COVARIANTS OF r-PARAMETER GROUPS*

BY

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1. **Introduction.** The theory of covariants of r-parameter groups has not had the development that the theory of invariants has had. Apparently this is because the methods used have been almost exclusively those of Lie, in which the concept of invariant is central. In projective geometry the covariant is surely of as great importance as the invariant, and in differential geometry the differential covariant or tensor is basic. It would appear desirable, then, to attempt some new approach to this subject, an approach having the covariant rather than the invariant as the fundamental concept.

The present purely algebraic treatment is based upon a somewhat novel concept of what a covariant is. The approach seems to be justified by the fact that the fundamental theorem of covariant theory (Theorem 6), namely that every invariantive property can be characterized by the vanishing of covariants, follows immediately from the definition of covariant. Perhaps there is some justification in claiming that the approach is *durchsichtig*.

The ordinary projective invariant theory arises upon specializing the r-parameter group to the linear homogeneous group. That every projective property can be characterized by the vanishing of absolute covariants in n cogredient sets of variables follows at once. The reason that exactly n sets are required is apparent.

The application of the theory to differential forms indicates that the tensor analysis is more restrictive than is necessary. There are covariants which do not obey the tensor law, and yet seem to be of as much use and importance as tensors. While some writers have employed such covariants, their use has not become general.

Many of the concepts in this paper were first applied in the projective theory by J. Deruyts and A. Capelli, whose works are cited.

2. **The parameter groups.** Let

$$a_i' = f_i(a_1, a_2, \cdots, a_n; \xi_1, \xi_2, \cdots, \xi_r) \qquad (i = 1, 2, \cdots, n),$$

or more simply

$$G_r:\quad a' = f(a; \xi), \qquad a = f_{-1}(a'; \xi), \tag{2.1}$$

be an r-parameter group of transformations where r and n are finite or de-

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numerably infinite. It is understood that the variables $a=(a_1, a_2, \cdots, a_n)$ and the parameters $\xi=(\xi_1, \xi_2, \cdots, \xi_r)$ range over a field $\mathfrak{F}$, and that the functional values f_i are also in $\mathfrak{F}$. We shall denote by ξ_0 values of the parameters which give the identity transformation, and by ξ_{-1} values which give a transformation inverse to that with parameters ξ. We shall ordinarily assume that the parameters are essential.

Consider the two transformations

$$(2.2) \qquad T: \quad a'' = f(a'; \eta), \qquad T': \quad a'' = f(a; \zeta),$$

such that T' is the resultant of T and G_r. Since these transformations belong to a group, there exists a functional relation

$$(2.3) \qquad \zeta = g(\eta; \xi)$$

among the parameters. We may look upon the ξ as numbers associated with the variables a', namely the parameters of a transformation which represents the a' in terms of the a. The ζ are similarly associated with the a''. Under transformation T, the ξ are subjected to the induced transformation (2.3). Hence all transformations (2.3) constitute a group on the ξ with parameters η.

Let $x_1, x_2, \cdots, x_r$ be new independent variables, and define $x_1', x_2', \cdots, x_r'$ by the equations

$$(2.4) \qquad P_r: \quad x' = g(\eta; x).$$

We shall call P_r the *first parameter-group* of G_r.*

Equations (2.1) and (2.2) can be looked upon from another angle. We may consider the η as numbers associated with the variables a', namely the parameters of a transformation which represents the a'' in terms of the a'. The ζ are similarly associated with the a. Under (2.1) the η are subjected to the induced transformation (2.3). Hence (2.3) constitute a group on the η with parameters ξ.

Let $u_1, u_2, \cdots, u_r$ be new independent variables, and define u_1', $u_2', \cdots, u_r'$ by the equations

$$(2.5) \qquad P_r': \quad u' = g(u; \xi).$$

We shall call P_r' the *second parameter-group* of G_r.†

If in particular $\xi=\eta_{-1}$, we shall say that the u are *contragredient* to the x. That is, if $x'=g(\eta; x)$, then

$$(2.6) \qquad u' = g(u; \eta_{-1}), \text{ or } u = g(u'; \eta).$$

* Lie-Engel, *Theorie der Transformationsgruppen*, Abs. 1, Teubner, 1888, p. 401.

† G. Kowalewski, *Einführung in die Theorie der kontinuierlichen Gruppen*, Leipzig, 1931, p. 131 L. P. Eisenhart, *Continuous Groups of Transformations*, Princeton, 1933, p. 31.

THEOREM 1. *The first parameter-group of P_r, P_r' is isomorphic with P_r, P_r', respectively. The second parameter-group of P_r, P_r' is isomorphic with P_r', P_r, respectively.**

Let us write

$$\begin{aligned} T&: \quad a' = f(a; \xi), & S&: \quad a'' = f(a'; \eta), \\ T'&: \quad a'' = f(a; \xi'), & S'&: \quad a''' = f(a'; \eta'), \\ T''&: \quad a''' = f(a; \xi''), & R&: \quad a''' = f(a''; \theta). \end{aligned}$$

From T, T', and S, we have

$$\xi' = g(\eta; \xi).$$

From T, T'', and S',

$$\xi'' = g(\eta'; \xi),$$

and from T', T'', and R,

$$\xi'' = g(\theta; \xi').$$

These are transformations on the ξ with parameters η, η', and θ. Their first parameter-group is

$$\eta' = g'(\theta; \eta).$$

But directly from S, S', and R we have

$$\eta' = g(\theta; \eta)$$

as the relation by which the η' are defined in terms of the η. Thus the first parameter-group of P_r is isomorphic with P_r.

The rest of the theorem may be proved similarly.

3. **Concomitants.** Let us consider the group

$$G_r: \quad a' = f(a; \xi) \tag{3.1}$$

and the parameter-groups

$$P_r: \quad x' = g(\xi; x), \tag{3.2}$$

$$P_r': \quad u' = g(u; \xi_{-1}), \tag{3.3}$$

with the variables u contragredient to the variables x.

The concept of invariant of a group is well established. If

$$F(a_1, a_2, \cdots, a_n) \equiv F(a)$$

is any function of the variables such that $F(a) \equiv F(a')$ is an identity in the

* Lie noted that the (first) parameter-group is its own parameter-group. S. Lie, Videnskabs-Selskabet i Christiania, Forhandlinger, 1884, No. 15.

a and the ξ when the a' are replaced by their values as given by (3.1), the function $F(a)$ is called an *absolute invariant* of G_r.

The following formulation of the concept of covariant is believed to be new. If $F(a;\ x) \equiv F(a';\ x')$ is an identity in the variables a and x and the parameters ξ when a' and x' are replaced by their values as given by (3.1) and (3.2), then we shall call F an *absolute covariant* of G_r.

Similarly, if $F(a;\ u) \equiv F(a';\ u')$ holds identically by virtue of G_r and P_r', we shall call F an *absolute contravariant* of G_r.

More generally, if $F(a;\ x;\ u) \equiv F(a';\ x';\ u')$ holds identically in all the letters involved after a', x', and u' have been replaced by their values as given by G_r, P_r, and P_r', we shall call F an *absolute concomitant* of G_r. Thus the concept of concomitant includes invariant, covariant, and contravariant as special instances.

A concomitant involving only the x and u is sometimes called an *identical concomitant*.

4. **Structure of covariants.** We prove the following theorem.

THEOREM 2. *Let $F(a)$ be any function. Let $F(a)$ become $G(a';\ \xi)$ under G_r. Then $G(a;x)$ is a covariant of G_r.*

The function $F(a)$ is called the *source** of $G(a;x)$, and we shall write

$$G(a;\ x) = [F(a)].$$

Let $F(a)$ become under (2.1) and (2.2) T'

$$F(a) = G(a';\ \xi) = G(a'';\ \zeta).$$

These expressions are identical by virtue of (2.2) T and (2.3). Since the x are cogredient with the η,

$$G(a';\ x) = G(a'';\ x')$$

holds identically in η by virtue of (2.2) T and (2.4). Hence by a change of variables

$$G(a;\ x) = G(a';\ x')$$

holds identically in ξ by virtue of (3.1) and (3.2).

The covariant $G(a;x) = [F(a)]$ is uniquely defined by $F(a)$.

THEOREM 3. *Every covariant has a source.*

Let $G(a;x)$ be a covariant, and denote $G(a;\ \xi_0)$ by $F(a)$, where ξ_0 are the values of the parameters which reduce (2.1) to the identity. We shall show that $G(a;x) = [F(a)]$.

* For a development of the projective theory along these lines, see J. Deruyts, *Essai d'une Théorie Générale des Formes Algébriques*, Liége, 1890.

If $G(a; x)$ is a covariant, then by the substitution of cogredient variables we obtain

$$G(a'; \xi) = G(a''; \zeta)$$

holding by virtue of (2.2) T and (2.3). In particular set $\zeta = \zeta_0$ so that $a'' = a$. Then

$$G(a'; \xi) = G(a; \zeta_0) = F(a)$$

holds by virtue of (2.1). This shows that $G(a; x)$ is obtainable from $F(a)$ according to the procedure of Theorem 2.

The source of $G(a; x)$ is unique.

In particular the n functions $a_1, a_2, \cdots, a_n$ are sources of the *elementary covariants*

$$[a_1], [a_2], \cdots, [a_n].$$

THEOREM 4. *Every covariant is a function of the elementary covariants.*

Let

$$G(a; x) = [F(a)] = [F(a_1, a_2, \cdots, a_n)].$$

Then

$$G(a; x) = F([a_1], [a_2], \cdots, [a_n]),$$

for one may use (2.1) on $F(a)$ and then replace the parameters by the x, or replace the ξ in (2.1) by the x and use the result on $F(a)$.

Similar results hold for contravariants. Let $F(a)$ be any function, and let $F(a')$ become $G(a; \xi)$ under G_r. Then $G(a; u)$ is an absolute contravariant of G_r. The analogs of Theorems 3 and 4 hold for contravariants.

THEOREM 5. *The functions $g_i(u; x)$ determined by P are identical concomitants, and every identical concomitant is a function of them.*

Let us take $x' = g(\xi; x)$ as G_r. By Theorem 1 the second parameter-group is still (2.6). Hence by the preceding paragraph every $g_i(u; x)$ is an absolute contravariant of G_r, and hence a concomitant of the original group.

Since the $g_i(u; x)$ are the elementary contravariants of G_r, every contravariant of G_r (that is, every identical concomitant of (2.1)) is a function of them.

The same concomitants are obtained by finding the elementary covariants of (2.5).

5. **Characterization of geometric properties.** One of the major problems in the application of invariant theory to geometry is the characterization of

geometric properties by means of invariants or covariants or tensors. The following theorem gives a general solution of this problem.

Theorem 6. *A necessary and sufficient condition in order that*

$$\phi_1(a) = 0, \phi_2(a) = 0, \cdots, \phi_k(a) = 0$$

shall hold for all coordinate systems is that the covariants

$$[\phi_1(a)], [\phi_2(a)], \cdots, [\phi_k(a)]$$

vanish identically in x.

If $\phi_i(a) = 0$ holds for all coordinate systems, then the functions $\psi_i(a; \xi)$ obtained by using (2.1) in $\phi_i(a')$ must vanish for all values of ξ. That is,

$$\psi_i(a; x) = [\phi_i(a)]$$

must vanish identically in x.

Since $\psi_i(a; \xi_0) = \phi_i(a)$, the condition is sufficient.

If a system of equations $\phi_i(a) = 0$ characterize a geometric property of G_r, it is not necessarily true that there exists a set of covariants having the functions $\phi_i(a)$ as their coefficients. But by Theorem 6 the coefficients of the covariants $[\phi_i(a)]$ when set equal to zero constitute a system of equations which also characterize the geometric property. It must be true, then, that this latter system is equivalent to the system $\phi_i(a) = 0$. Thus the problem of putting the system $\phi_i(a) = 0$ into covariant form has been solved.

It is thus evident that contravariants and mixed concomitants are not essential in geometry. Indeed, it is evident from the reciprocal relationship of P_r and P_r' that the theories of covariants and contravariants are coextensive. The use of contravariants, however, is often a matter of great convenience.

6. **Relative covariants and contravariants.** In this paragraph we shall assume that $\mathfrak{F}$ is the complex field, and that the functions f_i have differential coefficients of the first order.

In order that a transformation of type (2.1) be non-singular, that is, have an inverse, it is necessary and sufficient that the jacobian

$$J(a'; a) = \left| \frac{\partial a_r'}{\partial a_s} \right|$$

be different from zero. Since G_r consists only of non-singular transformations, those sets of values of $a_1, a_2, \cdots, a_n, \xi_1, \xi_2, \cdots, \xi_r$ which make $J = 0$ are excluded.

A function $F(a; x; u)$ such that

$$F(a'; x'; u') = [J(a'; a)]^\mu F(a; x; u)$$

is called a ***relative concomitant*** of ***weight*** μ. Relative covariants and relative contravariants are special instances. Since $J\neq 0$, the identical vanishing of a relative concomitant is invariantive.

THEOREM 7. *If $J(a'; a)$ is written as $C(a'; \xi)$ by means of* (2.1), *then $C(a; x)$ is a relative covariant of weight* 1. *If $J(a'; a)$ is written as $D(a; \xi)$ by means of* (2.1), *then $D(a; u)$ is a relative contravariant of weight* -1.

Consider transformations (2.1) and (2.2). Since $J(a''; a)$ is a function of a and ζ, we can use (2.2) T' to write

$$J(a''; a) = C(a''; \zeta).$$

Similarly from (2.1)

$$J(a'; a) = C(a'; \xi).$$

Now from the familiar relation

$$J(a''; a) = J(a''; a')J(a'; a),$$

we have

$$C(a''; \zeta) = J(a''; a')C(a'; \xi).$$

This is an identity by virtue of (2.2) and (2.3). Hence

$$C(a''; x') = J(a''; a')C(a'; x)$$

is an identity in η by virtue of (2.2) and (2.4). That is, by a change of variables,

$$C(a'; x') = J(a'; a)C(a; x)$$

is an identity in ξ by virtue of (3.1) and (3.2), and $C(a; x)$ is a relative covariant of weight 1.

Similarly if we write

$$J(a''; a) = D(a; \zeta), \qquad J(a''; a') = D(a'; \eta)$$

by means of (2.2), we have

$$D(a; \zeta) = D(a'; \eta)J(a'; a)$$

holding by virtue of (2.1) and (2.3). That is,

$$D(a'; u') = [J(a'; a)]^{-1}D(a; u)$$

holds by virtue of (2.1) and

$$u = g(u'; \xi).$$

Since these concomitants never vanish, multiplying an invariantive equation by a power of one of them does not alter the geometric nature of the equation.

THEOREM 8. *Every relative covariant of weight μ can be represented as a function of the elementary covariants multiplied by $[C(a; x)]^\mu$. Every relative contravariant of weight μ can be represented as a function of the elementary contravariants divided by $[D(a; u)]^\mu$.*

For if $F(a; x)$ is a covariant of weight μ, $F(a; x)/[C(a; x)]^\mu$ is absolute, and can by Theorem 4 be represented as a function of the elementary covariants.

7. **Covariants of the projective group.** Let G_r be

$$a_i' = \sum_{j=1}^{n} \xi_{ij} a_j \qquad (i = 1, 2, \cdots, n) \tag{7.1}$$

where the ξ_{ij} are independent except that $J(a'; a) = |\xi_{ij}| \neq 0$. Corresponding to (2.2) we have

$$a_i'' = \sum_{j=1}^{n} \eta_{ij} a_j', \qquad a_i'' = \sum_{j=1}^{n} \zeta_{ij} a_j.$$

Corresponding to (2.3) we have

$$\zeta_{ik} = \sum_{j=1}^{n} \eta_{ij} \xi_{jk}. \tag{7.2}$$

If we define the new independent variables $x_j^{(k)}$ in accordance with §2, we have

$$x_i^{(k)\prime} = \sum_{j=1}^{n} \eta_{ij} x_j^{(k)} \qquad (i, k = 1, 2, \cdots, n). \tag{7.3}$$

Thus for every k we have a group of the same form as (7.1). Since the $x^{(k)\prime}$ are functions only of the $x^{(k)}$, we say that (7.3) is intransitive, breaking up into n blocks or sets of intransitivity.* Thus

THEOREM 9. *The first parameter-group of* (7.1) *on the n^2 variables $x_i^{(k)}$ is intransitive, and each of its n sets of intransitivity is isomorphic with* (7.1).

This result does much to explain the simplicity of the projective covariant theory. It has long been known that the covariant theory using a single set of n variables is inadequate for geometry, while n cogredient sets are suffi-

* Miller, Blichfeldt and Dickson, *Theory and Applications of Finite Groups*, Wiley, 1916, p. 206.

cient. Indeed, if relative covariants are used, $n-1$ cogredient sets suffice.*

Since $J(a'; a)=|\xi_{rs}|$, the relative covariant $C(a; x)$ of §6 is $X=|x_r^{(s)}|$.

To obtain the elementary covariants, write (7.1) in the form

$$a_i = \frac{1}{J(a'; a)} \sum_{j=1}^{n} \Xi_{ji} a_j',$$

where Ξ_{ji} is the cofactor of ξ_{ji} in (ξ_{rs}). It is evident upon the replacement of §4 that

THEOREM 10. *The elementary covariants of* (7.1) *are* $[a_i]=P_i/X$ *where* $X=|x_r^{(s)}|$ *and* P_i *is obtained from* X *by replacing the elements of the ith column by the* a.

The second parameter-group of (7.1) may be written

$$u_k^{(i)} = \sum_{j=1}^{n} u_j^{(i)\prime} \xi_{jk}.$$

It follows now from Theorem 5 that

THEOREM 11. *The functions*

$$\sum_{j=1}^{n} u_j^{(i)} x_j^{(k)} \qquad (i, k = 1, 2, \cdots, n)$$

are identical concomitants, and every identical concomitant is a function of them.

8. Covariants of algebraic forms. Consider a system of l algebraic forms

$$\phi_i(a_{i1}, a_{i2}, \cdots, a_{iq_i}; x_1, x_2, \cdots, x_n) \equiv \phi_i \qquad (i = 1, 2, \cdots, l).$$

The linear homogeneous transformation

$$x_i' = \sum_{j=1}^{n} \xi_{ij} x_j \qquad (i = 1, 2, \cdots, n) \tag{8.1}$$

induces upon the coefficients of the ground forms ϕ_i a set of transformations

$$a_{ij}' = \sum_{k=1}^{q_i} \beta_{ijk} a_{ik} \qquad (i = 1, 2, \cdots, l; j = 1, 2, \cdots, q_i), \tag{8.2}$$

where the β_{ijk} are functions of the ξ.

By an invariant or covariant of the forms ϕ_i is meant an invariant of the induced group (8.2), or of (8.2) and its first parameter-group. The only role of the ground forms ϕ_i is to determine the transformation (8.2), which is our G_r.

* See, for instance, Clebsch, Abhandlungen, Gesellschaft der Wissenschaften zu Göttingen, vol. 17. Capelli, Atti, Accademia Nazionale dei Lincei, (3), vol. 12 (1882), pp. 529–598. Deruyts, loc. cit., introduction. A short proof was given by the writer, Bulletin of the American Mathematical Society vol. 29 (1923), p. 32.

Since the groups (8.1) and (8.2) are isomorphic, their first parameter-groups are identical, namely (7.3).

If

$$\phi' = \sum \frac{n!}{i!j!k!\cdots} a'_{ijk\cdots} x_1'^i x_2'^j x_3'^k \cdots$$

is one of the ground forms, that set of intransitivity of (8.2) which it induces is in solved form

$$a_{pqr\cdots} = \sum \frac{p!q!r!\cdots}{i!j!k!\cdots} B^{pqr\cdots}_{ijk\cdots}(\xi) a'_{ijk\cdots},$$

where $B^{pqr\cdots}_{ijk\cdots}(\xi)$ is a polynomial in the ξ of degree p in the ξ_{h1}, of degree q in the ξ_{h2}, etc., and of degree i in the ξ_{1h}, of degree j in the ξ_{2h}, etc. In fact, $[a_{n00\cdots}]$ is the ground form ϕ with each x_i replaced by $x_i^{(1)}$, $[a_{0n0\cdots}]$ is ϕ with each x_i replaced by $x_i^{(2)}$, etc. It is not difficult to see that $[a_{p,\, n-p,\, 0\cdots}]$ is, except for a numerical factor, the pth polar of $[a_{0n0\cdots}]$ with respect to the $x^{(1)}$, and the $(n-p)$th polar of $[a_{n00\cdots}]$ with respect to the $x^{(2)}$. Further, all the elementary covariants are, except for non-zero numerical factors, the ground forms in the n cogredient sets of variables and their polars. Every absolute covariant is a function of these.*

9. **Covariants of the general functional transformation.** Consider the group

$$x_i' = f_i(x_1, x_2, \cdots, x_n) \tag{9.1}$$

where the f_i range over all analytic functions of the complex variables $x_1, x_2, \cdots, x_n$ subject to the restriction that $J(x'; x) \neq 0$. This group may conveniently be written

$$x_i' = \sum_{p=0}^{\infty} \sum_{i_1+\cdots+i_n=p} \frac{p!}{i_1!i_2!\cdots i_n!} \overset{(i)}{c}_{i_1\cdots i_n}(x_1-\xi_1)^{i_1}\cdots(x_n-\xi_n)^{i_n} \tag{9.2}$$

where

$$\overset{(i)}{c}_{i_1\cdots i_n} = \left.\frac{\partial^p x_i'}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_n^{i_n}}\right]_\xi.$$

We may look upon this as a group in infinitely many parameters, the c and the ξ, of which the latter are unessential in the sense that the c are functions of them.

The transformation on the parameters corresponding to (2.3) is

$$\zeta_i = f_i(\xi_1, \xi_2, \cdots, \xi_n),$$

$$\left.\frac{\partial x_i''}{\partial x_k}\right]_\zeta = \sum_j \left.\frac{\partial x_i''}{\partial x_j'}\right]_\eta \left.\frac{\partial x_j'}{\partial x_k}\right]_\xi,$$

* A. Capelli, *Lezioni sulla Teoria delle Forme Algebriche*, Naples, 1902, p. 247.

$$
(9.3)\qquad \left.\frac{\partial^2 x_i''}{\partial x_{k_1}\partial x_{k_2}}\right]_\zeta = \sum_j \left.\frac{\partial x_i''}{\partial x_j'}\right]_\eta \left.\frac{\partial^2 x_j'}{\partial x_{k_1}\partial x_{k_2}}\right]_\xi + \sum_{j_1,j_2} \left.\frac{\partial^2 x_i''}{\partial x_{j_1}'\partial x_{j_2}'}\right]_\eta \left.\frac{\partial x_j'}{\partial x_{k_1}}\,\frac{\partial x_j'}{\partial x_{k_2}}\right]_\xi,
$$

$$
\left.\frac{\partial^3 x_i''}{\partial x_{k_1}\partial x_{k_2}\partial x_{k_3}}\right]_\zeta = \sum_j \left.\frac{\partial x_i''}{\partial x_j'}\right]_\eta \left.\frac{\partial^3 x_j'}{\partial x_{k_1}\partial x_{k_2}\partial x_{k_3}}\right]_\xi + \cdots,
$$

$$
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
$$

Out of deference to convention we shall call the new variables of the parameter-group *differentials*, and denote by the symbol

$$
d_{i_1}d_{i_2}\cdots d_{i_p}x_i
$$

that variable which corresponds to the parameter

$$
\left.\frac{\partial^p x_i'}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_p}}\right]_\xi .
$$

Then the first parameter-group of (9.1) is

$$
\begin{aligned}
x_i' &= f_i(x_1, x_2, \cdots, x_n),\\
d_k x_i' &= \sum_j \frac{\partial x_i'}{\partial x_j} d_k x_j,\\
(9.4)\qquad d_{k_1}d_{k_2}x_i' &= \sum_j \frac{\partial x_i'}{\partial x_j} d_{k_1}d_{k_2}x_j + \sum_{j_1,j_2} \frac{\partial^2 x_i'}{\partial x_{j_1}\partial x_{j_2}} d_{k_1}x_{j_1}d_{k_2}x_{j_2},\\
d_{k_1}d_{k_2}d_{k_3}x_i' &= \sum_j \frac{\partial x_i'}{\partial x_j} d_{k_1}d_{k_2}d_{k_3}x_j + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
$$

The theory of sources carries over intact to this situation. The elementary covariants have as their sources the partial derivatives (9.3).

THEOREM 12. *If $[\phi]$ is an absolute covariant whose source is ϕ, then $d_r\phi$ is an absolute covariant whose source is $\partial\phi/\partial x_r$.*

For

$$
\frac{\partial\phi}{\partial x_r} = \sum_p \frac{\partial\phi}{\partial x_p'}\,\frac{\partial x_p'}{\partial x_r},
$$

and therefore

$$
\left[\frac{\partial\phi}{\partial x_r}\right] = \sum_p \frac{\partial\phi}{\partial x_p} d_r x_p = d_r\phi .
$$

With Theorem 4 this gives

THEOREM 13. *Every covariant is a function of the elementary covariants* $[x_i]$ *and their differentials.*

10. Covariants of differential forms. A differential form of order k and degree n is a polynomial in the differentials (9.4), with coefficients which are functions of the x, such that the sum of the products of the orders and degrees of the differentials in each term is n, and at least one differential of order k is present, and none of higher order. Thus the right members of (9.4) are forms of orders $0, 1, 2, \cdots$, etc. If in these forms the partial derivatives are replaced by arbitrary functions, the results are general forms.

In the invariant theory of differential forms, certain forms and "associated functions" (which are merely forms of order 0) are taken as ground forms. The group G_r is the group induced on the coefficients of these ground forms and their differentials by (9.4). It is evident that every absolute covariant of this system is a function of the elementary covariants, whose sources are the coefficients of the ground forms and their differentials.*

Classical tensor analysis is the covariant theory of a system of ground forms of order 1, one of degree 2 and the rest of degree 0. The quadratic form may be written $\sum g_{ij}dx_idx_j$, and the associated functions ϕ_i. A covariant tensor is a system of functions of the g_{ij}, ϕ_i and their partial derivatives which are the coefficients of a differential covariant of order 1. It has not been the practice to recognize covariants of higher order,† although various differential operators d_1, d_2, etc., are employed. Upon differentiation of a covariant of order 1 and degree k, there results a covariant of order 2. By a process known as covariant differentiation, it is possible to form with this covariant of order 2 and others whose sources are functions of the g_{ij} and their derivatives a new covariant of order 1 and degree $k+1$.

The important property of a tensor is that the simultaneous vanishing of its components is invariantive. There seems to be no valid reason for demanding the "tensor law of transformation," which means that the covariant shall be of the first order, for the coefficients of differential covariants of all orders have the above mentioned property. That the use of such covariants is of great value in reducing computation and in expressing new relations has been conclusively shown.‡

* E. Pascal, Atti, Accademia Nazionale dei Lincei, Memorie, (5), vol. 8 (1910), pp. 3–99, treated the covariant theory of forms of order higher than the first by other methods.

† An exception to this statement is the work of E. Nöther, Nachrichten, Gesellschaft der Wissenschaften zu Göttingen, vol. 25 (1918), p. 37.

‡ E. Nöther, loc. cit.

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