ON CONVEX FUNCTIONS*

BY

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INTRODUCTION

0.1. The problems raised and solved in this paper were suggested by certain results concerning subharmonic functions, published jointly by Dr. E. F. Beckenbach and myself.†

0.2. According to F. Riesz, a function $u(x, y)$, continuous in a certain region R of the xy-plane, is called *subharmonic* if the inequality

$$u(x, y) \leqq \frac{1}{2\pi}\int_0^{2\pi} u(x + \rho\cos\theta, y + \rho\sin\theta)d\theta$$

holds for every point (x, y) in R and for every value of $\rho > 0$, such that the circular disc with center (x, y) and radius ρ is entirely comprised in R.‡

0.3. We have the following theorem:

A function $u(x, y)$, continuous in a region R, is subharmonic there if and only if the inequality

$$\frac{1}{\rho^2\pi}\iint_{\xi^2+\eta^2<\rho^2} u(x + \xi, y + \eta)d\xi d\eta \leqq \frac{1}{2\pi}\int_0^{2\pi} u(x + \rho\cos\theta, y + \rho\sin\theta)d\theta$$

holds for every point (x, y) in R and for every $\rho > 0$, such that the circular disc with center (x, y) and radius ρ is entirely comprised in R.§

0.4. In the second of the joint papers by Dr. Beckenbach and myself, referred to in 0.1, the following theorem was proved in connection with an investigation of the isoperimetric inequality.

The logarithm of a function $u(x, y)$, continuous and positive in a region R, is subharmonic there if and only if the inequality

$$\left[\frac{1}{\rho^2\pi}\iint_{\xi^2+\eta^2<\rho^2} u(x + \xi, y + \eta)^2 d\xi d\eta\right]^{1/2} \leqq \frac{1}{2\pi}\int_0^{2\pi} u(x + \rho\cos\theta, y + \rho\sin\theta)d\theta$$

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† *Subharmonic functions and minimal surfaces*, these Transactions, vol. 35 (1933), pp. 648–661. *Subharmonic functions and surfaces of negative curvature*, these Transactions, vol. 35 (1933), pp. 662–674.

‡ See F. Riesz, *Sur les fonctions subharmoniques* etc., in two parts, Acta Mathematica, vol. 48 1926), pp. 329–343, and vol. 54 (1930), pp. 321–360.

§ See the second paper quoted under † above.

holds for every point (x, y) *in* R *and for every* $\rho>0$, *such that the circular disc with center at* (x, y) *and with radius* ρ *is entirely comprised in* R.

0.5. The theorems in 0.3 and 0.4 are clearly two links in a chain of similar theorems. In the paper referred to in 0.4, some rather incomplete remarks were made concerning the character of these theorems. On account of the close analogy between subharmonic functions of two variables on the one hand and convex functions of one variable on the other,† there arose in this manner certain problems concerned with convex functions. We shall indicate briefly the character of the problems thus suggested.

0.6. Let $f(x)$ be a function, continuous and positive in a given open interval $x_1<x<x_2$. Denoting by α a real exponent, we define

$$I(f, x, h, \alpha) = \left[\frac{1}{2h}\int_{-h}^{h} f(x+\xi)^\alpha d\xi\right]^{1/\alpha}, \text{ if } \alpha \neq 0,$$

and

$$I(f, x, h, 0) = \exp\left[\frac{1}{2h}\int_{-h}^{h} \log f(x+\xi)d\xi\right],$$

where x and h are supposed to satisfy the inequalities

$$x_1 < x-h < x+h < x_2.$$

Then $I(f, x, h, \alpha)$ is a continuous and increasing function of α, for $-\infty<\alpha<+\infty$.‡

0.7. Given a real exponent γ, we define the class C_γ as the class of all functions $f(x)$ which are continuous and positive in $x_1<x<x_2$ and which are such that

$$f^\gamma \operatorname{sgn} \gamma \text{ is convex, if } \gamma \neq 0,$$

and

$$\log f \text{ is convex, if } \gamma = 0.$$

0.8. Given a real exponent δ, we define the class C_δ^* as the class of all functions $f(x)$ which are continuous and positive in $x_1<x<x_2$ and which satisfy the inequality

$$I(f, x, h, \delta) \leqq \frac{f(x-h)+f(x+h)}{2}$$

for all values of x and h such that $x_1<x-h<x+h<x_2$.

† See, for instance, P. Montel, *Sur les fonctions convexes et les fonctions sousharmoniques*, Journal de Mathématiques, (9), vol. 7 (1928), pp. 29–60.

‡ See, for instance, Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1 (Berlin, Springer, 1925), problem 82 on p. 54 and problem 83 on p. 55.

0.9. The theorems in 0.3 and 0.4 suggest, by implication, the problem of determining couples of exponents (γ, δ) such that $C_\gamma \equiv C_\delta^*$. In conversations between Dr. Beckenbach and myself, it was found† that

$$C_\delta^* \subset C_\gamma \text{ for } 2\gamma + \delta - 3 = 0.‡$$

Dr. Beckenbach proved then that

$$C_\gamma \subset C_\delta^* \text{ for } 2\gamma + \delta - 3 = 0 \text{ and } 0 \leqq \gamma \leqq 1.$$

0.10. Thus it was established that

$$C_\gamma \equiv C_\delta^* \text{ for } 2\gamma + \delta - 3 = 0 \text{ and } 0 \leqq \gamma \leqq 1.‡$$

In a seminar on convex functions, held at Ohio State University in 1933–34, the topics just described came up again for discussion. I found, against my own expectation, that the implication

$$(1) \qquad C_\gamma \subset C_\delta^* \text{ for } 2\gamma + \delta - 3 = 0$$

did not hold generally. On the other hand, members of that seminar verified (1) for a number of values of γ outside of the interval $0 \leqq \gamma \leqq 1$, considered by Dr. Beckenbach. The purpose of this paper is to present the results of an investigation suggested by the situation just described.

0.11. For the sake of brevity, we restrict ourselves to state in this introduction the main results only. A number of applications, in particular a complete discussion of the relation $C_\gamma \equiv C_\delta^*$, will be considered in §4.

0.12. Besides the mean $I(f, x, h, \alpha)$, defined in 0.6, we shall use also another mean value $A(f, x, h, \beta)$, defined as follows:

$$A(f, x, h, \beta) = \left[\frac{f(x-h)^\beta + f(x+h)^\beta}{2}\right]^{1/\beta}, \text{ if } \beta \neq 0,$$

and

$$A(f, x, h, 0) = \exp\left[\frac{\log f(x-h) + \log f(x+h)}{2}\right] = [f(x-h)f(x+h)]^{1/2}.$$

Again, β is a real exponent, $f(x)$ a function continuous and positive in $x_1 < x < x_2$, and x and h are supposed to satisfy $x_1 < x-h < x+h < x_2$. As is

† These conversations, as well as the investigations published in the papers quoted in foot note †, p. 266, were carried on in 1932–33 while Dr. Beckenbach worked as a National Research Fellow at Ohio State University.

‡ For the sake of accuracy, it should be mentioned that we only considered at that time the case of functions with continuous first and second derivatives. Concerning the case of general continuous functions, see the remarks made in 0.15.

well known, $A(f, x, h, \beta)$ is a continuous and increasing function of β for $-\infty<\beta<+\infty$.†

0.13. *Denote by E the set of all pairs* (α, β) *for which the following assertion is true: every function* $f(x)$, *which is continuous, positive and convex in* $x_1<x<x_2$ *satisfies the inequality* $I(f, x, h, \alpha) \leqq A(f, x, h, \beta)$ *for all values of* x *and* h *such that* $x_1<x-h<x+h<x_2$.

Denote by $\bar{E}$ *the set of all pairs* (α, β) *for which the following assertion is true: if a function* $f(x)$, *which is positive and continuous in* $x_1<x<x_2$, *satisfies the inequality* $I(f, x, h, \alpha) \leqq A(f, x, h, \beta)$ *for all values of* x *and* h *such that* $x_1<x-h<x+h<x_2$, *then* $f(x)$ *is convex in* $x_1<x<x_2$.

Our main result is the explicit determination of these sets E, $\bar{E}$. As we shall prove in §2, a pair (α, β) *belongs to* E *if and only if one of the following four conditions is satisfied.*‡

I. $\alpha \leqq -2$ *and* $\beta \geqq 0$.

II. $-2 \leqq \alpha \leqq -\dfrac{1}{2}$ *and* $\beta \geqq \dfrac{\alpha+2}{3}$.

III. $-\dfrac{1}{2} \leqq \alpha \leqq 1$ *and* $\beta \geqq \dfrac{\alpha \log 2}{\log(\alpha+1)}$.

IV. $1 \leqq \alpha$ *and* $\beta \geqq \dfrac{\alpha+2}{3}$.

As we shall prove in §3, a pair (α, β) *belongs to* $\bar{E}$ *if and only if* $3\beta-\alpha-2 \leqq 0$.

The corresponding results for concave functions will be given in §4.

0.14. The following remarks should be made concerning the method used in this paper. Whenever we shall be concerned with deriving an inequality of the form $I(f, x, h, \alpha) \leqq A(f, x, h, \beta)$ for a function $f(x)$ with certain convexity

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† See second footnote on p. 267.

‡ The following remarks may be helpful to the reader in making a picture of the set E. Define three functions $\psi_1(\alpha)$, $\psi_2(\alpha)$, $\psi_3(\alpha)$ as follows:

$$\psi_1(\alpha) \equiv 0, \quad -\infty<\alpha<+\infty;$$

$$\psi_2(\alpha) = \frac{\alpha+2}{3}, \quad -\infty<\alpha<+\infty;$$

$$\psi_3(\alpha) = \begin{cases} 0, \text{ if } -\infty<\alpha \leqq -1, \\ \dfrac{\alpha \log 2}{\log(\alpha+1)}, \text{ if } -1<\alpha<+\infty \text{ and } \alpha \neq 0, \\ \log 2, \text{ if } \alpha = 0. \end{cases}$$

Then E consists of all points (α, β) such that

$$\beta \geqq \max\,[\psi_1(\alpha), \psi_2(\alpha), \psi_3(\alpha)],$$

where the symbol max means the largest of the numbers which it precedes. The function $\psi_3(\alpha)$ is positive, increasing and concave for $-1<\alpha<+\infty$, as is easily seen by differentiation. The curves $\beta=\psi_2(\alpha)$ and $\beta=\psi_3(\alpha)$ intersect each other at the points $(-2, 0)$, $(-\frac{1}{2}, \frac{1}{2})$ and $(1, 1)$.

properties, it will be possible to reduce the discussion to the case when $f(x)$ is linear. The means I and A can be computed then explicitly, and our arguments will consist, generally speaking, of a more or less accurate discussion of the graphs of certain elementary functions. The result

$$C_\gamma \subset C_\delta^* \text{ for } 2\gamma + \delta - 3 = 0 \text{ and } 0 \leqq \gamma \leqq 1, \tag{2}$$

proved by Dr. Beckenbach, is a particular case which comes under this description (see 4.8–4.9 below). The proof of Dr. Beckenbach for (2) consisted of quite elementary, but rather elaborate, computations, and the same remark applies, unfortunately to an even greater extent, to the arguments used in the present paper. While the general appearance of our inequalities strongly suggests the use of the simple and fundamental inequalities named after Hölder and Minkowski,† I was unable to establish a connection in this direction.

0.15. Whenever we shall be concerned with deriving, from an inequality of the form $I(f, x, h, \alpha) \leqq A(f, x, h, \beta)$, certain properties of convexity for $f(x)$, the results will be quite trivial in case $f(x)$ has continuous first and second derivatives. On the other hand, I had to follow rather devious ways to deal with the case of general continuous functions.‡ While it seems quite natural to establish those results first under the assumption of continuous first and second derivatives and to handle the general case by approximation, I was unable to carry out this program in a generality sufficient for the purposes of this paper.

The purpose of the remarks made in 0.14 and 0.15 is to call the attention of the reader to situations which might quite possibly suggest some interesting investigations.

0.16. In Part 1 of this paper, certain elementary functions, used in the sequel, are discussed. In Parts 2 and 3 respectively, we shall give the explicit determination of the sets E and $\overline{E}$, defined in 0.13. Finally, Part 4 will be concerned with extensions and miscellaneous applications.

1. Lemmas

1.1. Let $\omega_1, \omega_2, \cdots, \omega_n$ and $a_1, a_2, \cdots, a_n$ be real numbers, and consider the function

$$Q(u) = a_1 u^{\omega_1} + a_2 u^{\omega_2} + \cdots + a_n u^{\omega_n}$$

of the variable $u > 0$. It is assumed that not all the a's vanish. Then the num-

† See the very elegant presentation by F. Riesz, *Su alcune disuguaglianze*, Bollettino della Unione Matematica Italiana, vol. 7 (1928), pp. 77–79.

‡ See 3.4 to 3.10.

ber of positive roots of the equation $Q(u)=0$ is $\leqq n-1$, every root being counted with the proper multiplicity.†

1.2. Let α, β denote real numbers such that

$$\alpha \neq 0,\ \alpha + 1 \neq 0,\ \beta \neq 0,\ 3\beta - \alpha - 2 \neq 0, \tag{3}$$

and consider the function

$$P(u) = -\frac{1}{\alpha}u^{\alpha+\beta+1} + u^{\alpha+2} - \frac{\alpha+1}{\alpha}u^{\alpha+1} + \frac{\alpha+1}{\alpha}u^{\beta+1} - u^{\beta} + \frac{1}{\alpha}u,$$

of the real variable $u>0$. We shall need a few simple properties of $P(u)$.

1.3. Given α and β, such that (3) is satisfied, there exists an $\epsilon>0$ such that

$$\operatorname{sgn} P(u) = \operatorname{sgn}\left[(\alpha+1)(3\beta-\alpha-2)\right] \text{ for } 1-\epsilon<u<1.$$

This follows immediately from

$$P(1) = P'(1) = P''(1) = 0,\ P'''(1) = -(\alpha+1)(3\beta-\alpha-2),$$

by using Taylor's formula.

1.4. The function $P(u)$, defined in 1.2, has at most one root in the interval $0<u<1$. If such a root exists, then it is a simple root.

Indeed, on account of $P(1)=P'(1)=P''(1)=0$, $P'''(1)\neq 0$, we have a root of multiplicity 3 at $u=1$. A direct computation shows that

$$u^{\alpha+\beta+2}P\left(\frac{1}{u}\right) = -P(u).$$

Hence, if there are σ roots in the interval $0<u<1$, then there are also σ roots in the interval $1<u<+\infty$. By 1.1, there are at most five positive roots. Hence, $3+2\sigma\leqq 5$, and consequently $\sigma\leqq 1$.

1.5. Supposing again that α, β satisfy conditions (3), we shall consider the function

$$\phi(u) = \log \frac{\left[\dfrac{1}{\alpha+1}\,\dfrac{u^{\alpha+1}-1}{u-1}\right]^{1/\alpha}}{\left(\dfrac{1+u^{\beta}}{2}\right)^{1/\beta}},\quad 0<u<1. \tag{4}$$

We shall need a few simple properties of this function $\phi(u)$.

1.6. Given α and β, such that (3) is satisfied, there exists an $\epsilon>0$ such that

$$\operatorname{sgn}\phi(u) = -\operatorname{sgn}(3\beta-\alpha-2) \text{ for } 1-\epsilon<u<1. \tag{5}$$

Indeed, we see from (4) that

† The change of variable $u=e^{x}$ reduces $Q(u)$ to an exponential polynomial for which the result is well known. Cf. J. Tamarkin, *Some general problems*, etc., Mathematische Zeitschrift, vol. 27 (1928), p. 28.

$$\phi(u) \to 0 \text{ for } u \to 1. \tag{6}$$

We also find, by a direct computation, that

$$(1 + u^{\beta})(u^{\alpha+1} - 1)(u - 1)u\phi'(u) = P(u), \tag{7}$$

where $P(u)$ is the function defined in 1.2. Since $0<u<1$, it follows that

$$\operatorname{sgn} \phi'(u) = \operatorname{sgn} [(\alpha + 1)P(u)], \quad 0 < u < 1. \tag{8}$$

By 1.3 there exists an $\epsilon>0$ such that

$$\operatorname{sgn} \phi'(u) = \operatorname{sgn} (3\beta - \alpha - 2) \text{ for } 1 - \epsilon < u < 1. \tag{9}$$

Relation (5) is derived from (6) and (9) by using the mean-value theorem.

1.7. By inspection of formula (4) it is seen that

$$\phi(+0) < 0 \text{ if } \begin{cases} \alpha + 1 < 0 \text{ and } \beta > 0, \text{ or} \\ \alpha + 1 > 0 \text{ and } \beta > \dfrac{\alpha \log 2}{\log (\alpha + 1)}. \end{cases}$$

On the other hand,

$$\phi(+0) > 0 \text{ if } \begin{cases} \alpha + 1 < 0 \text{ and } \beta < 0, \text{ or} \\ \alpha + 1 > 0 \text{ and } \beta < \dfrac{\alpha \log 2}{\log (\alpha + 1)}. \end{cases}$$

1.8. If $3\beta-\alpha-2>0$, then

$$\phi(u) < \max [\phi(+0), 0] \text{ for } 0 < u < 1. \tag{10}$$

Indeed, we have (see formula (9) in 1.6),

$$\operatorname{sgn} \phi'(u) = \operatorname{sgn} (3\beta - \alpha - 2) \text{ for } 1 - \epsilon < u < 1,$$

where ϵ is some positive constant. Since, by assumption, $3\beta-\alpha-2>0$, we have therefore

$$\phi'(u) > 0 \text{ for } 1 - \epsilon < u < 1. \tag{11}$$

Case I. $\phi'(u)\neq 0$ in $0<u<1$. By (11) $\phi'(u)>0$ in $0<u<1$. Hence $\phi(u)$ is increasing in $0<u<1$. Also, $\phi(u)\to 0$ for $u\to 1$. Thus $\phi(u)<0$ in $0<u<1$, and (10) is proved.

Case II. $\phi'(u)$ has some zero in $0<u<1$. From (7) it follows that $\phi'(u)$ and $P(u)$ have the same number of roots in $0<u<1$. But (see 1.4) $P(u)$ has at most one root in $0<u<1$. Thus $\phi'(u)$ has exactly one root in $0<u<1$, and this root, which we denote by u_0, is a simple root. Hence $\phi'(u)$ changes its sign at u_0. Since $\phi'(u)>0$ for u close to 1 (see (11)),

$$\phi'(u) < 0 \text{ for } 0 < u < u_0,$$

and

$$\phi'(u) > 0 \text{ for } u_0 < u < 1.$$

That is to say, $\phi(u)$ increases in $u_0<u<1$ and decreases in $0<u<u_0$. Consequently,

$$\phi(u) < \phi(+0) \text{ in } 0 < u \leqq u_0,$$

and

$$\phi(u) < \lim_{u\to 1} \phi(u) = 0 \text{ in } u_0 \leqq u < 1.$$

Thus (10) is proved.

1.9. In a similar way, we obtain the following lemma:

If $3\beta-\alpha-2<0$, *then*

$$\phi(u) > \min[\phi(+0), 0] \textit{ for } 0 < u < 1.$$

2. Determination of the set E

2.1. In 0.13, we defined the set E as the set of all pairs (α, β) such that the inequality

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta) \tag{12}$$

is satisfied by every function $f(x)$ which is positive, continuous and convex in $x_1<x<x_2$. The inequality (12) is supposed to be satisfied for all values of x and h such that $x_1<x-h<x+h<x_2$. We shall determine this set E explicitly.

2.2. Suppose that (α, β) satisfies one of the conditions I–IV of 0.13. Then, as we shall show presently, (α, β) is in E.

2.3. We first prove this under the assumption that $f(x)$ is linear:†

$$f(x) = l(x) = ax + b > 0 \text{ in } x_1 < x < x_2,$$

and

$$\alpha \neq 0, \alpha + 1 \neq 0, \beta \neq 0, 3\beta - \alpha - 2 \neq 0, \beta \neq \frac{\alpha \log 2}{\log(\alpha+1)}. \tag{13}$$

Without loss of generality, we can assume that $l(x-h)\neq l(x+h)$. Otherwise, $l(x)$ reduces to a constant and (12) is trivial.

We write the inequality (12) in the form

$$\log \frac{I(f, x, h, \alpha)}{A(f, x, h, \beta)} \leqq 0. \tag{14}$$

Since $f(x)$ has the special form $ax+b$, the mean $I(f, x, h, \alpha)$ can be computed explicitly. Thus we find that (14) is equivalent to the inequality

† Cf. 0.14.

$$\phi(u) \leqq 0 \text{ for } 0 < u < 1,$$

where $\phi(u)$ is the function defined in 1.5 and

$$u = \frac{\min\,[l(x - h), l(x + h)]}{\max\,[l(x - h), l(x + h)]}.$$

Using 1.7, we see that in every one of the four cases stated in 2.2 we have, with regard to (13),

$$\phi(+0) < 0$$

and

$$3\beta - \alpha - 2 > 0.$$

Hence (see 1.8)

$$\phi(u) < \max\,[\phi(+0), 0] = 0 \text{ in } 0 < u < 1.$$

Thus our assertion is proved under the assumption that $f(x)$ is linear and that α, β are further restricted by (13).

2.4. Let now $f(x)$ be a general continuous and positive convex function in $x_1 < x < x_2$. Let (α, β) satisfy one of the four conditions of 0.13, and suppose for a moment that (13) is also satisfied. Let x and h be such that $x_1 < x-h < x+h < x_2$, and denote by l the linear function which coincides with f at $x-h$ and $x+h$. Then we have

$$A(f, x, h, \beta) = A(l, x, h, \beta). \tag{15}$$

Since f is convex, we have $f \leqq l$ in the interval $(x-h, x+h)$. Hence:

$$I(f, x, h, \alpha) \leqq I(l, x, h, \alpha). \tag{16}$$

By 2.3 we have

$$I(l, x, h, \alpha) \leqq A(l, x, h, \beta). \tag{17}$$

(15), (16) and (17) yield

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta). \tag{18}$$

Thus the assertion in 2.2 is proved under the restriction (13). An easy continuity consideration will allow us, however, to remove this restriction and thus to complete the proof of our assertion in 2.2.

2.5. So far we have proved that every pair (α, β) which satisfies one of the four conditions of 0.13 belongs to E. We shall prove now that every pair (α, β) in E satisfies one of the four conditions 0.13 and then the theorem of 0.13 concerning E will be completely proved.

2.6. We shall need the following trivial remark: if $(\alpha, \beta) \subset E$, and $\eta \geqq 0$, then $(\alpha - \eta, \beta + \eta) \subset E$.

To see this, let $f(x)$ be any positive and continuous convex function in $x_1<x<x_2$, and let x and h be such that $x_1<x-h<x+h<x_2$. Since $(\alpha, \beta) \subset E$, we have then

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta). \tag{19}$$

We also have, since I and A are increasing functions of their last arguments, the inequalities

$$I(f, x, h, \alpha - \eta) \leqq I(f, x, h, \alpha), \tag{20}$$

$$A(f, x, h, \beta) \leqq A(f, x, h, \beta + \eta). \tag{21}$$

(19), (20) and (21) yield

$$I(f, x, h, \alpha - \eta) \leqq A(f, x, h, \beta + \eta).$$

Thus $(\alpha-\eta, \beta+\eta) \subset E$.

2.7. Suppose now that $(\alpha, \beta) \subset E$, and suppose that, in contradiction to the assertion made in 2.5, the pair (α, β) satisfies none of the four conditions of 0.13. We shall show that these assumptions lead to a contradiction.

Let η be a small positive number, and put

$$\bar{\alpha} = \alpha - \eta, \ \bar{\beta} = \beta + \eta.$$

Then, if $\eta>0$ is sufficiently small, $(\bar{\alpha}, \bar{\beta})$ will not satisfy any of the four conditions of 0.13 either. Furthermore, if $\eta>0$ is suitably chosen, $(\bar{\alpha}, \bar{\beta})$ will satisfy the relations

$$\bar{\alpha} \neq 0, \ \bar{\alpha} + 1 \neq 0, \ \bar{\beta} \neq 0, \ 3\bar{\beta} - \bar{\alpha} - 2 \neq 0.$$

Finally, on account of 2.6, $(\bar{\alpha}, \bar{\beta})$ will also belong to E.

2.8. Writing again (α, β) for $(\bar{\alpha}, \bar{\beta})$ to simplify the notation, we would have a pair (α, β) satisfying the following conditions:

$$\begin{gathered} (\alpha, \beta) \subset E; \\ \alpha \neq 0, \ \alpha + 1 \neq 0, \ \beta \neq 0, \ 3\beta - \alpha - 2 \neq 0. \end{gathered} \tag{22}$$

Besides, (α, β) satisfies none of the four conditions of 2.2. Clearly, (α, β) satisfies then one of the following three conditions:

$$\alpha + 1 < 0 \text{ and } \beta < 0, \tag{23}$$

$$\alpha + 1 > 0 \text{ and } \beta < \frac{\alpha \log 2}{\log (\alpha + 1)}, \tag{24}$$

$$3\beta - \alpha - 2 < 0. \tag{25}$$

2.9. From $(\alpha, \beta) \subset E$ we infer that the inequality

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta) \tag{26}$$

is satisfied, in particular, by every positive linear function $ax+b$. As observed in 2.3, this is equivalent to the fact that the function

$$\phi(u) = \log \frac{\left[\dfrac{1}{\alpha+1}\,\dfrac{u^{\alpha+1}-1}{u-1}\right]^{1/\alpha}}{\left(\dfrac{1+u^{\beta}}{2}\right)^{1/\beta}}$$

is $\leqq 0$ in $0<u<1$. On account of (22), we can use the lemmas developed in §1.

If either (23) or (24) holds, we have (see 1.7)

$$\phi(+0) > 0.$$

If (25) holds, then (see 1.6)

$$\operatorname{sgn} \phi(u) = -\operatorname{sgn}(3\beta-\alpha-2) = +1 \quad \text{for } 1-\epsilon < u < 1$$

where ϵ is some positive constant. Hence $\phi(u) \leqq 0$ is not satisfied. This contradicts (26), and the proof is complete.

3. Determination of the set $\overline{E}$

3.1. The set $\overline{E}$ has been defined in 0.13 as consisting of all pairs (α, β) such that the inequality

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta) \tag{27}$$

implies the convexity of the function $f(x)$. It is assumed that (27) holds for all values of x and h such that $x_1<x-h<x+h<x_2$, and that $f(x)$ is positive and continuous in $x_1<x<x_2$.

We shall now prove that $(\alpha, \beta) \subset \overline{E}$ if and only if $3\beta-\alpha-2 \leqq 0$.

3.2. We shall need the following trivial remark: if $(\alpha, \beta) \subset \overline{E}$, and $\eta>0$, then $(\alpha+\eta, \beta-\eta) \subset \overline{E}$.

Indeed, suppose that $f(x)$ is a positive and continuous function in $x_1<x<x_2$ which satisfies the inequality

$$I(f, x, h, \alpha+\eta) \leqq A(f, x, h, \beta-\eta) \tag{28}$$

for all values of x and h such that $x_1<x-h<x+h<x_2$. Since I and A are increasing functions of their last arguments, we have

$$I(f, x, h, \alpha) \leqq I(f, x, h, \alpha+\eta), \tag{29}$$

$$A(f, x, h, \beta-\eta) \leqq A(f, x, h, \beta). \tag{30}$$

(28), (29) and (30) yield

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta). \tag{31}$$

Since $(\alpha, \beta) \subset \bar{E}$, it follows from (31) that $f(x)$ is convex. Thus, starting from (28), we proved that $f(x)$ is convex. Hence (28) implies the convexity of $f(x)$, and consequently $(\alpha+\eta, \beta-\eta) \subset \bar{E}$.

3.3. We shall show first: if $3\beta-\alpha-2>0$, then (α, β) is not in $\bar{E}$. Suppose that

$$3\beta - \alpha - 2 > 0. \tag{32}$$

Let σ and η be two constants such that

$$\sigma > 1, \quad \eta > 0,$$

and put

$$\bar{\alpha} = \frac{\alpha + \eta}{\sigma}, \quad \bar{\beta} = \frac{\beta - \eta}{\sigma}\cdot \tag{33}$$

If σ is close enough to 1 and η close enough to zero and suitably chosen, then $(\bar{\alpha}, \bar{\beta})$ will satisfy the relations

$$\bar{\alpha} \neq 0,\ \bar{\alpha} + 1 \neq 0,\ \bar{\beta} \neq 0,\ 3\bar{\beta} - \bar{\alpha} - 2 \neq 0.$$

We set up the function

$$\bar{\phi}(u) = \log \frac{\left[\dfrac{1}{\bar{\alpha}+1}\,\dfrac{u^{\bar{\alpha}+1} - 1}{u - 1}\right]^{1/\bar{\alpha}}}{\left(\dfrac{1 + u^{\bar{\beta}}}{2}\right)^{1/\bar{\beta}}}, \quad 0 < u < 1.$$

On account of 1.6, we have an $\epsilon>0$, such that

$$\operatorname{sgn} \bar{\phi}(u) = -\operatorname{sgn}(3\bar{\beta} - \bar{\alpha} - 2) \text{ for } 1 - \epsilon < u < 1.$$

Hence, by (32),

$$\bar{\phi}(u) < 0 \text{ for } 1 - \epsilon < u < 1. \tag{34}$$

We define now a positive linear function l by the conditions

$$l(x_1) = 1 - \epsilon, \quad l(x_2) = 1,\ x_1 < x < x_2. \tag{35}$$

We assert that l satisfies the inequality

$$I(l, x, h, \bar{\alpha}) \leqq A(l, x, h, \bar{\beta}) \tag{36}$$

for all values of x and h such that $x_1<x-h<x+h<x_2$. If we put

$$u_0 = \frac{l(x - h)}{l(x + h)}, \tag{37}$$

then (see 2.3) this assertion is equivalent to the assertion

$$\bar{\phi}(u_0) \leqq 0. \tag{38}$$

But, from (35) and (37) and the linearity of $l(x)$ we have

$$1 - \epsilon < u_0 < 1. \tag{39}$$

Hence (38) and consequently (36) follow from (39) and (34).

Consider now the function

$$f(x) = l(x)^{1/\sigma}, \quad x_1 < x < x_2. \tag{40}$$

This function $f(x)$ is continuous and positive in $x_1<x<x_2$. If we raise (36) to the power $1/\sigma$, then by (33) and (40),

$$I(f, x, h, \alpha + \eta) \leqq A(f, x, h, \beta - \eta). \tag{40*}$$

On the other hand, we find from (40)

$$f''(x) = \frac{1}{\sigma}\left(\frac{1}{\sigma} - 1\right) l(x)^{(1-2\sigma)/\sigma}(l'(x))^2.$$

Since $\sigma>1$, we see that $f''(x)<0$ in $x_1<x<x_2$. Hence $f(x)$ is not convex in $x_1<x<x_2$.

To summarize, we have exhibited a function $f(x)$, which is continuous and positive in $x_1<x<x_2$ and which satisfies (40*) for all values of x and h such that $x_1<x-h<x+h<x_2$, and which is *not convex* in $x_1<x<x_2$. This shows that $(\alpha+\eta, \beta-\eta)$ is not in $\overline{E}$. From 3.2, it follows finally that (α, β) is not in $\overline{E}$ either.

3.4. We show next that if $3\beta-\alpha-2\leqq 0$, then $(\alpha, \beta) \subset \overline{E}$.† We first assume that the pair (α, β) also satisfies the conditions

$$3\beta - \alpha - 2 < 0, \alpha \neq 0, \alpha + 1 \neq 0, \beta \neq 0. \tag{41}$$

Let $f(x)$ be any positive continuous function in $x_1<x<x_2$ which satisfies the inequality

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta) \tag{42}$$

for all values of x and h such that $x_1<x-h<x+h<x_2$. We have to show that $f(x)$ is convex.

Suppose $f(x)$ is not convex. Then we have an x_0 and an h_0 such that $x_1<x_0-h_0<x_0+h_0<x_2$ and

$$f(x_0) > \frac{f(x_0 - h_0) + f(x_0 + h_0)}{2}.$$

† Cf. 0.15.

Clearly, we have then two numbers a_1, b_1 such that $x_0-h_0<a_1<b_1<x_0+h_0$ and such that the following is true: if $l_1(x)$ is the linear function defined by $l_1(a_1)=f(a_1)$, $l_1(b_1)=f(b_1)$, then $l_1<f$ in $a_1<x<b_1$.

3.5. By a simple reasoning whose details may be left to the reader,† we infer from this situation the existence of a number $\bar{x}$ and of two sequences a_n and b_n, such that the following hold:

I. $a_1<a_n<a_{n+1}<\bar{x}<b_{n+1}<b_n<b_1$.

II. $a_n\rightarrow\bar{x}$, $b_n\rightarrow\bar{x}$.

III. If l_n is the linear function defined by $l_n(a_n)=f(a_n)$, $l_n(b_n)=f(b_n)$, then

$$l_n \leqq f \text{ in } a_n \leqq x \leqq b_n.$$

IV. For $n=1$ we have, on account of 3.4, the stronger relation

$$l_1 < f \text{ in } a_1 < x < b_1.$$

V. All the lines $y=l_n(x)$ are parallel to each other.

3.6. From III in 3.5 we infer that

$$I(l_n, x_n, h_n, \alpha) \leqq I(f, x_n, h_n, \alpha), \tag{43}$$

where

$$x_n = \frac{a_n+b_n}{2}, \quad h_n = \frac{b_n-a_n}{2},$$

and also that

$$A(f, x_n, h_n, \beta) = A(l_n, x_n, h_n, \beta). \tag{44}$$

From (43), (44), (42) we see that

$$I(l_n, x_n, h_n, \alpha) \leqq A(l_n, x_n, h_n, \beta). \tag{45}$$

From IV in 3.5 it follows that for $n=1$ we have the stronger inequality

$$I(l_1, x_1, h_1, \alpha) < A(l_1, x_1, h_1, \beta). \tag{46}$$

Let us put

$$u_n = \frac{\min[l_n(a_n), l_n(b_n)]}{\max[l_n(a_n), l_n(b_n)]}.$$

Then

$$0 < u_n \leqq 1.$$

If $u_n=1$ for some n, then $l_n(x)$ is constant. Since all the lines $y=l_n(x)$ are

† Hint: Move the line $y=l_1(x)$ upward, parallel to itself, until it reaches a final position in which it still has some point in common with $y=f(x)$.

parallel to each other, we see that either $u_n=1$ for all values of n, or $0<u_n<1$ for all values of n.

3.7. Suppose first that $u_n=1$ for all values of n. Then, in particular, $l_1(x)$ reduces to some constant c, and (46) reduces to $c<c$. Thus this case is impossible.

3.8. We have therefore $0<u_n<1$ for all values of n. On account of II in 3.5 we have

$$u_n \to 1.$$

As observed in 2.3, the inequality (45) is equivalent to the fact that the function $\phi(u)$ of 1.5 satisfies

$$\phi(u_n) \leqq 0. \tag{47}$$

On account of 1.6, we have an $\epsilon>0$ such that

$$\operatorname{sgn} \phi(u) = -\operatorname{sgn}(3\beta-\alpha-2) \text{ for } 1-\epsilon<u<1.$$

Hence, with regard to (41),

$$\phi(u) > 0 \text{ for } 1-\epsilon<u<1. \tag{48}$$

Since $u_n \to 1$, (48) and (47) obviously contradict each other for large values of n. Thus the assumption that $f(x)$ is not convex is shown to lead to a contradiction.

3.9. We drop now the last three restrictions in (41) and we assume only that $3\beta-\alpha-2<0$. If $\eta>0$ is sufficiently small, then $\bar\alpha=\alpha-\eta$, $\bar\beta=\beta+\eta$ will satisfy all four conditions stated in (41). Hence, as proved above, $(\bar\alpha, \bar\beta) \subset \bar E$. From 3.2 it follows then that $(\alpha, \beta)=(\bar\alpha+\eta, \bar\beta-\eta)$ also belongs to $\bar E$.

3.10. It remains to show that if

$$3\beta-\alpha-2=0, \tag{49}$$

then $(\alpha, \beta) \subset \bar E$. Let $f(x)$ be any positive and continuous function in $x_1<x<x_2$ which satisfies

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta) \tag{50}$$

for all values of x and h such that $x_1<x-h<x+h<x_2$. Let γ be a constant such that

$$\gamma > 1,$$

and put

$$f(x)^\gamma = g(x).$$

Then it follows from (50) that $g(x)$ satisfies the inequality

$$I(g, x, h, \bar\alpha) \leqq A(g, x, h, \bar\beta), \tag{51}$$

where

$$\bar{\alpha} = \alpha/\gamma, \quad \bar{\beta} = \beta/\gamma.$$

From (49) we get

$$3\bar{\beta} - \bar{\alpha} - 2 = \frac{3\beta - \alpha - 2\gamma}{\gamma} = \frac{2(1-\gamma)}{\gamma} < 0. \tag{52}$$

But then, on account of 3.9, (51) implies that $g(x)$ is convex. That is to say: $f(x)^\gamma$ is convex, whenever $\gamma > 1$. Allowing $\gamma \to 1$ we conclude finally that $f(x)$ itself is also convex. Thus we see that (50) implies the convexity of $f(x)$, which proves that $(\alpha, \beta) \subset \bar{E}$.

4. Miscellaneous applications

4.1. In what precedes, we were concerned with convex functions. The lemmas developed in §1 cover however the case of concave functions also. Since the proofs result by obvious modifications of those we used for convex functions, we restrict ourselves to statements of results.

4.2. *Let us define the set E^* as consisting of all pairs (α, β) such that the following assertion is true: every function $f(x)$, which is positive, continuous and concave in $x_1 < x < x_2$, satisfies the inequality $I(f, x, h, \alpha) \geqq A(f, x, h, \beta)$ for all values of x and h such that $x_1 < x-h < x+h < x_2$.*

Theorem. *A pair (α, β) belongs to E^* if and only if one of the following five conditions is satisfied.*†

I. $\alpha \leqq -2$ *and* $\beta \leqq \dfrac{\alpha+2}{3}$.

II. $-2 \leqq \alpha \leqq -1$ *and* $\beta \leqq 0$.

III. $-1 \leqq \alpha \leqq -\dfrac{1}{2}$ *and* $\beta \leqq \dfrac{\alpha \log 2}{\log(\alpha+1)}$.

IV. $-\dfrac{1}{2} \leqq \alpha \leqq 1$ *and* $\beta \leqq \dfrac{\alpha+2}{3}$.

V. $1 \leqq \alpha$ *and* $\beta \leqq \dfrac{\alpha \log 2}{\log(\alpha+1)}$.

4.3. *Let us define the set $\bar{E}^*$ as consisting of all pairs (α, β) for which the following assertion is true: if a function $f(x)$, which is positive and continuous in $x_1 < x < x_2$, satisfies the inequality $I(f, x, h, \alpha) \geqq A(f, x, h, \beta)$ for all values of x and h such that $x_1 < x-h < x+h < x_2$, then $f(x)$ is concave in $x_1 < x < x_2$.*

† In terms of the functions $\psi_1(\alpha)$, $\psi_2(\alpha)$, $\psi_3(\alpha)$, used in the second footnote on p. 269, the set E^* may be described as consisting of all those points (α, β) for which

$$\beta \leqq \min\left[\psi_1(\alpha), \psi_2(\alpha), \psi_3(\alpha)\right].$$

THEOREM. *The pair* (α, β) *belongs to* $\bar{E}^*$ *if and only if* $3\beta-\alpha-2\geqq 0$.

4.4. We proceed now to present a few applications of the preceding results.

We shall say that the inequality

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta) \tag{53}$$

expresses a *characteristic property of positive and continuous convex functions*, if the following assertion is true: a function $f(x)$, positive and continuous in $x_1<x<x_2$, is convex there if and only if it satisfies the inequality (53) for all values of x and h such that $x_1<x-h<x+h<x_2$.

It is then clear that (53) expresses a characteristic property of positive and continuous convex functions if and only if the pair (α, β) is in both sets E and $\bar{E}$, defined in 0.13. From 0.13 we infer therefore the following

THEOREM. *The inequality*

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta)$$

expresses a characteristic property of positive and continuous convex functions if and only if (1) $3\beta-\alpha-2=0$ *and* (2) *either* $-2\leqq\alpha\leqq-\frac{1}{2}$ *or* $1\leqq\alpha<+\infty$.

4.5. In a similar fashion, from 4.2 and 4.3, we have the

THEOREM. *The inequality*

$$I(f, x, h, \alpha) \geqq A(f, x, h, \beta)$$

expresses a characteristic property of positive and continuous concave functions if and only if (1) $3\beta-\alpha-2=0$ *and* (2) *either* $\alpha\leqq-2$ *or* $-\frac{1}{2}\leqq\alpha\leqq 1$.

4.6. If we put $\beta=0$ in the theorem of 4.4, we see that

The inequality

$$\left[\frac{1}{2h}\int_{-h}^{h} f(x+\xi)^{-2}d\xi\right]^{-1/2} \leqq \text{geometric mean of } f(x-h) \text{ and } f(x+h)$$

is characteristic for positive and continuous convex functions. There does not exist any other characteristic inequality of the form

$$I(f, x, h, \alpha) \leqq (f(x-h)f(x+h))^{1/2}.$$

For $\beta=-1$, we obtain the following from 4.4:

There does not exist any inequality of the form

$$I(f, x, h, \alpha) \leqq \text{harmonic mean of } f(x-h) \text{ and } f(x+h),$$

which would be characteristic for positive and continuous convex functions.

We leave it to the reader to formulate the corresponding theorems for concave functions, on the basis of 4.5.

4.7. The results developed in this paper imply an infinity of *sharp inequalities for convex and for concave functions*. We wish to illustrate this point on the following special example. *Let us ask for the best inequality of the form*

$$I(f, x, h, 2) \leqq A(f, x, h, \beta), \tag{54}$$

which is satisfied by all positive and continuous convex functions. Since the right-hand side of (54) is an increasing function of β, our problem requires the determination of the smallest value of β such that (54) holds for every continuous and positive convex function. In other words, we have to determine the smallest β such that the pair $(2, \beta)$ is in the set E defined in 0.13. It follows from 0.13 that this smallest value is 4/3. Hence:

The inequality

$$\left[\frac{1}{2h}\int_{-h}^{h} f(x+\xi)^2 d\xi\right]^{1/2} \leqq \left[\frac{f(x-h)^{4/3}+f(x+h)^{4/3}}{2}\right]^{3/4} \tag{55}$$

is a sharp inequality for positive and continuous convex functions, in the following sense. Every positive and continuous convex function satisfies (55), *but for every* $\epsilon>0$ *there exist positive and continuous convex functions which do not satisfy the inequality*

$$\left[\frac{1}{2h}\int_{-h}^{h} f(x+\xi)^2 d\xi\right]^{1/2} \leqq \left[\frac{f(x-h)^{4/3-\epsilon}+f(x+h)^{4/3-\epsilon}}{2}\right]^{1/(4/3-\epsilon)}.$$

Clearly, our results permit us to determine, for every fixed α and for every fixed β, the sharp inequality of the form $I(f, x, h, \alpha) \leqq A(f, x, h, \beta)$ for convex functions, and to answer the corresponding questions for concave functions.

4.8. As a last application, we shall present and discuss a theorem which gives a complete answer to the question raised in 0.9 concerning the classes C_γ and C_δ^*.

THEOREM. *The relation* $C_\gamma \equiv C_\delta^*$ *holds if and only if* (1) $2\gamma+\delta-3=0$ *and* (2) *either* $\gamma \leqq 1$ *or* $\gamma \geqq 2$.

This theorem is an immediate consequence of our results concerning the sets E, $\overline{E}$, E^*, $\overline{E}^*$. We shall reproduce the reasoning in the case $\gamma>0$. The verification of the theorem for the cases $\gamma<0$ and $\gamma=0$ will be left to the reader.

4.9. Suppose then that $\gamma>0$ and let us ask for all those values of δ, if any, for which $C_\gamma \equiv C_\delta^*$.

If $f(x)$ is positive and continuous in $x_1<x<x_2$, then $f\subset C_\gamma$ means that $g=f^\gamma$ is convex. On the other hand, $f\subset C_\delta^*$ means that $I(f, x, h, \delta)\leqq A(f, x, h, 1)$ for all values of x and h such that $x_1<x-h<x+h<x_2$. This inequality can be written, in terms of the function $g=f^\gamma$, as

$$(56) \qquad I(g,\, x,\, h,\, \delta/\gamma) \leqq A(g,\, x,\, h,\, 1/\gamma).$$

Hence, $C_\gamma\equiv C_\delta^*$ is equivalent to the fact that (56) is a necessary and sufficient condition for the convexity of g. In other words, $C_\gamma\equiv C_\delta^*$ is equivalent to the fact that (56) expresses a characteristic property of positive and continuous convex functions. Putting

$$\alpha = \frac{\delta}{\gamma}, \quad \beta = \frac{1}{\gamma},$$

the theorem of 4.4 yields the following necessary and sufficient conditions:

$$\text{(i)} \qquad 3\beta - \alpha - 2 = \frac{3 - \delta - 2\gamma}{\gamma} = 0, \text{ and}$$

$$\text{(ii)} \qquad \text{either } -2 \leqq \alpha = \frac{\delta}{\gamma} \leqq -\frac{1}{2} \text{ or } 1 \leqq \alpha = \frac{\delta}{\gamma}.$$

These conditions are equivalent to the following set:

$$\text{(i}'\text{)} \qquad \delta = 3 - 2\gamma, \text{ and}$$

$$\text{(ii}'\text{)} \qquad \text{either } -2\gamma \leqq 3 - 2\gamma \leqq -\tfrac{1}{2}\gamma \text{ or } \gamma \leqq 3 - 2\gamma,$$

which is finally equivalent to the set

$$\text{(i}''\text{)} \qquad 2\gamma + \delta - 3 = 0, \text{ and}$$

$$\text{(ii}''\text{)} \qquad \text{either } \gamma \geqq 2 \text{ or } \gamma \leqq 1.$$

These are, however, exactly the conditions stated in 4.8.

4.10. It is interesting to compare the theorem in 4.8 with the remark made in 0.9. According to 0.9, we have

$$(57) \qquad C_\delta^* \subset C_\gamma \text{ for } 2\gamma + \delta - 3 = 0.$$

For what values of γ *does the converse hold?* In other words: for what values of γ is it true that

$$(58) \qquad C_\gamma \subset C_\delta^* \text{ for } 2\gamma + \delta - 3 = 0?$$

With regard to (57), if (58) holds for a certain γ, then $C_\gamma\equiv C_\delta^*$, where γ and

δ are related by the equation $2\gamma+\delta-3=0$. According to 4.8, we have this situation if and only if either $\gamma \leqq 1$ or $\gamma \geqq 2$. Summing up:† *we have*

$$C_\delta^* \subset C_\gamma \textit{ for } 2\gamma + \delta - 3 = 0$$

without any further restriction on γ. *The converse, namely the relation*

$$C_\gamma \subset C_\delta^* \textit{ for } 2\gamma + \delta - 3 = 0,$$

holds however if and only if $\gamma \leqq 1$ *or* $\gamma \geqq 2$. *For* $1 < \gamma < 2$ *the converse is false.*

† Cf. 0.9 for the origin of this theorem.

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