ON BOUNDED LINEAR FUNCTIONAL OPERATIONS*

BY

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The set of all bounded linear functional operations on a given vector space† plays an important role in the consideration of linear functional operations. For the sake of greater definiteness it is desirable to know the form of such operations and a space determined thereby. This problem has been solved for a number of spaces, for instance, all continuous functions on a bounded closed interval, all Lebesgue pth power ($p \geqq 1$) integrable functions, all sequences whose pth powers ($p \geqq 1$) form absolutely convergent series, all sequences having a limit, and so on.‡ All of these spaces have the property of separability. For non-separable spaces, there is a recent determination of the operation for the space of all bounded functions on a finite interval, having at most discontinuities of the first kind, by H. S. Kaltenborn.§

In this paper we give a determination of the linear operation for the space of (a) all bounded sequences, (b) all bounded measurable functions, (c) all bounded functions on the infinite interval having at most discontinuities of the first kind, (d) all bounded continuous functions on the infinite interval, (e) all almost bounded functions. With the least upper bound as norm, all of these spaces are not separable.

1. **Notations. The integral.** We shall denote by

(a) $\mathfrak{P}$ a set of elements p.

(b) ξ a real-valued function on $\mathfrak{P}$.

(c) $\mathfrak{X}$ a set of functions ξ.

(d) $\mathfrak{E}$ a set or class of subsets E of $\mathfrak{P}$, containing the null set and the set $\mathfrak{P}$.

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* Presented to the Society, April 7, 1934; received by the editors April 17, 1934.

† By a *linear vector space* $\mathfrak{X}$, we shall mean a so-called Banach space (see Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 55) of elements ξ in which there is defined addition, and multiplication by constants, subject to the usual laws of algebra, a unique zero, and a distance function or norm $\|\xi\|$ subject to the condition $\|c_1\xi_1+c_2\xi_2\| \leqq |c_1| \cdot \|\xi_1\| + |c_2| \cdot \|\xi_2\|$ for all c_1 and c_2. A *linear operation* L on $\mathfrak{X}$ transforms $\mathfrak{X}$ into real numbers and satisfies the condition $L(c_1\xi_1+c_2\xi_2)=c_1L(\xi_1)+c_2L(\xi_2)$. L is *bounded* and therefore *continuous* if there exists an M such that, for all ξ, $|L(\xi)| \leqq M\|\xi\|$. The smallest possible value for M is the *modulus* or norm M_L of L. We shall limit ourselves to real-valued linear operations since a complex-valued operation is expressible as the sum of two real-valued ones.

‡ See Banach, *Opérations Linéaires*, pp. 59–72; Hildebrandt, *Linear functional transformations in general spaces*, Bulletin of the American Mathematical Society, vol. 37 (1931), p. 189.

§ See Bulletin of the American Mathematical Society, vol. 40 (1934).

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It will be assumed that $\mathfrak{E}$ is additive and multiplicative, i.e., if E_1 and E_2 belong to $\mathfrak{E}$ so do E_1+E_2 and E_1E_2.

(e) Π a finite partition or subdivision of $\mathfrak{P}$ into mutually exclusive sets $E_1, \cdots, E_n$ belonging to $\mathfrak{E}$. $\Pi_1 \geqq \Pi_2$ shall mean that every set $E^{(1)}$ of Π_1 is a subset of some set $E^{(2)}$ of Π_2.

Because of the multiplicative property of $\mathfrak{E}$ the partitions Π satisfy the conditions of a range on which the general limit of E. H. Moore-H. L. Smith* is definable, i.e., if β_Π is any function defined for all partitions Π of $\mathfrak{P}$, then $\lim \beta_\Pi = b$ has the following meaning: for every $e>0$ there exists a Π_e, such that if $\Pi \geqq \Pi_e$ then $|\beta_\Pi - b| \leqq e$.

(f) $\alpha(E)$ a function on $\mathfrak{E}$. $\alpha(E)$ is *additive* if $\alpha(E_1+E_2)+\alpha(E_1E_2)=\alpha(E_1)+\alpha(E_2)$, for every E_1 and E_2 of $\mathfrak{E}$. $\alpha(E)$ is of bounded variation on $\mathfrak{P}$ if $\sum_\Pi |\alpha(E_i)|$ is bounded for all Π of $\mathfrak{P}$, and the least upper bound of this sum, which agrees with the limit in the Π sense if α is additive, is the total variation of α, $V(\alpha)$, on $\mathfrak{P}$. Obviously if α is additive, the boundedness of α on $\mathfrak{E}$ is necessary and sufficient that α be of bounded variation.

For a bounded function ξ and a function α it is possible to define the Stieltjes integral $S\int \xi d\alpha$:

$$S\int \xi d\alpha = \lim_\Pi \sum \xi(p_i)\alpha(E_i),$$

where $\Pi = E_1, \cdots, E_n$, and p_i is any point of E_i. We shall say that ξ is S-integrable relative to α if the limit on the left exists.†

For a bounded function ξ which is measurable relative to $\mathfrak{E}$, in the sense that for every c and d the set $E[c<\xi(p)\leqq d]$ belongs to $\mathfrak{E}$, it is possible to define the Lebesgue integral $L\int \xi d\alpha$ by the Lebesgue process, viz., if (a, b) is an interval containing the range of values of ξ, and $a=y_0<y_1< \cdots <y_n=b$ is any subdivision of (a, b) while $y_{i-1}<\eta_i\leqq y_i$, then

$$L\int \xi d\alpha = \lim \sum_i \eta_i \alpha(E_i),$$

where $E_i=E[y_{i-1}<\xi(p)\leqq y_i]$, and the limit is taken as the maximum of y_i-y_{i-1} approaches zero.

If α is additive and bounded on $\mathfrak{E}$, and ξ is measurable relative to $\mathfrak{E}$, then obviously $L\int \xi d\alpha$ exists. The $S\int \xi d\alpha$ exists also in this case and agrees with the L-integral. The S-integral may exist even though ξ be not measurable

* *A general theory of limits*, American Journal of Mathematics, vol. 44 (1922), p. 103.

† This is a type of integral suggested by Moore-Smith (loc. cit., p. 114) and considered by Kolmogoroff, *Untersuchungen ueber den Integralbegriff*, Mathematische Annalen, vol. 103 (1930), pp. 682 ff.

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relative to $\mathfrak{E}$. For example if $\mathfrak{P}=[0<p\leqq 1]$, $\mathfrak{E}$ consists of all finite sets of non-overlapping subintervals of $\mathfrak{P}$ open on the left, while $\alpha(E)$ is the length of E, then the set of all bounded Riemann integrable functions on $\mathfrak{P}$ is obviously S-integrable with respect to α, but includes functions not measurable with respect to $\mathfrak{E}$. The same is true to a lesser degree if $\mathfrak{E}$ is the set of all subsets of $\mathfrak{P}$ having Jordan content and $\alpha(E)=\text{cont } E$.*

2. **Bounded sequences.** Let $\mathfrak{P}$ be the set of all positive integers p. Let $\mathfrak{E}$ be the set of *all* subsets of $\mathfrak{P}$, i.e., E is any set of positive integers. Π is then any division of $\mathfrak{P}$ into a finite number of mutually exclusive sets of positive integers. At least one set in Π will contain an infinite number of elements, but they all may.

Let $\mathfrak{X}$ be the vector space consisting of all bounded sequences, i.e., of all bounded real-valued functions ξ on $\mathfrak{P}$, with $\|\xi\|$ the least upper bound of the values $|\xi(p)|$. Then we have the following

THEOREM. *Any bounded linear operation L on $\mathfrak{X}$ is expressible in the form $L(\xi)=\int\xi d\alpha$, the integral being taken in either the L or S sense, and α being a bounded additive function on $\mathfrak{E}$ whose total variation on $\mathfrak{P}$ is the modulus of L. Conversely every such integral is a linear bounded operation on $\mathfrak{X}$.*

Let $\chi(E, p)$ represent the characteristic function of the set E, i.e., zero for p not on E and unity for p on E. Then if $\alpha(E)=L(\chi(E, p))$ it is obviously additive and bounded on $\mathfrak{E}$.

Divide $\mathfrak{P}$ by the partition $\Pi=E_1, \cdots, E_n$. Define

$$\xi(\Pi) = \sum_{i=1}^{n}\xi(p_i)\chi(E_i, p).$$

Then $\lim\|\xi(\Pi)-\xi\|=0$. For suppose that the range of values of $\xi(p)$ is contained in the interval (a, b), and divide (a, b) into n equal parts by the points $a=y_0<y_1<y_2<\cdots<y_n=b$, so that $(b-a)/n$ is less than a given e. If E_i is the set $E[y_{i-1}<\xi(p)\leqq y_i]$ and Π_e consists of $E_1, \cdots, E_n$, then obviously $\|\xi(\Pi_e)-\xi\|\leqq e$. The same inequality will also hold for any repartition Π of Π_e, which demonstrates the assertion.

Now by the linearity of L,

$$L(\xi(\Pi)) = \sum_i\xi(p_i)L(\chi(E_i, p)) = \sum_i\xi(p_i)\alpha(E_i).$$

By the boundedness of L and the convergence of the right-hand side, it follows that

$$L(\xi) = \int \xi d\alpha,$$

* See J. Ridder, Nieuw Archiv der Wiskunde, (2), vol. 15 (1928), pp. 321–9; O. Frink, Annals of Mathematics, (2), vol. 34 (1933), pp. 518–527.

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where it is obvious that since ξ is measurable relative to $\mathfrak{E}$, the integral on the left may be defined in either the S or L sense. The fact that the total variation of α is the modulus of L follows from the fact that if

$$\xi(p) = \sum \chi(E_i, p) \operatorname{sgn} \alpha(E_i),$$

then $\|\xi\| = 1$ and $L(\xi) = \sum |\alpha(E_i)|$.

The converse theorem follows from obvious properties of the integral.

It is possible to give the result another form. Suppose p_i is the first integer in the set E_i. Define the sequence or function $\beta_\Pi(p) = 0$ if $p \neq p_i$, while $\beta_\Pi(p) = \alpha(E_i)$ if $p = p_i$. Then the approximating sum $\sum \xi(p_i)\alpha(E_i)$ can be written $\sum_p \beta_\Pi(p)\xi(p)$, where only a finite number of the $\beta_\Pi(p)$ are not zero for a given Π. We can consequently state the following:

To every linear functional operation L there corresponds a set of sequences $\beta_\Pi(p)$ whose elements are different from zero at most for a finite number of p, such that

(1) $$\lim_\Pi \sum_p \beta_\Pi(p)\xi(p) = L(\xi);$$

(2) $$\sum_p |\beta_\Pi(p)| \leqq M_L$$

and

(3) $$\lim_\Pi \sum_p |\beta_\Pi(p)| = M_L.$$

This result parallels a result due to Banach* for separable subspaces of the space $\mathfrak{X}$. While the limit involved in this result can be reduced to a sequential limit for each particular ξ, a non-sequential limit is needed for the whole space. The import of the Banach theorem is that for the case of a separable subspace $\mathfrak{X}_0$ of $\mathfrak{X}$, there exists a sequence of partitions Π_n, such that for every ξ of $\mathfrak{X}_0$,

$$\lim_n \|\xi(\Pi_n) - \xi\| = 0 \text{ and } \lim_n \sum_p \beta_{\Pi_n}(p)\xi(p) = L(\xi).$$

It is possible to deduce this result from our general considerations. For this purpose we note that if ξ_m is any sequence of functions of the space $\mathfrak{X}$, it is possible to select a sequence of partitions Π_n by the diagonal process, such that $\lim_n \xi(\Pi_n) = \xi$ for every ξ_m of the given sequence. If $\mathfrak{X}_0$ is a separable subspace of $\mathfrak{X}$ and ξ_n is dense in $\mathfrak{X}_0$, then if ξ belongs to $\mathfrak{X}_0$, there exists a sequence ξ_{n_m} approaching ξ. Let $\Pi_1, \cdots, \Pi_k, \cdots$ be the partitions such that for every n

$$\lim_k \xi_n(\Pi_k) = \xi_n.$$

* *Opérations Linéaires*, p. 72.

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Now

$$\lim_m \|\xi_{n_m}(\Pi_k) - \xi(\Pi_k)\| = 0$$

uniformly in k, since

$$\|\xi_{n_m}(\Pi_k) - \xi(\Pi_k)\| \leqq \|\xi_{n_m} - \xi\|.$$

Hence by the iterated limits theorem on double sequences it follows that $\lim_k \|\xi(\Pi_k) - \xi\| = 0$. Consequently, for every ξ of $\mathfrak{X}_0$,

$$\lim_k \sum_p \beta_{\Pi_k}(p)\xi(p) = \lim_k \sum\nolimits_{\Pi_k} \xi(p_i)\alpha(E_i) = \int \xi d\alpha = L(\xi).$$

We note that if ξ is a special sequence, it may not be necessary to use all of the values of α. For instance if ξ is a sequence converging to zero, it is sufficient to know the values of $\alpha(E_p)$ where E_p consists of the integer p only. Obviously in this case $L(\xi)$ reduces to $\sum_1^\infty \xi(p)\alpha(E_p)$, with $\sum |\alpha(E_p)| < \infty$, which is a well known result. Similarly for any sequence converging to a limit, the values $\alpha(E_p)$ and $\alpha(\mathfrak{P})$ suffice.

The effect of the fundamental theorem established is that a conjugate space to the space of all bounded sequences is the space of all additive bounded functions on subsets of integers, with norm the total variation of the function.

The question naturally arises whether additive functions on the set $\mathfrak{E}$ exist, which are not absolutely additive, i.e., whether this form of operation is effective. The instance of sequences approaching a limit must come from such a function. Banach's measure function* on subsets of positive integers gives a complete example.

3. **Bounded measurable functions.** The procedure in this case is entirely analogous to the preceding case.

We let $\mathfrak{P} = -\infty < p < \infty$, $\mathfrak{E}$ the set of all measurable subsets of $\mathfrak{P}$, $\mathfrak{X}$ the set of all bounded measurable functions ξ on $\mathfrak{P}$, with $\|\xi\|$ the *least* upper bound of $|\xi(p)|$. Then we have

THEOREM. *Any linear functional operation on the space of all bounded measurable functions is expressible in the form $\int \xi d\alpha$, where the integral is to be taken in either the S or L sense, the function α is additive and bounded on $\mathfrak{E}$, and the total variation of α is the modulus M_L of L. $\alpha(E)$ is the value of $L(\chi(E))$, where where $\chi(E)$ is the characteristic function of E.*

Obviously it is possible to give a theorem corresponding to the Banach result, viz., that the operation L is the Π-limit of a set of finite sums, each involving the function ξ at only a finite number of points.

* *Opérations Linéaires*, p. 231.

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4. **Bounded functions on the infinite interval having at most discontinuities of the first kind.** This class of functions is obviously a subclass of the set of bounded measurable functions. As a consequence it is to be expected that a smaller set $\mathfrak{E}$ will suffice. Let again $\mathfrak{P} = -\infty < p < \infty$. Then the set $\mathfrak{E}$ consists of all subsets E of $\mathfrak{P}$, which consist of a finite or infinite number of open intervals and single points, there being at most a finite number of intervals and individual points in the finite part of the fundamental interval, i.e., a set E consists of disjoint open intervals $a_n < p < b_n$, together with points p_n either end points of (a_n, b_n) or not belonging to any (a_n, b_n), where a_n, b_n, p_n have at most $+\infty$ and $-\infty$ as limiting points. The intervals $-\infty < p < a$ and $a < p < \infty$ will be considered open intervals.

With this definition we have the following

THEOREM. *Every bounded linear operation on the set of all bounded functions on $-\infty < p < \infty$ having at most discontinuities of the first kind is expressible in the form $\int f d\alpha$ where the integral is of the S type, α is additive and bounded on $\mathfrak{E}$ and has total variation M_L.*

In order to prove this theorem it is sufficient to show that the functions

$$\xi(\Pi) = \sum_i \xi(p_i)\chi(E_i, p)$$

approach ξ uniformly in the Π-sense. For this purpose we utilize the theorem of Lebesgue* that if ξ is bounded and has only discontinuities of the first kind and is limited to a finite interval (a, b) then there exists for any given $e>0$ a subdivision of (a, b) into a finite number of intervals such that on each open subinterval the oscillation of ξ is less than e. It follows that for any given ξ and any $e>0$, there exists a sequence of points $\cdots p_{-n} < \cdots < p_{-1} < p_0 < p_1 < \cdots < p_n < \cdots$ approaching $-\infty$ on the left and $+\infty$ on the right, such that interior to each interval (p_{i-1}, p_i) the oscillation of ξ is less than e. Suppose now (c, d) contains the region of variation of $\xi(p)$, i.e., $c < \xi(p) < d$ for all p. Divide (c, d) into a finite number of equal parts of length e_0, by the points $c = y_0 < y_1 < \cdots < y_n = d$. Let the set E_1 consist of all the intervals (p_i, p_{i+1}) containing in their interior a point p such that $y_0 < \xi(p) \leqq y_1$, together with all points p_i satisfying the same condition. Let E_2 consist of all the intervals not belonging to E_1 which contain a point p for which $y_1 < \xi(p) \leqq y_2$, and the points p_i satisfying the same condition, and so on. Then since the oscillation of ξ on any of the intervals (p_i, p_{i+1}) is at most e, it follows that the oscillation of $\xi(p)$ on any E_k is at most $e + e_0$. Consequently

$$\|\xi(p) - \sum_1^n \xi(\bar{p}_i)\chi(E_i, p)\| \leqq e + e_0,$$

* Annales de la Faculté des Sciences de Toulouse, (3), vol. 1 (1909), p. 60.

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$\bar{p}_i$ being any point of E_i. Since the same type of inequality will be valid for any partition $\Pi \geqq \Pi_e$, where Π_e consists of $E_1, \cdots, E_n$, we have the result desired.

The case in which the infinite interval is replaced by a finite interval has been considered by Kaltenborn.* In that case the infinite parts of our partitions drop away, and it can be shown that the integral depends only on a point function of bounded variation and a function zero except at a denumerable set of points, but it is simpler to proceed directly in this case.

5. **Bounded continuous functions on the infinite interval.** Obviously the class of functions considered in §4 contains the set of bounded continuous functions as a subset. As a consequence we can effect a further reduction in the set $\mathfrak{E}$. We shall assume that $\mathfrak{E}$ contains all sets E which consist of a finite or denumerable set of non-overlapping half open intervals $(a_n < p \leqq b_n)$, whose end points have at most $-\infty$ and $+\infty$ as limiting points. The intervals $-\infty < p \leqq a$ and $a < p < \infty$ will be considered to be half open intervals.

With this definition of $\mathfrak{E}$ we can state the same theorem as in the preceding paragraph. It is to be noted, however, that in this case the function $\alpha(E)$ is defined in terms of an extension of the linear operation L on continuous functions to functions having discontinuities of the first kind.

If we limit ourselves to a finite interval, the ordinary Stieltjes integral applies, since because of the continuity of ξ, the successive partition limit agrees with the limit as the maximum length of subdivisions approaches zero.

It is possible to give a form to the general theorem which is comparable to the Banach result for sequences. Let Π be any partition of $\mathfrak{P}$ into sets $E_1, \cdots, E_n$. Let p_i be any point in the interval of E_i nearest to $p=0$. Let $\beta_\Pi(p)$ be a point function such that $\beta_{\bar{\Pi}}(0)=0$, and constant except at the points $p=p_i$, where it has a break or saltus of magnitude $\alpha(E_i)$. Then obviously

$$L(\xi(\Pi)) = L\left(\sum_1^n \xi(p_i)\chi(E_i, p)\right) = \sum \xi(p_i)\alpha(E_i) = \int_{-\infty}^{\infty} \xi(p)d\beta_\Pi(p),$$

where the infinite limits could be replaced by any finite interval containing the points $p_1, \cdots, p_n$ in its interior. It follows that we have the following alternative theorem:

If L is any bounded linear operation on the class of bounded continuous functions on $-\infty < p < \infty$, then there exists a set of point functions $\beta_\Pi(p)$ constant except at a finite number of points such that

$$L(\xi) = \lim_\Pi \int_{-\infty}^{\infty} \xi(p)d\beta_\Pi(p),$$

* Loc. cit.

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the integral being an ordinary Stieltjes integral. The functions β_Π *are uniformly of bounded variation and* $\lim_\Pi V(\beta_{\bar\Pi}) = M_L$.

For any separable subset we can obviously proceed as in §2 and replace the limit in the Π-sense by a sequential limit, i.e., we can find a sequence Π_n of partitions which is effective in the limit for all functions of the set.

6. **Almost bounded measurable functions.** In agreement with common usage the measurable function is almost bounded if it is bounded except for a set of zero measure. The $\|\xi\|$ is defined as the greatest lower bound of positive numbers a such that the set $E[|\xi(p)|>a]$ is of zero measure.

The only difference between this case and that of §3 is that if E is a set of zero measure then $\alpha(E)=0$, since then $L(\chi(E))=0$. It cannot however be concluded that if $\alpha(E)=0$ for any set of zero measure then $\alpha(E)$ is absolutely continuous and consequently the indefinite integral of a Lebesgue integrable function.

An example on the finite interval $0\leqq p\leqq 1$ of an additive bounded function $\alpha(E)$ on measurable sets which satisfies the condition that $\alpha(E)=0$ for meas $E=0$, but is not absolutely continuous nor absolutely additive, can be constructed. Let (a_n, b_n) be a sequence of disjoint intervals whose end points have 1 as their only limiting point. If E_0 is any measurable subset of (a_n, b_n), then define $\beta(E_0)=mE_0/(a_n-b_n)$. If now E is any subset of $0\leqq p\leqq 1$, and E_n the part of E lying on (a_n, b_n), then $\beta(E_n)$ defines a bounded sequence of numbers. The function $\alpha(E)=\int\beta(E_n)d\mu$ (in the sense of §2), where μ is a measure function of Banach on subsets of positive integers,* will be additive on measurable subsets of (0, 1), will satisfy the condition

$$\alpha(E) = 0 \text{ if meas } E = 0,$$

but will not be absolutely additive, nor absolutely continuous. For if E is the set $(1-e\leqq p\leqq 1)$ then for all $e>0$, $\alpha(E)=1$. Incidentally it appears that if $\xi(p)$ is continuous on (0, 1) then $\int_0^1 \xi d\alpha=\xi(1)$, i.e., as far as the integration of continuous functions is concerned $\alpha(E)$ is equivalent to the function $\gamma(p)=0$ for $0\leqq p<1$, $\gamma(1)=1$.

It is obvious that the results of §§3, 5, and 6 can be extended to corresponding situations in n-dimensional space. Also that it would be possible to set up a general theorem reducing to the special cases considered by a proper choice of the set $\mathfrak{P}$ and $\mathfrak{E}$.

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* *Opérations Linéaires*, p. 231.

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