SOME INEQUALITIES FOR NON-UNIFORMLY BOUNDED ORTHO-NORMAL POLYNOMIALS*

BY

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1. Introduction. Let the set $\{\phi_n(x)\}$ be an ortho-normal set of functions on the interval (a, b) and let M be a constant such that

$$|\phi_n(x)| \leqq M \qquad (n = 0, 1, 2, \cdots; a \leqq x \leqq b);$$

then the Fourier expansion of any function $f(x)$ in terms of these functions is

$$f(x) \sim \sum_{n=0}^{\infty} c_n\phi_n(x), \text{ where } c_n = \int_a^b f(t)\phi_n(t)dt.$$

For sets of functions which satisfy the above assumptions the following two theorems of F. Riesz† are well known.

Theorem A. *Let the set $\{\phi_n(x)\}$ of ortho-normal functions defined on the interval (a, b) satisfy the condition*

$$|\phi_n(x)| \leqq M \qquad (n = 0, 1, 2, \cdots; a \leqq x \leqq b);$$

and let $f(x) \subset L_p$ $(1 < p \leqq 2)$. Then

$$\left(\sum_{n=0}^{\infty} |c_n|^{p'}\right)^{1/p'} \leqq M^{(2-p)/p}\left(\int_a^b |f(t)|^p dt\right)^{1/p},$$

where p' is determined by the relation $1/p + 1/p' = 1$.

Theorem B. *If the series $\sum |c_n|^p$ is convergent, then the constants c_n are the Fourier coefficients of a function $f(x) \subset L_{p'}$ $(p' \geqq 2)$, relative to a set of uniformly bounded ortho-normal functions; and moreover*

$$\left(\int_a^b |f(t)|^{p'} dt\right)^{1/p'} \leqq M^{(2-p)/p}\left(\sum_{n=0}^{\infty} |c_n|^p\right)^{1/p},$$

where p and p' satisfy the relation $1/p + 1/p' = 1$.

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† F. Riesz, *Über eine Verallgemeinerung der Parsevalschen Formel*, Mathematische Zeitschrift, vol. 18 (1923), pp. 117–124. For the case of trigonometric series see F. Hausdorff, *Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen*, Mathematische Zeitschrift, vol. 16 (1923), pp. 163–169. In this paper is also given a list of W. H. Young's papers on the subject.

As was called to my attention by Professors Hille and Tamarkin, in the case of the expansion of the function

$$f(x) = \left(\frac{2}{1-x}\right)^{\alpha} \tag{1}$$

in normalized Legendre polynomials, F. Riesz's theorems do not hold. Stieltjes† considered this function and showed that for the convergence of its Legendre series, besides assuming $-1<x<1$, it is necessary to take $\alpha<\frac{3}{4}$; from the asymptotic value of the coefficients this is easily seen, since

$$c_n = \frac{2^{1/2}(2n+1)^{1/2}\Gamma(\alpha+n)\Gamma(1-\alpha)}{\Gamma(\alpha)\Gamma(n-\alpha+2)} \sim \frac{2\Gamma(1-\alpha)}{\Gamma(\alpha)} n^{2\alpha-3/2}.$$

The function (1) belongs to L_p for every $p<1/\alpha$, whereas the series $\sum|c_n|^{p'}$ diverges whenever $\alpha \geqq \frac{3}{4}$; thus it is seen F. Riesz's first theorem does not apply to Legendre series.

The problem now is to modify the inequalities which appear in Theorems A and B so that the Legendre coefficients of a certain class of functions would satisfy a new inequality. In particular it is desirable to obtain an inequality which would take care of this function of Stieltjes. In the first part of the present paper this problem is solved not only for the case of normalized Legendre, Jacobi and Hermite polynomials but also for a general class of ortho-normal polynomials possessing certain properties.

The end of the paper contains theorems for our general class of ortho-normal polynomials, which were suggested by a publication of R. E. A. C. Paley‡ in which he extended some results of Hardy and Littlewood§ from Fourier series to the case of a set of uniformly bounded ortho-normal functions. The following theorem is typical of Paley's results.

THEOREM C. *Let $c_0, c_1, c_2, \cdots$ denote a bounded set of numbers such that $c_n \to 0$ as $n\to\infty$, and let*

$$c_0^* \geqq c_1^* \geqq c_2^* \geqq \cdots$$

denote the set $|c_0|, |c_1|, |c_2|, \cdots$ rearranged in descending order of magnitude. If the series $\sum c_n^{*p'} n^{p'-2}$ converges, where $p' \geqq 2$, and if the ortho-normal set $\{\theta_n(x)\}$ satisfies the condition

† *Correspondence d'Hermite et Stieltjes*, Paris, Gauthier-Villars, 1905, vol. 2, letter 249, p. 46.

‡ R. E. A. C. Paley, *Some theorems on orthogonal functions* (1), Studia Mathematica, vol. 3 (1931), pp. 226–238.

§ G. H. Hardy and J. E. Littlewood, *Some new properties of Fourier constants*, Mathematische Annalen, vol. 97 (1926), pp. 159–209. *Notes on the theory of series* (XIII): *Some new properties of Fourier constants*, Journal of the London Mathematical Society, vol. 6 (1931), pp. 3–9.

$$|\theta_n(x)| \leqq M \qquad (n = 0, 1, 2, \cdots; 0 \leqq x \leqq 1),$$

then the function $f(x) \sim \sum c_n \theta_n(x)$ *is of class* $L_{p'}$ *and*

$$\int_0^1 |f(t)|^{p'} dt \leqq A_{p'} \sum_{n=0}^{\infty} c_n^{*p'} (n+1)^{p'-2} \tag{2}$$

where $A_{p'}$ *depends only on* p' *and* M.

As in the case of Theorems A and B modifications must be made in the inequality (2) in order to arrive at the theorems for our general class of ortho-normal polynomials.

2. Lemmas of M. Riesz. In this section we state two theorems of M. Riesz† which will be of fundamental importance in the proofs of our theorems. For convenience of reference we shall designate them as Lemmas 1 and 2. First we define a certain class of functions; the function $f(x)$ will be said to belong to the class L_ϕ^a if the following Lebesgue-Stieltjes integrals of $f(x)$ with respect to the non-decreasing function $\phi(x)$, defined on the interval (a, b), exist and are finite:

$$\int_a^b f(t) d\phi(t), \quad \int_a^b |f(t)|^a d\phi(t) \qquad (a \geqq 1).$$

Similarly we can define the class of functions L_ψ^c $(c \geqq 1)$, corresponding to the non-decreasing function $\psi(x)$ defined on the interval (a', b').

LEMMA 1. *Let* $T = T(f)$ *be a linear limited functional transformation of certain classes* L_ϕ^a *into certain corresponding classes* L_ψ^c; *i.e.*,

(1) *the transformation is distributive, so that for arbitrary constants* λ_1, λ_2,

$$T(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 T(f_1) + \lambda_2 T(f_2);$$

(2) *there exists a constant* M^* *such that*

$$\left(\int |T(f)|^c d\psi \right)^{1/c} \leqq M^* \left(\int |f|^a d\phi \right)^{1/a}.$$

Denote by $M^*(\alpha, \gamma)$ *the least upper bound of the ratio*

$$\left(\int |T(f)|^c d\psi \right)^{1/c} \Big/ \left(\int |f|^a d\phi \right)^{1/a}$$

for every couple of exponents a *and* c, *where* $a\alpha = c\gamma = 1$. *If the relation between* a *and* c *is such that one always has* $c \geqq a$, *and if the point* (α, γ) *describes a straight line segment in the triangle* $0 \leqq \gamma \leqq \alpha \leqq 1$, *then* $\log M^*(\alpha, \gamma)$ *is a convex function of the points of the line segment.*

† M. Riesz, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires*, Acta Mathematica, vol. 49 (1926), pp. 465–497. In particular see Theorems V and VI.

LEMMA 2. *Every time that one has a linear limited functional transformation of $L_\phi^{a_1}$ into $L_\psi^{c_1}$ and of $L_\phi^{a_2}$ into $L_\psi^{c_2}$, with $c_1 \geqq a_1$, $c_2 \geqq a_2$, the transformation can be extended to every couple of exponents corresponding to the points (α, γ) of the line segment joining the points (α_1, γ_1) and (α_2, γ_2).*

3. Notation and definitions. Let

$$A_0(x), A_1(x), \cdots, A_n(x), \cdots$$

be polynomials which are of exactly the nth degree for each value of n; let $p(x)$ be a non-negative weight function, integrable and not identically equal to zero in the interval (a, b). This set of polynomials will be said to be orthogonal if

$$\int_a^b p(t)A_n(t)A_m(t)dt = 0 \qquad (n \neq m)$$

and normal if

$$\int_a^b p(t)A_n^2(t)dt = 1.$$

If the Fourier coefficients of a function $f(x)$ relative to these polynomials,

$$c_n = \int_a^b p(t)f(t)A_n(t)dt,$$

exist, the expansion of $f(x)$ in terms of these polynomials is

$$f(x) \sim \sum_{n=0}^{\infty} c_n A_n(x).$$

Let the function $\alpha(x)$ be absolutely continuous and such that

$$\alpha'(x) = \beta(x) \geqq 0;$$

in addition let $\beta(x) > 0$ except for a set of measure zero. Set

$$W(x) = [p(x)/\beta(x)]^{1/2} \geqq 0; \tag{3}$$

for convenience we shall write

$$J_p(f) \equiv J_p = \left(\int_a^b |W(t)f(t)|^p d\alpha(t)\right)^{1/p}, \; S_{p'}(f) \equiv S_{p'} = \left(\sum_{n=0}^{\infty} |c_n|^{p'}\right)^{1/p'}.$$

If $J_p^p(f)$ exists, we shall write $W(x)f(x) \subset L_\alpha^p$.

Throughout the paper we shall understand by p and p' two numbers which satisfy the relations $1 \leqq p \leqq 2$, $p' \geqq 2$, $1/p + 1/p' = 1$; hence when $p = 1$, the corresponding value of p' is ∞. Furthermore, A will be used in the generic sense to denote a constant independent of n and x.

We postulate the following properties of the set of polynomials $\{A_n(x)\}$:

(1) the $A_n(x)$ are ortho-normal in the above sense;

(2) $|W(x)A_n(x)| \leqq A$, for all $(n=0, 1, 2, \cdots)$ and $a \leqq x \leqq b$.

Property (2) is also a condition for the function $\beta(x)$ since it appears in $W(x)$.

It is interesting to see how the function $W(x)$ is introduced. Bessel's inequality for the polynomials $A_n(x)$ suggests putting

$$J_2^2 = \int_a^b p(t)\,|\,f(t)\,|^2 dt;$$

on the other hand, in order to use M. Riesz's lemmas we must have

$$J_2^2 = \int_a^b |\,W(t)f(t)\,|^2 d\alpha(t) = \int_a^b |\,W(t)f(t)\,|^2 \beta(t) dt.$$

Comparison of these two expressions for J_2^2 leads to setting

$$|\,W(t)\,|^2 \equiv p(t)/\beta(t).$$

4. **Generalizations of F. Riesz's theorems.** Having agreed upon the above notation and properties of our ortho-normal polynomials, we can prove the following theorems.

THEOREM I. *If*

$$W(x)f(x) \subset L_\alpha^p, \tag{4}$$

then

$$S_{p'} \leqq A^{(2-p)/p} J_p,$$

where A is a constant.

Set

$$F(x) = W(x)f(x), \tag{5}$$

and let $\phi(x) = \alpha(x)$, $\psi(x) = [x]$, where the symbol $[x]$ denotes the greatest integer in x. By definition the linear transformation T is

$$T(F) = c_n = \int_a^b p(t)f(t)A_n(t)dt$$

for all integral values n of x, and of arbitrary value for non-integral values of x.

In terms of this notation and the notation of Lemmas 1 and 2 what we wish to prove is that

$$M^*\left(\frac{1}{p}, \frac{1}{p'}\right) = \sup\,(S_{p'}/J_p) \leqq A^{(2-p)/p}. \tag{6}$$

To prove the theorem it will suffice to show that $M^*(\frac{1}{2}, \frac{1}{2})$ and $M^*(1, 0)$ are bounded; then to interpolate for other values of $1/p$ and $1/p'$ on the line segment joining the two points $(\frac{1}{2}, \frac{1}{2})$ and $(1, 0)$ by Lemma 2. The desired inequality will result from Lemma 1.

Now $M^*(\frac{1}{2}, \frac{1}{2}) \leqq 1$, by Bessel's inequality for our ortho-normal polynomials. For $M^*(1, 0)$ we write†

$$M^*(1, 0) = \sup \frac{\max\limits_{0 \leqq n < \infty} |c_n|}{J_1}.$$

By Property (2) we have‡

$$|c_n| \leqq \int_a^b p(t)\,|f(t)|\,|A_n(t)|\,dt \leqq A\int_a^b p^{1/2}(t)\beta^{1/2}(t)\,|f(t)|\,dt = AJ_1;$$

hence we have $M^*(1, 0) \leqq A$.

The line segment with end points $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ has the equation $\gamma = 1 - \alpha$, and lies in the triangle $0 \leqq \gamma \leqq \alpha \leqq 1$. Consequently Lemma 2 applies and whenever $F(x) \subset L_\alpha{}^p$, the series $\sum |c_n|^{p'}$ converges. The desired inequality (6) results from the convexity of $\log M^*(1/p, 1/p')$, assured by Lemma 1. Indeed, one has§

$$\begin{aligned} M^*\left(\frac{1}{p}, \frac{1}{p'}\right) &\leqq [M^*(1, 0)]^{(1/p-1/2)/(1-1/2)} \left[M^*\left(\frac{1}{2}, \frac{1}{2}\right)\right]^{(1-1/p)/(1-1/2)} \\ &\leqq A^{(2-p)/p}. \end{aligned} \tag{7}$$

Theorem II. *If the series $\sum |c_n|^p$ converges, then the numbers c_n are the Fourier coefficients of a function $f(x)$ such that*

$$W(x)f(x) \subset L_\alpha^{p'},$$

and

$$J_{p'} \leqq A^{(2-p)/p} S_p,$$

where A is a constant.

† The usual convention is made here. When $p' = \infty$, in order to compute $M^*(1, 0)$, the numerator is replaced by max $|c_n|$ over all values $(n=0, 1, 2, \cdots)$. In the case of an integral appearing in the numerator, it is replaced by the upper measurable bound (in the sense of Lebesgue) of the integrand, which we shall designate simply as the maximum. Cf. M. Riesz, loc. cit., footnote 2, p. 477.

‡ The author is indebted to the referee for a simplification of the argument at this point.

§ For the origin of this inequality see M. Riesz, loc. cit., p. 484.

The notation of Lemmas 1 and 2 becomes $\psi(x)=\alpha(x)$, $\phi(x)=[x]$. In the case of Theorem I it was $f(x)$ which was varied but now the c_n are the quantities varied. By definition the transformation T will be such a transformation on the space of elements $c=(c_0, c_1, c_2, \cdots, c_n, \cdots)$, $\sum |c_n|^p<\infty$, which associates with c the Fourier expansion of the function $[W(x)]^{-1}F(x)$ which has the components of c for Fourier coefficients. We set

$$T(c) = F(x) \sim W(x)\sum_{n=0}^{\infty} c_n A_n(x);$$

then

$$M^*\left(\frac{1}{p}, \frac{1}{p'}\right) = \sup\ (J_{p'}/S_p).$$

Now $M^*(\frac{1}{2}, \frac{1}{2}) \leqq 1$, by the Riesz-Fischer theorem. For $M^*(1, 0)$ we must write

$$M^*(1, 0) = \sup \frac{\max\limits_{a\leqq x\leqq b} |F(x)|}{S_1},$$

but by Property (2) the numerator is bounded by AS_1; hence $M^*(1, 0)\leqq A$. Using Lemmas 1 and 2 and the inequality (7), the statement of the theorem follows.

5. Jacobi polynomials. In order to show that Theorems I and II hold for normalized Jacobi polynomials we have only to prove that they possess Properties (1) and (2).

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x), \alpha>-1, \beta>-1$, with the weight function $p(x)\equiv(1-x)^\alpha(1+x)^\beta$ and $a=-1$ and $b=1$, are orthogonal in the sense of Property (1). In fact†

$$\int_{-1}^{1} (1-t)^\alpha(1+t)^\beta P_n^{(\alpha,\beta)}(t)P_m^{(\alpha,\beta)}(t)dt = \begin{cases} 0, & n\neq m; \\ \dfrac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\, \dfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, & n=m; \end{cases}$$

then the set of polynomials

$$Q_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{(k_n(\alpha,\beta))^{1/2}} \qquad (n=0, 1, 2, \cdots),$$

† G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, Julius Springer, 1925, vol. II, pp. 93, 292.

where

$$k_n(\alpha, \beta) = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

is an ortho-normal set of polynomials.

The function $\alpha(x) = \arcsin x$; this $\alpha(x)$ is obviously absolutely continuous in the interval $(-1, 1)$, and the function $\beta(x) = (1-x^2)^{-1/2}$ is positive throughout this interval. The function $W(x)$ takes the form

$$W(x) = (1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4}.$$

It can be easily verified by applying Stirling's formula for $\Gamma(x)$ to the normalizing factor $[k_n(\alpha, \beta)]^{-1/2}$ that

$$Q_n^{(\alpha,\beta)}(x) = O(n^{1/2})P_n^{(\alpha,\beta)}(x). \tag{8}$$

Suppose $\alpha \geqq -\frac{1}{2}$, $\beta \geqq -\frac{1}{2}$; then the following result of S. Bernstein* is just a statement of the fact that the $Q_n^{(\alpha,\beta)}(x)$ possess Property (2).

LEMMA 3. *Suppose that*

$$\max_{-1 \leqq x \leqq 1} (1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4} \left| P_n^{(\alpha,\beta)}(x) \right| = M_n(\alpha, \beta).$$

Then

$$\lim_{n\to\infty} n^{1/2} M_n(\alpha, \beta) = 2^{(\alpha+\beta)/2} M(\alpha, \beta)$$

exists and

$$M(\alpha, \beta) = \begin{cases} \left(\dfrac{2}{\pi}\right)^{1/2}, \text{ if } -\dfrac{1}{2} \leqq \alpha \leqq \dfrac{1}{2}, \quad -\dfrac{1}{2} \leqq \beta \leqq \dfrac{1}{2}; \\ \text{finite and } > \left(\dfrac{2}{\pi}\right)^{1/2}, \text{ if } \alpha > \dfrac{1}{2}, \ \beta \geqq -\dfrac{1}{2} \\ \qquad\qquad \text{or if } \alpha \geqq -\dfrac{1}{2}, \ \beta > \dfrac{1}{2}; \\ +\infty, \text{ if } \alpha \text{ or } \beta < -\dfrac{1}{2}. \end{cases}$$

THEOREM I$'$. *The result of Theorem* I *is valid in the case of normalized Jacobi polynomials when* $\alpha > -1$, $\beta > -1$, *provided that* $f(x)$ *satisfies the further condition* $(1-x)^\alpha(1+x)^\beta f(x) \subset L$.

* S. Bernstein, *Sur les polynomes orthogonaux relatifs à un segment fini*, Journal de Mathématiques, (9), vol. 9 (1930), pp. 127–177; vol. 10 (1931), pp. 219–286; see pp. 225–232. These results are proved in a very simple way by G. Szegö, *Asymptotische Entwicklungen der Jacobischen Polynome*, Schriften der Königsberger Gelehrten Gesellschaft, vol. 10 (1933), pp. 35–110, p. 79.

The analysis is the same as that already given except that in showing that $|c_n|$ is bounded independently of n a discussion of the case $-1<\alpha<-\frac{1}{2}$, $-1<\beta<-\frac{1}{2}$ must be given. For this purpose we shall use the following bound for $P_n^{(\alpha,\beta)}(x)$ found by Szegö*,

$$|P_n^{(\alpha,\beta)}(x)| \leqq An^{\max(\alpha,-1/2)} \qquad (-1+\epsilon \leqq x \leqq 1), \tag{9}$$

the classical inequality for $P_n^{(\alpha,\beta)}(x)$ due to Darboux†,

$$|P_n^{(\alpha,\beta)}(\cos\theta)| \leqq An^{-1/2}(\sin\theta)^{-1/2}\left(\sin\frac{\theta}{2}\right)^{-\alpha}\left(\cos\frac{\theta}{2}\right)^{-\beta} \tag{10}$$
$$\left(\epsilon \leqq \theta \leqq \pi-\epsilon, 0<\epsilon<\frac{\pi}{2}; n=1,2,\cdots\right);$$

and the well known relation,

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \tag{11}$$

Making use of (8), we write

$$\begin{aligned}
|c_n| &\leqq O(n^{1/2})\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}|f(t)||P_n^{(\alpha,\beta)}(t)|\,dt \\
&= O(n^{1/2})\left\{\int_{-1}^{-1+\delta}+\int_{-1+\delta}^{1-\delta}+\int_{1-\delta}^{1}(1-t)^{\alpha}(1+t)^{\beta}|f(t)||P_n^{(\alpha,\beta)}(t)|\,dt\right\} \\
&= I_1+I_2+I_3,
\end{aligned}$$

where $0<\delta<1$. We have by (9) that

$$I_3 = O(1)\int_{1-\delta}^{1}(1-t)^{\alpha}(1+t)^{\beta}|f(t)|\,dt; \tag{12}$$

using (11) and then (9), we can write

$$I_1 = O(1)\int_{-1}^{-1+\delta}(1-t)^{\alpha}(1+t)^{\beta}|f(t)|\,dt. \tag{13}$$

By the added condition on $f(x)$ the integrals (12) and (13) tend to zero as $\delta\to 0$. There remains to be considered I_2; in estimating it we make use of (10),

$$\begin{aligned}
I_2 &= O(1)\int_{-1+\delta}^{1-\delta}(1-t)^{\alpha/2-1/4}(1+t)^{\beta/2-1/4}|f(t)|\,dt \\
&\leqq A\int_{-1}^{1}(1-t)^{\alpha/2-1/4}(1+t)^{\beta/2-1/4}|f(t)|\,dt;
\end{aligned}$$

* G. Szegö, loc. cit., p. 77.

† G. Darboux, *Sur l'approximation des fonctions de très grands nombres et sur une classe étendue de développements en série*, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5–56, 377–416; p. 50.

hence

$$|c_n| \leqq I_1 + I_2 + I_3 \leqq A \int_{-1}^{1} (1-t)^{\alpha/2-1/4}(1+t)^{\beta/2-1/4}|f(t)|\,dt = AJ_1.$$

This completes the proof that $|c_n|$ is bounded independently of n when $-1<\alpha<-\frac{1}{2}$, $-1<\beta<-\frac{1}{2}$; the proof in the case that $\alpha \geqq -\frac{1}{2}$, $\beta \geqq -\frac{1}{2}$ follows as before.

Our proof depends fundamentally on the applicability of Lemmas 1 and 2. The space of functions satisfying the conditions of Theorem I′ is a sub-space of the space of functions satisfying the conditions of Theorem I. Hence what we have assumed is that M. Riesz's theorems hold in every sub-space of their space of definition. That this is true is apparent from the way in which M. Riesz derives his results.

That a similar extension of Theorem II is possible is not at all obvious. The lemma of S. Bernstein would seem to preclude that.

6. **Legendre polynomials.** The normalized Legendre polynomials defined on the interval $(-1, 1)$,

$$\left\{\left(\frac{2n+1}{2}\right)^{1/2} P_n(x)\right\}, \quad \left(\frac{2n+1}{2}\right)^{1/2}\left(\frac{2m+1}{2}\right)^{1/2} \int_{-1}^{1} P_n(t)P_m(t)dt = \delta_{nm},$$

automatically possess Properties (1) and (2) since they correspond to the values $\alpha=\beta=0$ of the parameters in Jacobi polynomials. The function $\alpha(x)$ has the same definition.

It can be shown that the Theorems I and II for Legendre polynomials sift out the correct intervals of convergence and divergence of the sum and integral involved in the inequalities for the Stieltjes function (1).

7. **Hermite polynomials.** The set of normalized Hermite polynomials,

$$\left\{\frac{1}{(2^n n!\pi^{1/2})^{1/2}} H_n(x)\right\},$$

with $p(x) \equiv e^{-x^2}$, $a=-\infty$, $b=\infty$, possesses Property (1); indeed*

$$\int_{-\infty}^{\infty} e^{-t^2} H_n(t)H_m(t)dt = \begin{cases} 0, & n \neq m, \\ 2^n n!\pi^{1/2}, & n = m. \end{cases}$$

If $\alpha(x)=x$, it is easily seen that it is absolutely continuous and that $\beta(x)>0$; by use of the inequality†

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* E. Hille, *A class of reciprocal functions*, Annals of Mathematics, (2), vol. 27 (1925–26), pp. 427–464; pp. 431, 436.

† E. Hille, loc. cit., p. 435.

$$| H_n(x) | \leqq A 2^{n/2}(n!)^{1/2} e^{x^2/2},$$

it is easy to see that normalized Hermite polynomials possess Property (2).

8. **Generalizations of Paley's theorems.** Let the set of polynomials $\{A_n(x)\}$ and the function $\alpha(x)$ have Properties (1) and (2).

Throughout we shall denote by $c_0, c_1, c_2, \cdots$ a bounded set of numbers such that $c_n \to 0$ as $n \to \infty$, and by

$$c_0^* \geqq c_1^* \geqq c_2^* \geqq \cdots$$

the set $|c_0|, |c_1|, |c_2|, \cdots$ rearranged in descending order of magnitude.

THEOREM III. *If the series* $\sum c_n^{*p'} n^{p'-2}$ *converges where* $p' \geqq 2$, *then*

$$W(x)f(x) \sim W(x) \sum_{n=0}^{\infty} c_n A_n(x) \tag{14}$$

is of class $L_\alpha^{p'}$, *and*

$$(J_{p'})^{p'} \leqq A \sum_{n=0}^{\infty} c_n^{*p'} (n+1)^{p'-2},$$

where A *is a constant.*

It is observed that the series in the right member of (14) converges in the mean of order 2 and hence represents some function of the Lebesgue class L_2; for

$$\sum_{n=0}^{\infty} c_n^{*2} \leqq \left(\sum_{n=0}^{\infty} c_n^{*p'} (n+1)^{p'-2} \right)^{2/p'} \left(\sum_{n=0}^{\infty} (n+1)^{-2} \right)^{(p'-2)/p'} < \infty .$$

Consider the inequality

$$\int_a^b | W(t)f(t) |^{p'} d\alpha(t) \leqq A \sum_{n=0}^{\infty} | c_n |^{p'} (n+1)^{p'-2}; \tag{15}$$

for the case $p'=2$ it is well known. If Lemmas 1 and 2 applied, it would be sufficient to prove the theorem for positive even values of p'. Let us show first that these lemmas do apply.

Let the linear transformation T be by definition

$$T\{(n+1)c_n\} = W(x)f(x) \sim W(x) \sum_{n=0}^{\infty} (n+1) c_n \left\{ \frac{A_n(x)}{n+1} \right\};$$

let $\psi(x) = \alpha(x)$, and

$$\phi(x) = 1 + \frac{1}{2^2} + \cdots + \frac{1}{([x]+1)^2} \quad (n \leqq x < n+1; n = 0, 1, 2, \cdots).$$

For the bound $M^*(1/p', 1/p')$ we have

$$M^*\left(\frac{1}{p'}, \frac{1}{p'}\right) = \sup \frac{\left(\int_a^b | W(t)f(t) |^{p'} d\alpha(t)\right)^{1/p'}}{\left(\sum_{n=0}^{\infty} | c_n(n+1) |^{p'}(n+1)^{-2}\right)^{1/p'}}.$$

Therefore Lemmas 1 and 2 will apply, and it suffices to prove the theorem when p' is an even integer.

To fix the ideas take $p'=4$; the proof for other even integral values of p' is similar. In this case (15) becomes

$$\int_a^b | W(t)f(t) |^4 d\alpha(t) \leqq A \sum_{n=0}^{\infty} | c_n |^4 (n+1)^2.$$

Consider the sequence of functions,

$$f_0(x) = W(x)c_0A_0(x), \quad f_1(x) = W(x)c_1A_1(x),$$

$$f_m(x) = W(x) \sum_{n=2^{m-1}}^{2^m-1} c_nA_n(x), \quad m \geqq 2;$$

and let

$$\epsilon_0 = c_0^4, \quad \epsilon_1 = c_1^4 2^2, \quad \epsilon_m = \sum_{n=2^{m-1}}^{2^m-1} c_n^4(n+1)^2, \quad m \geqq 2.$$

Then, if μ, ν are any two integers such that $0 < \mu \leqq \nu$,

$$\begin{aligned}
&\int_a^b f_\mu^2(t) f_\nu^2(t) d\alpha(t) \\
&\leqq \int_a^b W^2(t)\left[\sum_{n=2^{\nu-1}}^{2^\nu-1} c_nA_n(t)\right]^2 d\alpha(t) \cdot \max_{a\leqq t\leqq b}\left[W(t)\sum_{n=2^{\mu-1}}^{2^\mu-1} c_nA_n(t)\right]^2 \\
&\leqq \left[\sum_{n=2^{\nu-1}}^{2^\nu-1} c_n^2\right]\left[A\sum_{n=2^{\mu-1}}^{2^\mu-1} | c_n |\right]^2 \leqq A\left[\sum_{n=2^{\nu-1}}^{2^\nu-1} c_n^4(n+1)^2\right]^{1/2}\left[\sum_{n=2^{\nu-1}}^{2^\nu-1}(n+1)^{-2}\right]^{1/2} \\
&\qquad \times\left[\sum_{n=2^{\mu-1}}^{2^\mu-1} c_n^4(n+1)^2\right]^{1/2}\left[\sum_{n=2^{\mu-1}}^{2^\mu-1}(n+1)^{-2/3}\right]^{3/2} \\
&\leqq A\epsilon_\nu^{1/2}\epsilon_\mu^{1/2}2^{(\nu-\mu)/2} \leqq A(\epsilon_\nu+\epsilon_\mu)2^{-|\mu-\nu|/2},
\end{aligned}$$

where use has been made of Hölder's inequality and Properties (1) and (2). Since this result is symmetric in μ and ν, it holds also if $\mu > \nu$.

It follows from the above equation that if m_1, m_2, m_3, m_4 are arbitrary positive integers, then

$$\begin{aligned}\int_a^b & \left| f_{m_1}(t) f_{m_2}(t) f_{m_3}(t) f_{m_4}(t) \right| d\alpha(t) \\ & \leqq \left(\int_a^b f_{m_1}^2 f_{m_2}^2 d\alpha(t) \right)^{1/6} \left(\int_a^b f_{m_1}^2 f_{m_3}^2 d\alpha(t) \right)^{1/6} \\ & \quad \times \left(\int_a^b f_{m_1}^2 f_{m_4}^2 d\alpha(t) \right)^{1/6} \left(\int_a^b f_{m_2}^2 f_{m_3}^2 d\alpha(t) \right)^{1/6} \\ & \quad \times \left(\int_a^b f_{m_2}^2 f_{m_4}^2 d\alpha(t) \right)^{1/6} \left(\int_a^b f_{m_3}^2 f_{m_4}^2 d\alpha(t) \right)^{1/6} \\ & \leqq A(\epsilon_{m_1} + \epsilon_{m_2} + \epsilon_{m_3} + \epsilon_{m_4}) \\ & \qquad \times 2^{-(1/12)(|m_1-m_2|+|m_1-m_3|+|m_1-m_4|+|m_2-m_3|+|m_2-m_4|+|m_3-m_4|)}. \end{aligned}$$

Using this result we obtain,

$$\begin{aligned} & \int_a^b (|f_1| + |f_2| + \cdots + |f_m| + \cdots)^4 d\alpha(t) \\ & \leqq 6 \sum \int_a^b \left| f_{m_1}(t) f_{m_2}(t) f_{m_3}(t) f_{m_4}(t) \right| d\alpha(t) \\ & \leqq A \sum (\epsilon_{m_1} + \epsilon_{m_2} + \epsilon_{m_3} + \epsilon_{m_4}) 2^{-(1/12)(|m_1-m_2|+|m_1-m_3|+|m_1-m_4|+|m_2-m_3|+|m_2-m_4|+|m_3-m_4|)}. \end{aligned}$$

In the summation over m_4 the coefficient of an ϵ_m in the above sum is

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} 2^{-(1/12)(|m-m_1|+|m-m_2|+|m-m_3|+|m_1-m_2|+|m_1-m_3|+|m_2-m_3|)} \leqq A;$$

it follows that

$$\int_a^b (|f_1| + |f_2| + \cdots + |f_m| + \cdots)^4 d\alpha(t) \leqq A \sum_{m=1}^{\infty} \epsilon_m.$$

Now,

$$\int_a^b f_0^4(t) d\alpha(t) = c_0^4 \int_a^b [W(t) A_0(t)]^4 d\alpha(t) \leqq A c_0^4 \int_a^b d\alpha(t) \leqq A c_0^4 = A\epsilon_0,$$

by Property (2); consequently

$$\int_a^b \left[\sum_{m=0}^{\infty} |f_m(t)| \right]^4 d\alpha(t) \leqq A \sum_{m=0}^{\infty} \epsilon_m = A \sum_{n=0}^{\infty} |c_n|^4 (n+1)^2. \tag{16}$$

From this we infer that the series

$$\sum_{m=0}^{\infty} f_m(x) = W(x) \sum_{n=0}^{\infty} c_n A_n(x)$$

converges almost everywhere, but the series $\sum c_n A_n(x)$ converges in the mean of order 2 to $f(x)$ as was remarked at the beginning of the proof. Since these two limits must be the same, we have finally

$$\int_a^b |W(t)f(t)|^4 d\alpha(t) \leqq A \sum_{n=0}^{\infty} |c_n|^4 (n+1)^2.$$

It will be noticed that the inequality stated in the theorem was not proved but the inequality (15) was proved instead. The only point of the proof which depends on n is the use of Property (2); furthermore this estimate does not depend at all on the order in which it is made; hence we can assume the c_n are already rearranged in decreasing order of magnitude.

THEOREM IV. *If*

$$W(x)f(x) \sim W(x) \sum_{n=0}^{\infty} c_n A_n(x) \subset L_\alpha^p,$$

where $1 < p \leqq 2$, *then*

$$\sum_{n=0}^{\infty} c_n^{*p} (n+1)^{p-2} \leqq A J_p^p,$$

where A is a constant.

In view of our last remark above it suffices to prove

$$\sum_{n=0}^{\infty} |c_n|^p (n+1)^{p-2} \leqq A J_p^p.$$

Let us write

$$d_n = |c_n|^{p-1} (n+1)^{p-2} \overline{\operatorname{sgn} c_n};$$

then

$$c_n = |d_n|^{p'-1} (n+1)^{p'-2} \overline{\operatorname{sgn} d_n};$$

where $p' \geqq 2$, $1/p + 1/p' = 1$. Now

$$\sum_{n=0}^{N} |c_n|^p (n+1)^{p-2} = \sum_{n=0}^{N} c_n d_n = \sum_{n=0}^{N} |d_n|^{p'} (n+1)^{p'-2};$$

let

$$g_N(x) = \sum_{n=0}^{N} d_n A_n(x).$$

Using Hölder's inequality and Theorem III, we have

$$\sum_{n=0}^{N} |c_n|^p(n+1)^{p-2} = \sum_{n=0}^{N} c_n d_n = \int_a^b p(t)f(t)g_N(t)dt$$

$$\leqq J_p(f)J_{p'}(g_N) \leqq A\,J_p(f)\left(\sum_{n=0}^{N} |d_n|^{p'}(n+1)^{p'-2}\right)^{1/p'}$$

$$= A\,J_p(f)\left(\sum_{n=0}^{N} |c_n|^p(n+1)^{p-2}\right)^{1/p'};$$

therefore

$$\sum_{n=0}^{N} |c_n|^p(n+1)^{p-2} \leqq A\,J_p^p(f).$$

Since A is independent of N, the theorem follows by making N tend to infinity.

The form of the expression (16) suggests the stronger inequality of the following theorem, the details of whose proof are analogous to those of the proof of Paley's† Theorem III. The only change is the introduction of the integrator function $\alpha(x)$ and the factor $W(x)$.

THEOREM V. *Let $S(x)$ denote the upper bound*

$$S(x) = \sup_{0\leqq m\leqq\infty}\left|\sum_{n=0}^{m} c_n A_n(x)\right|.$$

Then, if $p'>2$,

$$\int_a^b [W(t)S(t)]^{p'}d\alpha(t) \leqq A\sum_{n=0}^{\infty} c_n^{*p'}(n+1)^{p'-2},$$

where A is a constant.

† R. E. A. C. Paley, loc. cit., pp. 232–238.

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