DIFFERENTIAL GEOMETRY OF A CERTAIN TYPE OF SURFACE IN S_4*

BY

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1. Introduction

In this paper we shall study the sustaining surface of an orthogonal conjugate net immersed in a space of four dimensions . We set up a defining system of partial differential equations. Associated with each point of the surface is a unique plane containing all of the normals to the surface. We define certain unique normals and pairs of normals to the surface and characterize them geometrically. For this purpose we study the sustaining surfaces of the orthogonal projections of the given net on certain geometrically defined spaces of three dimensions. We call these surfaces *normal projection surfaces*. A normal determines a unique normal projection surface. Among the normal projection surfaces there are two, one possessing maximum total curvature, the other minimum total curvature. The normals determining these particular projection surfaces are perpendicular. We have called them *the principal normals*. An analogue is given of the well known theorem that if a line of curvature is a geodesic it is a plane curve.

Let the curves of the given orthogonal conjugate net N_x be taken as the parametric curves. The non-homogeneous cartesian coordinates (x_1, x_2, x_3, x_4) of the point x on the given surface S_x will therefore satisfy the equations

$$x_{uv} = ax_u + bx_v, \qquad \sum x_u x_v = 0. \tag{1}$$

We shall call the plane containing all of the normals to S_x at x *the normal plane* to S_x at x. Select in the normal plane two perpendicular lines λ and μ with direction cosines $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\mu_1, \mu_2, \mu_3, \mu_4)$ respectively. It follows that the functions λ and μ satisfy the equations

$$\begin{aligned} &\sum \lambda x_u = 0, \qquad \sum \lambda x_v = 0, \qquad \sum \mu x_u = 0, \qquad \sum \mu x_v = 0,\\ &\sum \lambda^2 = 1, \qquad \sum \mu^2 = 1, \qquad \sum \lambda\mu = 0. \end{aligned} \tag{2}$$

We see readily that the functions x, λ, μ satisfy a system of differential equations of the form

$$\begin{aligned} x_{uu} &= \alpha x_u + \beta x_v + D_1\lambda + D_2\mu,\\ x_{uv} &= ax_u + bx_v,\\ x_{vv} &= \gamma x_u + \delta x_v + D_1''\lambda + D_2''\mu, \end{aligned} \tag{3}$$

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$$\lambda_u = m_1 x_u + A_1\mu, \qquad \mu_u = m_2 x_u + A_2\lambda,$$
$$\lambda_v = n_1 x_u + B_1\mu, \qquad \mu_v = n_2 x_u + B_2\lambda,$$

wherein

$$(4)\quad \begin{aligned} &\alpha = \tfrac{1}{2}E_u/E, && \beta = -\tfrac{1}{2}E_v/G, && D_1 = \sum \lambda x_{uu}, && D_2 = \sum \mu x_{uu},\\ &a = \tfrac{1}{2}E_v/E, && b = \tfrac{1}{2}G_u/G, && && \\ &\gamma = -\tfrac{1}{2}G_u/E, && \delta = \tfrac{1}{2}G_v/G, && D_1'' = \sum \lambda x_{vv}, && D_2'' = \sum \mu x_{vv},\\ &m_1 = -D_1/E, && n_1 = -D_1''/G, && m_2 = -D_2/E, && n_2 = -D_2''/G, \end{aligned}$$

and wherein

$$E = \sum x_u^2, \qquad G = \sum x_v^2.$$

The integrability conditions of system (3) are

$$(5)\quad \begin{aligned} &\tfrac{1}{2}E_v\left(\frac{D_1}{E} + \frac{D_1''}{G}\right) = D_{1v} + B_2 D_2, && \tfrac{1}{2}E_v\left(\frac{D_2}{E} + \frac{D_2''}{G}\right) = D_{2v} + B_1 D_1,\\ &\tfrac{1}{2}G_u\left(\frac{D_1}{E} + \frac{D_1''}{G}\right) = D_{1u}'' + A_2 D_2'', && \tfrac{1}{2}G_u\left(\frac{D_2}{E} + \frac{D_2''}{G}\right) = D_{2u}'' + A_1 D_1'', \end{aligned}$$

$$(6)\quad \begin{aligned} &G_u(E_u/E + G_u/G) + E_v(E_v/E + G_v/G)\\ &\qquad = 2(E_{vv} + G_{uu}) + 4(D_1 D_1'' + D_2 D_2''), \end{aligned}$$

and

$$(7)\quad A_{1v} = B_{1u}, \qquad A_1 + A_2 = 0, \qquad B_1 + B_2 = 0.$$

2. Power series expansions for the surface

If we use the tangent lines to C_u and to C_v and the lines λ and μ for the axes of a local system of reference, we find that the coordinates of a point y with general coordinates (y_1, y_2, y_3, y_4) will have local coordinates $(\xi_1, \xi_2, \xi_3, \xi_4)$ defined by the expression

$$(8)\quad y = x + \frac{\xi_1 x_u}{E^{1/2}} + \frac{\xi_2 x_v}{G^{1/2}} + \xi_3\lambda + \xi_4\mu.$$

Let y be a point on S_x with curvilinear coordinates $(u+\Delta u, v+\Delta v)$, where (u, v) are the curvilinear coordinates of x. The coordinates of y are of the form

$$(9)\quad y = x + x_u\Delta u + \tfrac{1}{2}(x_{uu}\Delta u^2 + 2x_{uv}\Delta u\Delta v + x_{vv}\Delta v^2) + \cdots.$$

If use be made of (3), (8), and (9) we find that the local coordinates of y are defined by the expressions

$$
(10)\qquad
\begin{aligned}
\xi_1 &= E^{1/2}[\Delta u + \tfrac{1}{2}(\alpha\Delta u^2 + 2a\Delta u\Delta v + \gamma\Delta v^2) + \cdots],\\
\xi_2 &= G^{1/2}[\Delta v + \tfrac{1}{2}(\beta\Delta u^2 + 2b\Delta u\Delta v + \delta\Delta v^2) + \cdots],\\
\xi_3 &= \tfrac{1}{2}(D_1\Delta u^2 + D_1''\Delta v^2) + \tfrac{1}{6}[(\alpha D_1 + D_{1u} + A_2D_2)\Delta u^3 + 3aD_1\Delta u^2\Delta v\\
&\qquad + 3bD_1''\Delta u\Delta v^2 + (\delta D_1'' + D_{1v}'' + B_2D_2'')\Delta v^3] + \cdots,\\
\xi_4 &= \tfrac{1}{2}(D_2\Delta u^2 + D_2''\Delta v^2) + \tfrac{1}{6}[(\alpha D_2 + D_{2u} + A_1D_1)\Delta u^3 + 3aD_2\Delta u^2\Delta v\\
&\qquad + 3bD_2''\Delta u\Delta v^2 + (\delta D_2'' + D_{2v}'' + B_1D_1'')\Delta v^3] + \cdots.
\end{aligned}
$$

From (10) we find the following expansions in local coordinates:

$$
(11)\qquad
\begin{aligned}
\xi_3 = {}& \frac{1}{2}\frac{D_1}{E}\xi_1^2 + \frac{1}{2}\frac{D_1''}{G}\xi_2^2 + \frac{1}{6}\frac{D_{1u} + A_2D_2 - 2\alpha D_1}{E^{3/2}}\xi_1^3\\
&- \frac{1}{2}\frac{aD_1 + \beta D_1''}{EG^{1/2}}\xi_1^2\xi_2 - \frac{1}{2}\frac{bD_1'' + \gamma D_1}{GE^{1/2}}\xi_1\xi_2^2\\
&+ \frac{1}{6}\frac{D_{1v}'' + B_2D_2'' - 2\delta D_1''}{G^{3/2}}\xi_2^3 + \cdots,\\
\xi_4 = {}& \frac{1}{2}\frac{D_2}{E}\xi_1^2 + \frac{1}{2}\frac{D_2''}{G}\xi_2^2 + \frac{1}{6}\frac{D_{2u} + A_1D_1 - 2\alpha D_2}{E^{3/2}}\\
&- \frac{1}{2}\frac{aD_2 + \beta D_2''}{EG^{3/2}}\xi_1^2\xi_2 - \frac{1}{2}\frac{bD_2'' + \gamma D_2}{GE^{1/2}}\xi_1\xi_2^2\\
&+ \frac{1}{6}\frac{D_{2v}'' + B_1D_1'' - 2\delta D_2''}{G^{3/2}}\xi_2^3 + \cdots.
\end{aligned}
$$

Equations (11) may be interpreted as follows. The first of equations (11) and $\xi_4 = 0$ are the equations of the sustaining surface of the orthogonal projection of the given net on to the S_3 determined by the tangent plane and the normal λ. A similar statement holds for the second of (11) and $\xi_3 = 0$. We shall call these surfaces *the normal projection surfaces of S_x determined by λ and by μ* respectively, and shall denote them by S_λ and S_μ respectively.

From (11) we find that the principal radii of normal curvature of S_λ are

$$
(12)\qquad R_1 = \frac{E}{D_1}, \qquad R_1'' = \frac{G}{D_1''},
$$

and the principal radii of normal curvature of S_μ are

$$
(13)\qquad R_2 = \frac{E}{D_2}, \qquad R_2' = \frac{G}{D_2''}.
$$

The total curvature of S_λ and S_μ are respectively

$$K_1 = \frac{D_1 D_1''}{EG}, \qquad K_2 = \frac{D_2 D_2''}{EG}, \tag{14}$$

and the mean curvatures are respectively

$$M_1 = \frac{D_1}{E} + \frac{D_1''}{G}, \qquad M_2 = \frac{D_2}{E} + \frac{D_2''}{G}. \tag{15}$$

From (6) we observe that *the sum of the total curvatures of the normal projection surfaces determined by perpendicular normals is a constant at a point of the surface.*

3. A canonical form of the defining differential equations

Let us now make the transformation

$$\begin{aligned} \lambda &= A\bar{\lambda} + B\bar{\mu}, \\ \mu &= -B\bar{\lambda} + A\bar{\mu}, \qquad A^2 + B^2 = 1, \end{aligned} \tag{16}$$

on system (3). The transformation (16) is equivalent to a rotation of axes in the normal plane. Let the coefficients of the transformed differential equations be denoted by $\bar{\alpha}$, $\bar{\beta}$, $\cdots$. We find that these new coefficients are given by the following formulas:

$$\begin{aligned} &\bar{\alpha} = \alpha, \quad \bar{\beta} = \beta, \quad \bar{a} = a, \quad \bar{b} = b, \quad \bar{\gamma} = \gamma, \quad \bar{\delta} = \delta, \\ &\bar{D}_1 = AD_1 - BD_2, \qquad \bar{D}_1'' = AD_1'' - BD_2'', \\ &\bar{D}_2 = BD_1 + AD_2, \qquad \bar{D}_2'' = BD_1'' + AD_2'', \\ &\bar{m}_1 = Am_1 - Bm_2, \qquad \bar{m}_2 = Bm_1 + Am_2, \\ &\bar{n}_1 = An_1 - Bn_2, \qquad \bar{n}_2 = Bn_1 + An_2, \\ &\bar{A}_1 = A_1 + A_uB - AB_u, \qquad \bar{B}_1 = B_1 + A_vB - AB_v, \\ &\bar{A}_2 = A_2 + AB_u - A_uB, \qquad \bar{B}_2 = B_2 + AB_v - A_vB. \end{aligned} \tag{17}$$

The total curvature $\bar{K}_1$ of the surface of normal projection $S_{\bar{\lambda}}$ is determined by the expression

$$EG\bar{K}_1 = (AD_1 - BD_2)(AD_1'' - BD_2'').$$

This surface has maximum or minimum total curvature if and only if A and B satisfy the quadratic equation

$$LA^2 - 2MAB - LB^2 = 0, \tag{18}$$

wherein

$$L = D_1 D_2'' + D_1'' D_2, \qquad M = D_1 D_1'' - D_2 D_2''. \tag{19}$$

Since the product of the roots of (18) as a quadratic in A/B is minus one, the two normals determined by these roots are perpendicular. We shall call these normals the *principal normals* of S_x at x, and the normal projection surfaces determined by the principal normals, the *principal normal projection surfaces*. We may state our results in the following form: *Through the point x there exist two normals with the property that the normal projection surfaces determined by them have maximum and minimum total curvatures.*

Let us suppose that the transformation (16) with values of A and B determined by (18) has been effected on the system (3). The resulting differential equations assume a canonical form in which

$$D_1'' = lD_1, \qquad D_2'' = -lD_2, \tag{20}$$

that is, a form for which $L=0$. For this form the normals λ and μ are the principal normals.

From (20) we observe that *if one of the principal normal surfaces is isothermic, the other has the same property.*

4. Other unique normals

The general coordinates of the principal centers of normal curvature of the surface of normal projection $S_{\bar{\lambda}}$ determined by

$$\bar{\lambda} = A\lambda - B\mu$$

are

$$x + \frac{E(A\lambda - B\mu)}{AD_1 - BD_2}, \qquad x + \frac{G(A\lambda - B\mu)}{AD_1'' - BD_2''}. \tag{21}$$

The local coordinates of these points are

$$\xi_1 = 0, \qquad \xi_2 = 0, \qquad \xi_3 = \frac{AE}{AD_1 - BD_2}, \qquad \xi_4 = -\frac{BE}{AD_1 - BD_2}, \tag{22}$$

and

$$\xi_1 = 0, \qquad \xi_2 = 0, \qquad \xi_3 = \frac{AG}{AD_1'' - BD_2''}, \qquad \xi_4 = -\frac{BG}{AD_1'' - BD_2''}, \tag{23}$$

respectively.

The locus of the centers of principal normal curvature for all normal projection surfaces are therefore straight lines.

The equations of these lines are

$$\begin{aligned} &\xi_1 = 0, \qquad \xi_2 = 0, \qquad D_1\xi_3 + D_2\xi_4 = E, \\ &\xi_1 = 0, \qquad \xi_2 = 0, \qquad D_1''\xi_3 + D_2''\xi_4 = G. \end{aligned} \tag{24}$$

We shall call these lines *the central lines* of S_x at x. The *central lines* of S_x *at* x *are orthogonal if and only if*

$$D_1 D_1'' + D_2 D_2'' = 0. \tag{25}$$

Hence *the central lines are perpendicular if and only if the total curvatures of the normal projection surfaces in two perpendicular directions differ only in sign.* Moreover from the integrability condition (6) we observe that *if the total curvatures of the normal projection surfaces of a surface sustaining an orthogonal conjugate net in perpendicular directions differ only in sign, all such normal projection surfaces have the same property.*

The tangent plane to S_x at x and the osculating plane to the curve C_u at x determine a space of three dimensions. This space intersects the normal plane in a line with direction cosines proportional to

$$D_1\lambda + D_2\mu. \tag{26}$$

A similar statement holds for the curve C_v and the line through x with direction cosines proportional to

$$D_1'\lambda + D_2''\mu. \tag{27}$$

We shall call the lines through x and with directions defined by (26) and (27) the *intersector normals* of the curves C_u and C_v respectively. We see readily that the *intersector normals are perpendicular if and only if the central lines are perpendicular.*

If we define a geodesic as an extremal curve of the integral

$$\int (Eu'^2 + Gv'^2)^{1/2} dt, \qquad u' = du/dt, \qquad v' = dv/dt,$$

we find that the differential equation of the geodesics on S_x is

$$u'v'' - u''v' = \gamma v'^3 - (\delta - 2a)u'v'^2 + (\alpha - 2b)u'^2 v' - \beta u'^3. \tag{28}$$

Consider now the curve $u = u(t)$, $v = v(t)$ on S_x. We may show very readily that the equations in local coordinates of the osculating plane of the curve are

$$G^{1/2}v'(D_1 u'^2 + D_1'' v'^2)\xi_1 - E^{1/2}u'(D_1 u'^2 + D_1'' v'^2)\xi_2 + J\xi_3 = 0,$$
$$G^{1/2}v'(D_2 u'^2 + D_2'' v'^2)\xi_1 - E^{1/2}u'(D_2 u'^2 + D_2'' v'^2)\xi_2 + J\xi_4 = 0,$$

wherein

$$J = u'v'' - u''v' - \gamma v'^3 + (\delta - 2a)u'v'^2 - (\alpha - 2b)u'^2 v' + \beta u'^3.$$

Hence *the osculating plane at the point of a curve on S_x intersects the normal plane to the surface at the point of the curve in a line if and only if the curve is a geodesic.*

It follows that the curve C_u is a geodesic if and only if $E_v=0$, and C_v is a geodesic if and only if $G_u=0$. Moreover from (6) we see that *if the curves of the given net are geodesics the central lines (and intersector normals) are perpendicular.*

Suppose that in the system (3) the following condition is satisfied:

$$\Delta = D_1D_2'' - D_2D_1'' = 0. \tag{29}$$

If we make the transformation

$$\bar{\lambda} = \frac{D_1\lambda + D_2\mu}{(D_1^2 + D_2^2)^{1/2}}, \qquad \bar{\mu} = -\frac{D_2\lambda - D_1\mu}{(D_1^2 + D_2^2)^{1/2}},$$

on system (3), we find that the system assumes the form

$$\begin{aligned}
x_{uu} &= \alpha x_u + \beta x_v + (D_1^2 + D_2^2)^{1/2}\bar{\lambda},\\
x_{uv} &= a x_u + b x_v,\\
x_{vv} &= \gamma x_u + \delta x_v + \frac{D_1D_1'' + D_2D_2''}{(D_1^2 + D_2^2)^{1/2}}\bar{\lambda},\\
\bar{\lambda}_u &= -\frac{(D_1^2 + D_2^2)^{1/2}}{E} x_u + \left[A_1 + \frac{D_1D_{2u} - D_2D_{1u}}{D_1^2 + D_2^2}\right]\bar{\mu}, \qquad \bar{\mu}_u = 0,\\
\bar{\lambda}_v &= -\frac{(D_1''^2 + D_2''^2)^{1/2}}{G} x_v + \left[B_1 + \frac{D_1D_{2v} - D_2D_{1v}}{D_1^2 + D_2^2}\right]\bar{\mu}, \qquad \bar{\mu}_v = 0.
\end{aligned} \tag{30}$$

But if use be made of (29) and the integrability conditions (5), we may readily show that the coefficients of $\bar{\mu}$ in (30) vanish. It follows therefore that *a necessary and sufficient condition that S_x be immersed in a space of three dimensions is that*

$$\Delta = D_1D_2'' - D_2D_1'' = 0.$$

Or we may say that *a necessary and sufficient condition that the surface S_x be immersed in a space of three dimensions is that the intersector normals coincide or that the central lines be parallel.*

From (17) and (15) we note that the surface of normal projection $S_{\bar{\lambda}}$ defined by (16) is a minimal surface if and only if

$$\overline{M}_1 = \overline{D}_1/E + \overline{D}_1''/G = AM_1 - BM_2. \tag{31}$$

It follows therefore that *if two surfaces of normal projection taken in perpendicular directions are both minimal surfaces, all surfaces of normal projection are minimal, and moreover the surface is a minimal surface immersed in a space of three dimensions. If not both of M_1 and M_2 are zero, there exists just one normal to the surface at x which determines a minimal surface of normal projection.*

The central lines are parallel if and only if (29) holds. Hence if the surface is not immersed in a space of three dimensions the central lines intersect in a point. This point and the point x determine a unique normal which determines a surface of normal projection which has the point x as an umbilical point.

Again from (17) we see that *there exist exactly two surfaces of normal projection which are developables.* The normals determining these developable surfaces have direction cosines proportional to

$$D_2\lambda \pm D_1\mu, \tag{32}$$

wherein D_1 and D_2 are the coefficients of the canonical system defined by (20). We shall call these normals the *developable normals. The developable normals are each perpendicular to one or the other of the intersector normals. The pair of developable normals and the pair of intersector normals each make equal angles with the principal normals, and hence are paired in involution with the principal normals as double lines.*

5. Congruences conjugate to the given net

A congruence is said to be conjugate to a given conjugate net if the developables of the congruence intersect the sustaining surface of the net in the curves of the net. It is well known that if a line g generates a congruence G conjugate to a net, any point y on g generates a surface S_y such that the tangents to the curves on S_y corresponding to the curves of the given net and the tangents to the curves of the given net are coplanar.

Let us find the condition that the two-parameter family of lines generated by λ generate a congruence conjugate to N_x. Let y be any point on λ. The coordinates of y are defined by an expression of the form

$$y = x + \lambda d, \qquad d \neq 0.$$

We find readily that

$$\begin{aligned} y_u &= (1 + m_1 d)x_u + d_u(y - x)/d + A_1\mu d, \\ y_v &= (1 + n_1 d)x_v + d_v(y - x)/d + B_1\mu d. \end{aligned} \tag{33}$$

Hence the lines λ form a congruence G conjugate to the given net if and only if

$$A_1 = 0, \qquad B_1 = 0. \tag{34}$$

But from (7), we see if (34) is satisfied that $A_2 = 0$, $B_2 = 0$. We see therefore that *if a given normal generates a congruence conjugate to the given orthogonal conjugate net, the unique normal perpendicular to the given normal also generates a congruence conjugate to the net.*

The normals $\bar{\lambda}$ defined by (16) will generate a congruence conjugate to N_x if and only if

$$\begin{aligned} \bar{A}_1 &= A_1 + A_u B - A B_u = 0, \\ \bar{B}_1 &= B_1 + A_v B - A B_v = 0. \end{aligned} \tag{35}$$

If we let

$$A = \cos\theta, \qquad B = \sin\theta,$$

we find that (35) may be written in the form

$$A_1 - \theta_u = 0, \qquad B_1 - \theta_v = 0. \tag{36}$$

The integrability condition of system (36) is

$$A_{1v} - B_{1u} = 0. \tag{7 bis}$$

Hence (36) may be solved by a quadrature. Suppose the transformation (16) with A and B determined by (36) has been effected on system (3). The resulting system will be of the same form as (3) with

$$A_1 = 0, \qquad B_1 = 0, \qquad A_2 = 0, \qquad B_2 = 0. \tag{37}$$

The differential equations (3) characterized by (37) are left unchanged in form by the transformation (16) with constant A and B. We may state our results in the following form:

All congruences 2I conjugate to the given orthogonal conjugate net may be found by quadratures. If each of the lines of a congruence conjugate to the given net is rotated through the same constant angle in the normal plane at each point x of S_x, the resulting two-parameter family of lines forms a congruence conjugate to the net.*

Let us suppose that S_x is not immersed in a space of three dimensions. From (3) we find that

$$\begin{aligned} \Delta\lambda &= D_2'' x_{uu} - D_2 x_{vv} + (\gamma D_2 - \alpha D_2'')x_u + (\delta D_2 - \beta D_2'')x_v, \\ \Delta\mu &= -D_1'' x_{uu} + D_1 x_{vv} + (\alpha D_1'' - \gamma D_1)x_u + (\beta D_1'' - \delta D_1)x_v. \end{aligned} \tag{38}$$

If we differentiate the first and last of (3) with respect to u and v respectively, and use (38) and (3), we find that the four functions x satisfy a system of third-order differential equations of the form

$$\begin{aligned} x_{uuu} &= p_1 x_{uu} + q_1 x_{vv} + p_1' x_u + q_1' x_v, \\ x_{vvv} &= p_2 x_{uu} + q_2 x_{vv} + p_2' x_u + q_2' x_v, \end{aligned} \tag{39}$$

wherein q_1 and p_2 are defined by the expressions

* L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, p. 168.

$$\Delta q_1 = D_1 D_{2u} - D_2 D_{1u} + A_1(D_1^2 + D_2^2),$$
(40)
$$\Delta p_2 = D_1'' D_{2v}'' - D_2'' D_{1v}'' + B_1(D_1''^2 + D_2''^2).$$

Two of the integrability conditions for the system composed of (39) and the second of (3) are

$$q_1 q_2 + q_{1v} + q_1' = a q_1, \qquad p_1 p_2 + p_{2u} + p_2' = b p_2. \tag{41}$$

It follows therefore that the curve C_u is a plane curve if and only if $q_1 = 0$, $\Delta \neq 0$, and the curve C_v is a plane curve if and only if $p_2 = 0, \Delta \neq 0$.

Suppose both C_u and C_v are plane curves. We may write the conditions $q_1 = 0$ and $p_2 = 0$ in the following form:

$$\frac{\partial}{\partial u} \arctan \frac{D_2}{D_1} = - A_1,$$
(42)
$$\frac{\partial}{\partial v} \arctan \frac{D_2''}{D_1''} = - B_1.$$

It follows from (7) that

$$\frac{\partial^2}{\partial u \partial v}\left[\arctan \frac{D_2}{D_1} - \arctan \frac{D_2''}{D_1''}\right] = 0. \tag{43}$$

Let the angle between the intersector normals be denoted by I. We may readily verify that equation (43) may be written in the form

$$I_{uv} = 0. \tag{44}$$

Suppose the congruence of normals λ is conjugate to the given net, and let C_u be a plane curve. We may write the first of (42) in the form

$$I_{1u} = 0,$$

where I_1 is the angle between the intersector normal and the normal λ. Hence *the angle between the intersector normal of a plane curve of an orthogonal conjugate net and any line generating a congruence conjugate to the net is a constant for points on S_x along the plane curve.*

The intersector normal corresponding to the curve C_u has direction cosines $\bar{\lambda}$ defined by expressions of the form

$$\bar{\lambda} = \frac{D_1 \lambda + D_2 \mu}{(D_1^2 + D_2^2)^{1/2}}.$$

We find readily that

$$\bar{\lambda}_u = -\frac{(D_1^2 + D_2^2)^{1/2} x_u}{E} - \frac{q_1(D_2\lambda - D_1\mu)}{\Delta(D_1^2 + D_2^2)^{3/2}},$$

$$(45)\qquad \bar{\lambda}_v = -\frac{(D_1 D_1'' + D_2 D_2'')x_v}{G} + \frac{[D_2 D_{1v} - D_1 D_{2v} + B_2(D_1^2 + D_2^2)](D_2\lambda - D_1\mu)}{(D_1^2 + D_2^2)^{3/2}}.$$

This intersector normal is therefore conjugate to the given net if and only if

$$(46)\qquad q_1 = 0, \qquad D_2 D_{1v} - D_1 D_{2v} + B_2(D_1^2 + D_2^2) = 0.$$

But if use be made of the integrability conditions (5) the second of (46) may be written in the form

$$\Delta E_v = 0.$$

The intersector normal of a curve of an orthogonal conjugate net on a surface not immersed in a space of three dimensions generates a congruence conjugate to the net if and only if the given curve is a plane curve and a geodesic on S_x. If moreover the central lines are orthogonal the intersector normals of one curve of the net are parallel as x moves along the other curve of the net.

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