INTEGRATION IN GENERAL ANALYSIS*

BY

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L. M. Graves† and others have defined and developed the theory of the Riemann integral for functions whose values are in a complete linear vector space. T. H. Hildebrandt‡ and S. Bochner§ have defined the Lebesgue integral for the same type of function. The present paper, which approaches the theory of the integral in a manner analogous to the Cantor definition of a real number, is concerned chiefly with the convergence of a sequence of integrals and is not as extensive in scope as that of Bochner which contains certain results pertaining to multiple integrals, Fourier series, and the class L_p. In what follows no use is made of the theory of integration for numerically valued functions other than a knowledge of the properties of an additive class of point sets and of a completely additive function on such a class. In fact the method when applied to such functions seems in many ways more direct than the classical one. The proofs in the section on types of convergence are omitted since they may be carried through exactly as in the case of real-valued functions. In the last section it is shown how the theory holds, with slight modifications, for a function having an arbitrary metric space as its domain and a Banach space for its range.

1. **Basis.** A class of point sets is said to be *additive* if for every pair of sets E, D and every sequence $\{E_n\}$ of disjoint sets in the class the sets $E-D$, $\sum E_n$ are also in the class. A function $\alpha(E)$ on an additive class of sets A is said to be *completely additive* if for every sequence $\{E_n\}$ of disjoint sets in A, $\alpha(\sum E_n)=\sum\alpha(E_n)$. In what follows, A will be used to denote an additive class of point sets which contains all Borel measurable sets belonging to a fundamental bounded and closed interval J of a euclidean space of n dimensions. The notation $\alpha(E)$ will always be used for a completely additive function on A to the real number system and $\beta(E)$ will stand for the total variation of α on E. Radon‖ has constructed such systems (A, α, β) corresponding to a

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† Graves, *Riemann integration and Taylor's theorem in general analysis*, these Transactions, vol. 29 (1927), pp. 163–177.

‡ Hildebrandt, *Lebesgue integration in general analysis*, this Bulletin, vol. 33 (1927), p. 646.

§ Bochner, *Integration von Funktionen deren Werte die Elemente eines Vektorraumes sind*, Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276.

‖ Radon, *Theorie und Anwendungen der absolut additiven Mengenfunktionen*, Sitzungsberichte der Akademie der Wissenschaften, Wien, Mathematisch-Naturwissenschaftliche Klasse, IIa, vol. 122, pp. 1295–1438.

given function of bounded variation. The properties of α and A which are used below are consequences of their additive nature and not of any particular method of construction, and hence, since we postulate the existence of α, Radon's procedure is not needed in what follows.

2. **Types of convergence.** The functions $f(P)$ to be considered throughout are functions on a subset of J to a complete linear vector space X. The sequence $\{f_n(P)\}$ is said to approach the function $f(P)$ (converge) *almost uniformly with respect to α on E* in case for every $\epsilon>0$ there exist sets E_ϵ, E_ϵ' such that E_ϵ' is in A, $E_\epsilon' \supset E-E_\epsilon$, $\beta(E_\epsilon')<\epsilon$ and $f_n(P)\rightarrow f(P)$ ($\{f_n(P)\}$ converges) uniformly on E_ϵ. The sequence $\{f_n(P)\}$ is said to approach $f(P)$ *approximately with respect to α on E* if for every integer n and every $\epsilon>0$ there exists a set $E'(n, \epsilon)$ in A containing the set $E(n, \epsilon)=E[\|f_n(P)-f(P)\|>\epsilon]$ and such that $\lim_n \beta[E'(n, \epsilon)]=0$. The *convergence of a sequence approximately with respect to α on E* is defined similarly by replacing the sets $E'(n, \epsilon)$, $E(n, \epsilon)$ by the sets $E'(m, n, \epsilon)$ and $E(m, n, \epsilon)=E[\|f_m(P)-f_n(P)\|>\epsilon]$ respectively. In what follows, the notations $E(n, \epsilon)$, $E(m, n, \epsilon)$ are used as above. The notation O_α will sometimes be used for a set contained in one on which $\beta=0$. It is assumed that all such sets are in A.

LEMMA 1. *Of the three types of convergence (applied either to the approach to a function or to the convergence of a sequence)*

(1) *almost uniformly with respect to α on E,*

(2) *almost everywhere with respect to α on E,*

(3) *approximately with respect to α on E,*

(1) *implies* (2) *and* (3), *and if $E(n, \epsilon)$ and $E(m, n, \epsilon)$ are in A, then* (2) *implies* (1) *and* (3).

LEMMA 2. *If E is in A, $f_n(P)\rightarrow f(P)$ on $E-O_\alpha$, and the set $E[\|f_n(P)\|>\epsilon]$ is in A for every n and $\epsilon>0$, then the set $E[\|f(P)\|>\epsilon]$ is in A.*

LEMMA 3. *If $f_n(P)\rightarrow f(P)$ and $f_n(P)\rightarrow f'(P)$ approximately with respect to α on E then $f=f'$ on $E-O_\alpha$ and $\{f_n(P)\}$ converges approximately with respect to α on E.*

LEMMA 4. *If the sequence $\{f_n(P)\}$ converges approximately with respect to α on E, E being a set of A, then there exists a function $f(P)$ on E to X and a subsequence $\{f_{n_i}(P)\}$ such that $f_{n_i}(P)\rightarrow f(P)$ almost uniformly with respect to α on E. Furthermore if $\{f_{m_i}(P)\}$ is any subsequence and $f^*(P)$ any function on E to X such that $f_{m_i}(P)\rightarrow f^*(P)$ approximately with respect to α on E then $f=f^*$ on $E-O_\alpha$. In case the functions f_m are uniformly continuous on E there exists a set F belonging to A and contained in E such that F is closed in E and $\beta(E-F)$ is arbitrarily small, and f as on F is uniformly continuous on F.*

Let $U=(u)$, $U'=(u')$, be arbitrary sets of elements and R, R' be transitive order relations on UU, $U'U'$ respectively having the composition property as defined by Moore and Smith.† Then with limits defined in terms of these order relations we have

LEMMA 5. *If $x(u, u')$ is on UU' to X and $\lim_u x(u, u')$ exists for each u' and $\lim_{u'} x(u, u')$ exists uniformly with respect to u then the following limits exist and are equal*: $\lim_{u,u'} x(u, u')$, $\lim_u \lim_{u'} x(u, u')$, $\lim_{u'} \lim_u x(u, u')$.

3. **The integral.** Let $S_0(E)$ denote the class composed of all functions uniformly continuous on E. Let $\pi_E=(E_1, \cdots, E_k)$ represent a partition of the set E which is supposed to be in A, and (P, P') represent the euclidean distance from P to P'. The norm, $n(\pi_E)$, of the partition π_E is defined as the least upper bound of (P_i, P_i') for P_i and P_i' in E_i and for $i=1, \cdots, k$. Then the distance function

$$\|f\| = \int_E \|f(P)\| d\beta = \lim_{n(\pi)=0} \sum_{\pi_E} \|f(P_k)\|\beta(E_k)$$

(P_k being any point in E_k) is surely defined for f in $S_0(E)$. For let

$$S = \sum_{\pi} \|(fP_i)\|\beta(E_i), \quad S' = \sum_{\pi'} \|f(P_k')\|\beta(E_k')$$

be the sums corresponding to the partitions $\pi=(E_i)$, $\pi'=(E_k')$ respectively. Then

$$S = \sum_i \|f(P_i)\| \sum_k \beta(E_iE_k') = \sum_{i,k} \|f(P_i)\|\beta(E_iE_k')$$

and

$$|S - S'| = \left|\sum_{i,k} (\|f(P_i)\| - \|f(P_k')\|)\beta(E_iE_k')\right|$$
$$\leqq \omega[n(\pi) + n(\pi')]\beta(E)$$

where

$$\omega[\delta] = \sup_{(P,P')\leqq\delta} \left| \|f(P)\| - \|f(P')\| \right|.$$

Thus the $\lim_{n(\pi)=0} S$ exists for f in $S_0(E)$. The integrals

$$\int_E f(P)d\alpha, \qquad \int_E f(P)d\beta, \qquad \int_E \|f(P)\|d\alpha$$

are defined in a similar manner.

By a *Cauchy sequence* of functions in $S_0(E)$ is meant one for which $\|f_m - f_n\| \to 0$, and two Cauchy sequences $\{f_m\}$ and $\{g_m\}$ are said to be *equivalent* in case $\|f_m - g_m\| \to 0$.

† Moore and Smith, *A general theory of limits*, American Journal of Mathematics, vol. 44 (1922), p. 103.

LEMMA 6. *To each class of equivalent Cauchy sequences of functions in $S_0(E)$ corresponds uniquely except for a set on which $\beta=0$, a function $f(P)$ on E to X such that if $\{f_m\}$ is any sequence of functions in the class defining f then there exists a subsequence $\{f_{m_i}\}$ with $f_{m_i}(P)\to f(P)$ almost uniformly with respect to α on E. Furthermore the limits*

$$\lim_n \int_E f_n(P)d\alpha, \qquad \lim_n \int_E f_n(P)d\beta,$$

$$\lim_n \int_E \|f_n(P)\|d\alpha, \qquad \lim_n \int_E \|f_n(P)\|d\beta$$

all exist and are independent of the particular Cauchy sequence in the class of equivalent Cauchy sequences.

The set $E(m, n, \epsilon)$ is the product of a region and the set E, and is thus in A. Now

$$\epsilon\beta[E(m, n, \epsilon)] \leqq \int_{E(m,n,\epsilon)} \|f_m(P) - f_n(P)\|d\beta$$

$$\leqq \int_E \|f_m(P) - f_n(P)\|d\beta \to 0$$

so that $\{f_m\}$ converges approximately with respect to α on E. The existence of $f(P)$ follows from Lemma 4, and its uniqueness may be established in a manner similar to that used in the proof of Lemma 4. Since

$$\left\|\int_E f_n(P)d\alpha - \int_E f_m(P)d\alpha\right\| \leqq \int_E \|f_m(P) - f_n(P)\|d\beta \to 0,$$

it follows that the $\lim_n \int_E f_n(P)d\alpha$ exists. It is independent of the sequence $\{f_m\}$ since

$$\left\|\int_E f_n(P)d\alpha - \int_E g_n(P)d\alpha\right\| \leqq \|f_n - g_n\| \to 0$$

if $\{f_n\}$ and $\{g_n\}$ are equivalent. In the same manner the other limits may be shown to exist.

A function $f(P)$ is said to be *summable with respect to α on E* in case it is the correspondent in the sense of Lemma 6 of some class of equivalent Cauchy sequences of functions in $S_0(E)$. The class of such functions will be denoted by $S(E)$. The *integral* $\int_E f(P)d\alpha$ of a function in $S(E)$ is defined as the $\lim_n \int_E f_n(P)d\alpha$ where $\{f_n(P)\}$ is any sequence in the class defining f. The integrals $\int_E\|f(P)\|d\alpha$, $\int_E\|f(P)\|d\beta$, $\int_E f(P)d\beta$ are defined similarly.

Note that $\|\int_E f(P)d\alpha\| \leqq \int_E\|f(P)\|d\beta$.

THEOREM 1. *If f is in $S(E)$ then the set $E[\|f(P)\|>\epsilon]$ is in A for every $\epsilon>0$.*

This is a corollary of Lemmas 1, 2.

THEOREM 2. *Every function $f(P)$ in $S(E)$ is approachable almost uniformly with respect to α on E by a sequence $f_m(P)$ of functions uniformly continuous on E and such that $\int_E\|f_m(P)-f(P)\|d\beta\to 0$.*

The first part of the conclusion is a corollary of Lemma 6, and the second part follows from the fact that for a fixed value of m the sequence $f_m(P)-f_n(P)$ is a Cauchy sequence of functions in $S_0(E)$ belonging to the class defining $f_m(P)-f(P)$. Thus

$$\int_E \|f_m(P) - f(P)\| d\beta = \lim_n \int_E \|f_m(P) - f_n(P)\| d\beta$$

and the conclusion is immediate.

THEOREM 3. *If f is in $S(E)$ and $\epsilon>0$, then there exists a set F belonging to A and contained in E such that F is closed in E, f as on F is uniformly continuous on F and $\beta(F)\geqq\beta(E)-\epsilon$.*

This is a corollary of Theorem 2, Lemma 1, and Lemma 4.

THEOREM 4. *The space $S(E)$ of functions summable with respect to α on E with the norming operation $\|f\|=\int_E\|f(P)\|d\beta$ is a complete linear vector space.*

Let $\{f_m\}$ be a Cauchy sequence of functions in $S(E)$; then by Theorem 2, for every m there exists a function g_m in $S_0(E)$ such that $\|f_m-g_m\|<1/m$. Now

$$\|g_m - g_n\| \leqq \|g_m - f_m\| + \|f_m - f_n\| + \|f_n - g_n\|,$$

so that $\{g_m\}$ is a Cauchy sequence of functions in $S_0(E)$ and thus defines a function $f(P)$ in $S(E)$ such that $\|g_n-f\|\to 0$. Thus

$$\|f_m - f\| \leqq \|f_m - g_m\| + \|g_m - f\| \to 0,$$

and $S(E)$ is complete. The rest of the proof follows immediately from the definition of the terms involved.

A function $h(E)$ on an additive class contained in A to a linear vector space is said to be *absolutely continuous with respect to α* in case $\lim_{\beta(E)=0} h(E)=0$.

THEOREM 5. *If $f(P)$ is in $S(E)$ then the integrals $\int_e f(P)d\alpha$, $\int_e f(P)d\beta$, $\int_e\|f(P)\|d\alpha$, $\int_e\|f(P)\|d\beta$ are all absolutely continuous with respect to α.*

If $f(P)$ is in $S_0(E)$ then $\|f(P)\|$ is bounded on E and so

$$\left\| \int_e f(P)d\alpha \right\| \leqq M\beta(e).$$

Now if f_m is a Cauchy sequence in $S_0(E)$ with $\|f_m - f\| \to 0$, we have for each m

$$\lim_{\beta(e)=0} \int_e f_m(P)d\alpha = 0,$$

and since

$$\left\| \int_e (f_m(P) - f(P))d\alpha \right\| \leqq \|f_m - f\| \to 0,$$

it follows that

$$\lim_m \int_e f_m(P)d\alpha = \int_e f(P)d\alpha \text{ uniformly with respect to } e.$$

Thus by Lemma 5

$$\lim_{\beta(e)=0} \int_e f(P)d\alpha = 0.$$

THEOREM 6. *If $f(P)$ is in $S(E)$ then each of the integrals listed in Theorem 5 is a completely additive function on the class $A(E)$ composed of all sets e in A and such that $e \subset E$.*

If $f(P)$ is in $S_0(E)$ and e_i, $i=1, \cdots, k$, are disjoint sets in $A(E)$, it follows from the definition of the integral on the class $S_0(E)$ that

$$\int_{\sum_{i=1}^k e_i} f(P)d\alpha = \sum_{i=1}^k \int_{e_i} f(P)d\alpha.$$

Thus for any $f(P)$ in $S(E)$ the same equation holds. Now for a sequence $\{e_i\}$ of disjoint sets with $\sum e_i = e$,

$$\left\| \int_e f(P)d\alpha - \sum_{i=1}^m \int_{e_i} f(P)d\alpha \right\| = \left\| \int_{e - \sum_{i=1}^m e_i} f(P)d\alpha \right\| \to 0$$

by Theorem 5.

THEOREM 7. *If f and f_m are in $S(E)$ for every integer m and if $\|f_m - f\| \to 0$, then $f_m \to f$ approximately with respect to α on E, $\|f_m\|$ is bounded, and $\int_e \|f_m(P)\| d\beta$ is absolutely continuous with respect to α uniformly with respect to m.*

Since $f_m - f$ is in $S(E)$ the set $E(m, \epsilon)$ is by Theorem 1 in A. Now

$$\epsilon\beta[E(m, \epsilon)] \leqq \int_{E(m,\epsilon)} \|f_m(P) - f(P)\| d\beta \leqq \|f_m - f\| \to 0$$

so that $f_m \to f$ approximately with respect to α on E. Also since $\|f_m\| \leqq \|f_m - f\| + \|f\|$, $\|f_m\|$ is bounded. The remaining part of the conclusion follows from the inequality

$$\int_e \|f_n(P)\| d\beta \leqq \|f_n - f\| + \int_e \|f(P)\| d\beta$$

and Theorem 6.

THEOREM 8. *If* $\{f_m(P)\}$ *is a sequence of functions in* $S(E)$ *and* $f_n(P) \to f(P)$ *approximately with respect to* α *on* E, *and if* $\int_e \|f_m(P)\| d\beta$ *is absolutely continuous with respect to* α *uniformly with respect to* m, *then* f *is in* $S(E)$ *and* $\int_e \|f_n(P) - f(P)\| d\beta \to 0$ *uniformly for* e *in* $A(E)$.

By Lemma 3 the sequence $\{f_m(P)\}$ converges approximately. By Theorem 1, $E(m, n, \epsilon)$ is in A. Now

$$\int_E \|f_m(P) - f_n(P)\| d\beta = \int_{E-E(m,n,\epsilon)} + \int_{E(m,n,\epsilon)}$$

$$\leqq \epsilon\beta[E - E(m, n, \epsilon)] + \int_{E(m,n,\epsilon)}$$

Thus $\|f_m - f_n\| \to 0$ and by Theorem 4 there exists a function f' such that $\|f_m - f'\| \to 0$. By Theorem 7 and Lemma 3 it is seen that $f = f'$ on $E - O_\alpha$ and thus f is in $S(E)$. Since $\int_e \|f_m(P) - f(P)\| d\beta \leqq \|f_m - f\|$ the proof is complete.

COROLLARY 1. *If the sequence* $\{f_n(P)\}$ *of functions in* $S(E)$ *approach* $f(P)$ *approximately with respect to* α *on* E, *and if there exists a function* $g(P)$ *in* $S(E)$ *such that* $\|f_n(P)\| \leqq \|g(P)\|$ *on* $E - O_\alpha$ *for every* n, *then* f *is in* $S(E)$ *and* $\int_e \|f_m(P) - f(P)\| d\beta \to 0$ *uniformly for* e *in* $A(E)$.

COROLLARY 2. *If* f *is in* $S(E)$ *and* $\phi(P)$ *is a real-valued function summable with respect to* α *on* E *and bounded on* $E - O_\alpha$, *then* $\phi(P)f(P)$ *is in* $S(E)$.

THEOREM 9. *If* f_m *and* f *are in* $S(E)$ *and* $f_m \to f$ *approximately with respect to* α *on* E, *then the following assertions are equivalent*:

(1) $\lim_m \int_e f_m d\alpha = \int_e f d\alpha$ *on* $A(E)$.
(2) $\lim_m \int_e f_m d\alpha = \int_e f d\alpha$ *uniformly on* $A(E)$.
(3) *The* $\lim_m \int_e f_m d\alpha$ *exists on* $A(E)$.
(4) $\lim_{\beta=0} \overline{\lim}_m \|\int_e f_m d\alpha\| = 0$.
(5) $\lim_{\beta=0} \int_e f_m d\alpha = 0$ *uniformly*.

The proof will be made by demonstrating the following implications: (4)→(1)→(3)→(5)→(2)→(4), where each arrow is directed from hypothesis to conclusion. To see that (4)→(1) first note that (4) is equivalent to the statement that for every $\epsilon>0$ there exists a $\delta>0$ such that for each e in $A(E)$ with $\beta(e)<\delta$ there is an n_e such that $\|\int_e f_n d\alpha\|<\epsilon$ for all $n\geqq n_e$. Now let it be supposed that $f_n\rightarrow f$ almost uniformly with respect to α on E. To show that $\int_e f_m d\alpha\rightarrow\int_e f d\alpha$, note that

$$\left\|\int_E (f_m - f)d\alpha\right\| \leqq \left\|\int_{E-e} (f_m - f)d\alpha\right\| + \left\|\int_e (f_m - f)d\alpha\right\|.$$

Fix e such that $\beta(E-e)<\delta$ and $f_m\rightarrow f$ uniformly on e. For this e there is an m_1 such that

$$\left\|\int_{E-e} (f_m(P) - f(P))d\alpha\right\| < \epsilon, \quad \left\|\int_e (f_m(P) - f(P))d\alpha\right\| < \epsilon, \quad m \geqq m_1.$$

Thus $\int_E f_m d\alpha\rightarrow\int_E f d\alpha$. The same proof holds for an arbitrary e in $A(E)$. The same conclusion follows if $f_n\rightarrow f$ approximately since by Lemma 4 and the above proof every subsequence of $\{\int_e f_m d\alpha\}$ contains a subsequence approaching $\int_e f d\alpha$. That (1)→(3) is obvious. For the proof of the implication (3)→(5) we refer to a paper by Saks.* Now (5)→(2), for

$$\left\|\int_e (f_m - f)d\alpha\right\| \leqq \left\|\int_{e(m,\epsilon)} (f_m - f)d\alpha\right\| + \left\|\int_{e-e(m,\epsilon)} (f_m - f)d\alpha\right\|.$$

Now

$$\left\|\int_{e-e(m,\epsilon)} (f_m - f)d\alpha\right\| \leqq \epsilon\beta(E),$$

and there exists an m_0 and a $\delta>0$ such that $\beta[E(m, \epsilon)]<\delta$ for $m\geqq m_0$ and

$$\left\|\int_e (f_m(P) - f(P))d\alpha\right\| < \epsilon$$

for all m and e with $\beta(e)<\delta$. Thus since $e(m, \epsilon)\subset E(m, \epsilon)$ it follows that

$$\left\|\int_e (f_m(P) - f(P))d\alpha\right\| \leqq \epsilon(1 + \beta(E)) \text{ for } m \geqq m_0.$$

Finally in view of Theorem 6 and the fact that (2) implies that

$$\lim_m \left\|\int_e f_m d\alpha\right\| = \left\|\int_e f d\alpha\right\|,$$

the implication (2)→(4) is obvious.

* *Addition to the note on some functionals*, these Transactions, vol. 35 (1933), p. 967.

THEOREM 10. *If E is in A; α_m, α are completely additive functions on $A(E)$; $\alpha_m(E_i^k)\to\alpha(E_i^k)$ for each set E_i^k found in a sequence of partitions $\pi_i(E)=(E_i^k)$ with $n(\pi_i)\to 0$; and if the sequence $\{\beta_m(E)\}$ is bounded; then*

$$\lim_m \int_E f d\alpha_m = \int_E f d\alpha$$

for any function $f(P)$ in $S_0(E)$.

From the proof given at the beginning of §3 and the boundedness of the sequence $\{\beta_m(E)\}$ it follows that

$$\lim_{i=\infty} \sum_k f(P_i^k)\alpha_m(E_i^k) = \int_E f d\alpha_m$$

uniformly with respect to m. Also for each i

$$\lim_m \sum_k f(P_i^k)\alpha_m(E_i^k) = \sum_k f(P_i^k)\alpha(E_i^k).$$

The conclusion follows from Lemma 5.

THEOREM 11. *If E is in A; α_m, α are completely additive on $A(E)$; $f(P)$ summable with respect to α_m and α on E; $\alpha_n(e)\to\alpha(e)$ on $A(E)$; and if the sequence $\{\beta_m(E)\}$ is bounded; then*

$$\int_e f d\alpha_m \to \int_e f d\alpha \text{ on } A(E)$$

provided

$$\lim_{\beta(e)=0} \overline{\lim_{m=\infty}} \left\| \int_e f d\alpha_m \right\| = 0.$$

We have

$$\left\| \int_E f d\alpha_m - \int_E f d\alpha \right\| \leqq \left\| \int_{E-e} f d\alpha_m \right\| + \left\| \int_e f d\alpha_m - \int_e f d\alpha \right\| + \left\| \int_{E-e} f d\alpha \right\|.$$

Now for $\epsilon > 0$ there is a $\delta > 0$ such that for every $E-e$ with $\beta(E-e)<\delta$ there is an integer m_0 such that $m \geqq m_0$ implies

$$\left\| \int_{E-e} f d\alpha_m \right\| < \epsilon, \qquad \left\| \int_{E-e} f d\alpha \right\| < \epsilon.$$

Fix e with $\beta(E-e)<\delta$ so that f as on e is uniformly continuous on e. Then for m sufficiently large (Theorem 10)

$$\left\| \int_e f d\alpha_m - \int_e f d\alpha \right\| < \epsilon$$

and thus for m sufficiently large

$$\left\| \int_E f d\alpha_m - \int_E f d\alpha \right\| < 3\epsilon.$$

The conclusion follows for an arbitrary e in $A(E)$ since all the hypotheses are satisfied by e when they are by E.

COROLLARY. *Suppose E in A; α_m, α, γ completely additive on $A(E)$; $\alpha_m(e)\to\alpha(e)$ on $A(E)$; $\beta_m(e)\leqq\gamma(e)$ on $A(E)$ for every m; and f is summable with respect to γ on E; then f is summable with respect to α_m, α on E and $\int_e f d\alpha_m \to \int_e f d\alpha$ on $A(E)$.*

4. **The generalization to the case where J is a metric space.** The above theory of the integral holds if the domain of the function $f(P)$ is a general metric space rather than a euclidean space of n dimensions. The few alterations and additions in the arguments that are necessary will be enumerated here.

J will now be interpreted as an arbitrary metric space not necessarily of bounded diameter. The class $S_0(E)$ will be the class of all functions $f(P)$ *uniformly continuous and bounded on E*. By a partition of the set E in J will be meant a set (E_η) of disjoint sets (possibly non-denumerable in number) such that $E=\sum E_\eta$ and which is found in the following manner. Let ϵ be an arbitrary positive number and $(P)_\epsilon$ denote all points P' in E for which $(P,P')<\epsilon$. Take any point P_1 in E; then $E_1=(P_1)_\epsilon$. In general P_η is any point in $E-\sum_{\zeta<\eta}E_\zeta$ and $E_\eta=(P_\eta)_\epsilon-\sum_{\zeta<\eta}E_\zeta$. The sets E_η form a partition $\pi(E)$ of E with $n(\pi(E))\leqq 2\epsilon$. It should be mentioned perhaps that such partitions will only be allowed in the definition to be given of $\int_E f(P)d\alpha$ and not in the definition of $\beta(E)$, the latter remaining unaltered. The partition just defined is devised to avoid assuming the separability of J as well as to eliminate the possibility of using a partition of certain connected sets each set of which consists of a single point.

Since β is completely additive, $\beta(E_\eta)=0$ for all except at most a denumerable number of the sets E_η in any partition $\pi(E)=(E_\eta)$. This follows from the fact that for an arbitrary integer m there can be at most a finite number of the sets E_η for which $\beta(E_\eta)>1/m$. If those sets E_η of the partition $\pi(E)$ for which $\beta(E_\eta)\neq 0$ are arranged into a sequence $\{E_i\}$ in *any order*, it is possible to associate with the partition $\pi(E)$ a sum

$$\sum_{i=1}^{\infty} f(P_i)\alpha(E_i)$$

where P_i now stands for any point in E_i. If $f(P)$ is in $S_0(E)$, the above sum exists and is independent of the particular arrangement of the terms in the sequence $\{E_i\}$. First the sum exists for any particular arrangement, since for $m'>m$

$$\left\| \sum_{i=m}^{m'} f(P_i)\alpha(E_i) \right\| \leqq \sup_{P\epsilon E} \|f(P)\| \sum_{i=m}^{m'} \beta(E_i).$$

Now let

$$x_1 = \sum_{i=1}^{\infty} f(P_i)\alpha(E_i), \qquad x_2 = \sum_{i=1}^{\infty} f(P_i')\alpha(E_i')$$

be the sums for two different arrangements of the sequence $\{E_i\}$. For every $\epsilon>0$ there is an integer m_1 such that, for $m \geqq m_1$,

$$\left\| x_1 - \sum_{i=1}^{m} f(P_i)\alpha(E_i) \right\| < \epsilon,$$

$$\left\| x_2 - \sum_{i=1}^{m} f(P_i')\alpha(E_i') \right\| < \epsilon,$$

$$\sum_{i=m_1}^{\infty} \beta(E_i) < \epsilon / \sup_{P\epsilon E} \|f(P)\|.$$

Now suppose m_2 the largest value of i for which the set E_i' is one of the sets $E_1, \cdots, E_{m_1}$; and E_k'', $k=1, \cdots, m_2-m_1$, are those E_i' $(1 \leqq i \leqq m_2)$ which are not found among the sets $E_1, \cdots, E_{m_1}$. Then $m_2 \geqq m_1$ and

$$\left\| \sum_{i=1}^{m_2} f(P_i')\alpha(E_i') - \sum_{i=1}^{m_1} f(P_i)\alpha(E_i) \right\| = \left\| \sum_{k=1}^{m_2-m_1} f(P_k'')\alpha(E_k'') \right\|$$

$$\leqq \sup_{P\epsilon E} \|f(P)\| \cdot \sum_{i=m_1}^{\infty} \beta(E_i) \leqq \epsilon.$$

Thus $\|x_1 - x_2\| \leqq 3\epsilon$, $x_2 = x_1$.

The integral $\int_E f(P) d\alpha$ is now defined as

$$\lim_{n(\pi)=0} \sum_{\pi(E)} f(P_i)\alpha(E_i).$$

The proof given in the text for the existence of this limit for f in $S_0(E)$ holds verbatim with the additional point involved in the justification of the equality

$$\sum_i f(P_i) \sum_k \alpha(E_i E_k') = \sum_{i,k} f(P_i)\alpha(E_i E_k').$$

This equality is established immediately with the use of Lemma 5.

In the proof of Theorem 6 another argument must be added. It is necessary to show that if $f(P)$ is in $S_0(E)$ and e', e'' are disjoint subsets of E, then

$$\sum_i f(P_i)\alpha(e_i) = \sum_i f(P_i')\alpha(e_i') + \sum_i f(P_i'')\alpha(e_i''),$$

where e_i' (e_i'') are those sets of a partition of $e'(e'')$ on which $\beta \neq 0$ and the partition of the set $e = e' + e''$ is formed by a combination of the two partitions. Let

$$x = \sum_{i=1}^{\infty} f(P_i)\alpha(e_i), \qquad x' = \sum_{i=1}^{\infty} f(P_i')\alpha(e_i'), \qquad x'' = \sum_{i=1}^{\infty} f(P_i'')\alpha(e_i'');$$

then for $\epsilon > 0$ there is an m_1 such that, for $m \geqq m_1$,

$$\left\| \sum_{i=1}^{m} f(P_i)\alpha(e_i) - x \right\| < \epsilon,$$

$$\left\| \sum_{i=1}^{m} f(P_i')\alpha(e_i') - x' \right\| < \epsilon,$$

$$\left\| \sum_{i=1}^{m} f(P_i'')\alpha(e_i'') - x'' \right\| < \epsilon,$$

$$\sum_{i=m_1}^{\infty} \beta(e_i' + e_i'') < \epsilon / \sup_{P \epsilon E} \| f(P) \|.$$

Let m_2 be the largest value of i for which the set e_i is one of the sets e_i', e_i'', $i = 1, \cdots, m_1$, and let e_i''', $i = 1, \cdots, m_2 - 2m_1$, be those of the sets e_i, $i = 1, \cdots, m_2$, which are not found among the sets e_i', e_i'', $i = 1, \cdots, m_1$; then

$$\left\| \sum_{i=1}^{m_1} [f(P_i')\alpha(e_i') + f(P_i'')\alpha(e_i'')] - \sum_{i=1}^{m_2} f(P_i)\alpha(e_i) \right\|$$

$$= \left\| \sum_{i=1}^{m_2 - 2m_1} f(P_i''')\alpha(e_i''') \right\| \leqq \sup_{P \epsilon E} \| f(P) \| \sum_{i=m_1}^{\infty} \beta(e_i' + e_i'') < \epsilon,$$

and so $\| x' + x'' - x \| < 4\epsilon$, $x = x' + x''$.

We see no way of proving Theorem 10 unless the hypothesis is strengthened to the extent that $\alpha_m \rightarrow \alpha$ on $A(E)$. It is then possible to show that

$$\lim_m \sum_\pi f(P_i)\alpha_m(E_i) = \sum_\pi f(P_i)\alpha(E_i)$$

for any partition π of the set E, where now the $\sum_\pi$ is to be taken over all sets E_η of the partition π for which any one of the inequalities $\beta(E_\eta)\neq 0$, $\beta_m(E_\eta)\neq 0$ hold. We have

$$\lim_m \sum_{i=1}^{n} f(P_i)\alpha_m(E_i) = \sum_{i=1}^{n} f(P_i)\alpha(E_i)$$

for each n, and

$$\lim_n \sum_{i=1}^{n} f(P_i)\alpha_m(E_i) = \sum_{i=1}^{\infty} f(P_i)\alpha_m(E_i)$$

uniformly with respect to m. To see this, note that the functions α_m are absolutely continuous with respect to the completely additive function

$$\gamma(e) = \sum_{n=1}^{\infty} \frac{\beta_n(e)}{2^n(\beta_n(E)+1)},$$

and thus by a theorem of Saks* they are absolutely continuous uniformly with respect to m. Thus for $\epsilon>0$ there is a $\delta>0$ such that

$$|\alpha_m(e)| < (\epsilon/2)\sup_{P\epsilon E}\|f(P)\|$$

for every m provided $\gamma(e)<\delta$. Now there is an n_1 such that

$$\sum_{i=n_1}^{\infty} \gamma(E_i) = \gamma\left(\sum_{i=n_1}^{\infty} E_i\right) < \delta.$$

If s_+ denotes the set of all integers $i\geqq n_1$ for which $\alpha_m(E_i)\geqq 0$ and s_- those $i\geqq n_1$ for which $\alpha_m(E_i)<0$,

$$\sum_{i=n_1}^{\infty} |\alpha_m(E_i)| = \alpha_m\left(\sum_{s+} E_i\right) - \alpha_m\left(\sum_{s-} E_i\right) \leqq \epsilon/\sup_{P\epsilon E}\|f(P)\|,$$

and for $n\geqq n_1$,

$$\left\|\sum_{i=n}^{\infty} f(P_i)\alpha_m(E_i)\right\| \leqq \sup_{P\epsilon E}\|f(P)\| \sum_{i=n_1}^{\infty} |\alpha_m(E_i)| \leqq \epsilon.$$

Thus, by Lemma 5,

$$\lim_m \sum_\pi f(P_i)\alpha_m(E_i) = \sum_\pi f(P_i)\alpha(E_i).$$

* Saks, loc. cit.

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