambda^{\alpha+\beta}.$$

For

$$\lim_{n\gamma\to(\alpha+\beta)\gamma} \Lambda^n = \lim_{m\gamma\to\alpha\gamma,n\gamma\to\beta\gamma} \Lambda^{m+n} = \lim_{m\gamma\to\alpha\gamma} \Lambda^m \lim_{n\gamma\to\beta\gamma} \Lambda^n.$$

COROLLARY 1. *Any two proper pseudo-powers of an a.p. point are permutable.*

COROLLARY 2. *If Λ^α is a proper pseudo-power of the a.p. point Λ, then $(\Lambda^\alpha)^n=\Lambda^{n\alpha}$.*

THEOREM V. *If γ is a base for the a.p. [space function $\Theta(t)$; space sequence Γ_n; point Λ] and [α and β are any scalars; Γ_α and Γ_β are proper pseudo-elements; Λ^α is a proper pseudo-power], then $[\Theta(t\alpha+\beta);\ \Gamma_{n\alpha+\beta};\ \Lambda^\alpha]$ is a.p. and has the reduced upper Fourier sequence $\alpha\gamma$.*

For in the case of the function, as $t\alpha\gamma\to 0$, $[(s+t)\alpha+\beta]\gamma\to[s\alpha+\beta]\gamma$ uniformly in s; and since $\Theta(\alpha)$ is uniformly continuous in $\alpha\gamma$ it follows that uniformly in s

$$\lim_{t\alpha\gamma\to 0}\Theta[(s+t)\alpha+\beta]=\Theta[s\alpha+\beta].$$

The proof is essentially the same for the functions and sequences.

8. Mono-basal functions, sequences and points

Notation. Let ω_p denote the scalar which has its pth element equal to unity and all other elements zero. Its length will be indicated by the context.

Definition. A [space function; space sequence; point] is called *mono-basal* if it is a.p. and has a base consisting of but one element.

Theorem I. *Any pure periodic [space function; space sequence; point] is mono-basal, having unity as a base.*

Theorem II. *If the a.p. [space function $\Theta(t)$; space sequence Γ_n; point Λ] has the [— —; proper; proper] base $\gamma: c_1, c_2, \cdots$ and [β is any scalar; $(\Gamma_\beta)_\gamma$ is any proper pseudo-element; — —], then $[\Theta(t\omega_p+\beta)_\gamma;\ (\Gamma_{n\omega_p+\beta})_\gamma;\ \Lambda_\gamma{}^{\omega_p}]$ is mono-basal and has c_p as a minimal base.*

For it has $\omega_p\gamma$ as a reduced upper Fourier sequence.

Definition. A space [function; multiple sequence] of any countable set of [variables; indices] will be called *mono-basal* in any one of its [variables t; indices n] if it is a mono-basal [function of t; sequence in n] for each set of constant values of the other [variables; indices]. The *diagonal [function; sequence]* of the [function $\Theta(t_1, t_2, \cdots)$; multiple sequence $\Gamma_{n_1,n_2}, \cdots$] is the [function $\Theta(t, t, \cdots)$; sequence $\Gamma_{n,n}, \cdots$].

Theorem III. *An a.p. space [function; sequence] can be expressed as the diagonal [function; sequence] on a mono-basal space [function; multiple sequence] of a countable set of variables.*

For let $\Theta(t)$ be the a.p. [function; sequence] of the real [number; integer] t. Then $\Theta(t)=\Theta(t\iota)=\Theta(t\omega_1+t\omega_2+\cdots)$ is the diagonal of $\Theta(t_1\omega_1+t_2\omega_2+\cdots)$, which is mono-basal in each of its arguments.

Theorem IV. *If Λ is an a.p. point, then with respect to any proper base the infinite product $\Lambda^{\omega_1}\Lambda^{\omega_2}\cdots$ or $\cdots\Lambda^{\omega_2}\Lambda^{\omega_1}$ converges absolutely and exponentially uniformly to the value Λ.*

For $n\omega_1+n\omega_2+\cdots$ converges uniformly in n to $n\iota$, and hence $\Lambda^{n\omega_1+n\omega_2+\cdots}$ converges uniformly in n to Λ^n. Moreover changing the order of the $\omega_1, \omega_2, \cdots$ would not destroy the convergence.

Because of its importance in this work, the following theorem will be stated in terms of both points and transformations.

THEOREM V. *A necessary and sufficient condition that a [transformation; point] be a.p. is that it be the exponentially uniformly convergent infinite product of permutable mono-basal [transformations; points].*

9. EXAMPLES

I will bring this paper to a close by giving two examples of a.p. transformations.

I. Let A_k be a two-way sequence of non-negative real numbers such that $\sum_{k=-\infty}^{\infty} A_k$ converges. Let the space $\mathfrak{C}$ have as its points the complex functions $f(x)$ of a real variable x having the Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx},$$

where the a_k are numbers satisfying $|a_k| \leqq A_k$. Let the distance between any two transformations $F(x) = \Theta_1[f(x)]$ and $F(x) = \Theta_2[f(x)]$ be

$$\max_{x,f} \left| \Theta_1[f(x)] - \Theta_2[f(x)] \right|.$$

Let c_k be any two-way sequence of complex numbers each having the absolute value 1, and let

$$g(y, t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k e^{ikv} t^{|k|}$$

be a function of the real variables y and t. *Then the transformation*

$$\Phi[f(x)] = F(x) = \lim_{t \to 1^-} \int_0^{2\pi} f(x - y) g(y, t) dy \tag{1}$$

is a.p.

For the set of transformations of the form (1) which is obtained by using all possible sets of values for the coefficients c_k of the function $g(y, t)$ is a $\mathfrak{T}$-space. Moreover it can be shown that

$$\Phi[f(x)] = \sum_{-\infty}^{+\infty} c_k a_k e^{ikx}$$

and hence that

$$\frac{\log c_0}{2\pi i}, \quad \frac{\log c_1}{2\pi i}, \quad \frac{\log c_{-1}}{2\pi i}, \quad \frac{\log c_2}{2\pi i}, \quad \frac{\log c_{-2}}{2\pi i}, \cdots$$

is a reduced upper Fourier sequence of Φ.

II. Let $c_1, c_2, \cdots$ be an infinite sequence of distinct complex numbers each having the absolute value 1, and let $\sum A_k$ be a convergent series of non-negative real numbers. Let $\mathfrak{C}$ have as its points all functions $f(z)$ of the complex variable z of the form

$$(1) \qquad f(z) = \sum_{k=1}^{\infty} a_k e^{c_k z},$$

where the a_k are any complex numbers satisfying $|a_k| \leqq A_k$.

Let the distance between two transformations

$$F(z) = \Theta_1[f(z)] \qquad \text{and} \qquad F(z) = \Theta_2[f(z)]$$

be the least upper bound for all functions $f(z)$ in $\mathfrak{C}$ of $\sum_{k=1}^{\infty} |a_k' - a''|$, where a_k' and a_k'' are the coefficients of the series of the form (1) for $\Theta_1[f(z)]$ and $\Theta_2[f(z)]$. *Then the transformation*

$$\Phi[f(z)] = F(z) = \frac{d}{dz} f(z)$$

is a.p.

For it can be shown that each function of $\mathfrak{C}$ has a unique representation of the form (1). Let us associate with each sequence of numbers $c_1', c_2', \cdots$, each of whose absolute values is unity the transformation which takes each function of the form (1) into the corresponding function

$$F(z) = \sum_{k=1}^{\infty} c_k' a_k e^{c_k z}.$$

This set of transformations is a $\mathfrak{T}$-space, and it contains the transformation Φ which corresponds to $c_1, c_2, \cdots$ and has the reduced upper Fourier sequence

$$\frac{\log c_1}{2\pi i}, \quad \frac{\log c_2}{2\pi i}, \cdots.$$

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