

CONVERSE THEOREMS OF SUMMABILITY FOR DIRICHLET'S SERIES*

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1. Let the Dirichlet series

$$(1) \quad F(t) = \sum_{\nu=1}^{\infty} c_{\nu} e^{-\lambda_{\nu} t}, \quad 0 < \lambda_1 < \lambda_2 < \dots, \lambda_n \rightarrow \infty,$$

be convergent for $t > 0$. Let, in addition, the limit

$$(2) \quad \lim_{t \rightarrow +0} F(t) = s$$

exist. It is well known that this is certainly the case whenever the series $\sum c_{\nu}$ converges. But the converse, in general, is not true, as is shown for instance by the example $\lambda_{\nu} = \nu$, $c_{\nu} = (-1)^{\nu}$,

$$F(t) = \sum_1^{\infty} (-1)^{\nu} e^{-\nu t} = -e^{-t}(1 + e^{-t})^{-1} \rightarrow -\frac{1}{2} \text{ as } t \rightarrow 0.$$

Thus we are led to the problem of finding additional conditions which together with the assumption (2) would assure the convergence of $\sum c_{\nu}$.

Such conditions are the following:

$$(3) \quad c_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \text{ as } n \rightarrow \infty. \dagger$$

$$(4a) \quad \limsup_{m \rightarrow \infty} \max_{\lambda_m(1+\delta)^{-1} \leq \lambda_n \leq \lambda_m(1+\delta)} |s_n - s_m| = \psi(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0, \quad \frac{\lambda_{n+1}}{\lambda_n} \rightarrow 1,$$

and

$$(4b) \quad s_n = \sum_{\nu=1}^n c_{\nu} = O(1) \text{ as } n \rightarrow \infty$$

[Landau, 6].

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† See Littlewood [7] in case $\lambda_{n+1}/\lambda_n \rightarrow 1$, Hardy and Littlewood [4] in case

$$\liminf_{n \rightarrow \infty} \frac{\lambda_{n+1}}{\lambda_n} > 1,$$

Ananda-Rau [1] for the rest. The numbers in brackets refer to the list at the end of this paper.

$$(5) \quad \lim_{x \rightarrow \infty} \sup_{\substack{\leq (1+\delta)x \\ x \leq \lambda_k}} |c_k| = \eta(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0, \quad \frac{\lambda_{n+1}}{\lambda_n} \rightarrow 1$$

[Neder, 8].

$$(6) \quad \sum_{\nu=2}^n |c_\nu|^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n), \quad p > 1, \quad \frac{\lambda_{n+1}}{\lambda_n} \rightarrow 1$$

[Szász, 9].

$$(7) \quad \liminf_{x \rightarrow \infty} \text{minimum}_{x \leq \lambda_\nu \leq (1+\delta)x} \sum c_\nu = \phi(\delta) \rightarrow \gamma \geq 0 \text{ as } \delta \rightarrow 0$$

[Szász, 10].

2. The proof of (5) can be reduced to the special case $\lambda_\nu = \nu$ without using the condition $\lambda_{n+1}/\lambda_n \rightarrow 1$. Thus condition $\lambda_{n+1}/\lambda_n \rightarrow 1$ can be omitted in (5). Indeed, let

$$\sum_{\nu-1 < \lambda_k}^{\leq \nu} c_k = b_\nu, \quad \sum_{\nu-1 < \lambda_k}^{\leq \nu} |c_k| = \beta_\nu \quad (\nu = 1, 2, 3, \dots).$$

Then for any $\delta > 0$

$$|b_\nu| \leq \beta_\nu < \eta(\delta) + \delta, \text{ for } \nu > n(\delta),$$

so that

$$(8) \quad \lim_{\nu \rightarrow \infty} b_\nu = \lim_{\nu \rightarrow \infty} \beta_\nu = 0,$$

and the series

$$f(t) = \sum_{\nu=1}^{\infty} b_\nu e^{-\nu t}$$

converges for $t > 0$. Moreover

$$F(t) - |f(t)| = \left| \sum_{\nu=1}^{\infty} \sum_{\nu-1 < \lambda_k}^{\leq \nu} c_k (e^{-\lambda_k t} - e^{-\nu t}) \right| \leq \sum_{\nu=1}^{\infty} \sum_{\nu-1 < \lambda_k}^{\leq \nu} |c_k| (e^{-(\nu-1)t} - e^{-\nu t}).$$

Thus by (8)

$$|F(t) - f(t)| \leq (1 - e^{-t}) \sum_{\nu=1}^{\infty} \beta_\nu e^{-(\nu-1)t} \rightarrow 0 \text{ as } t \rightarrow 0,$$

and, by (2),

$$(2^*) \quad f(t) \rightarrow s \text{ as } t \rightarrow 0.$$

Since

$$\sum_{x \leq \nu}^{\leq (1+\delta)x} |b_\nu| \leq \sum_{x-1 \leq \lambda_k}^{\leq (1+\delta)x} |c_k|$$

and

$$(1 + \delta)x \leq (1 + 2\delta)(x - 1) \text{ for } x \geq \frac{1}{\delta} + 2,$$

$$\limsup_{x \rightarrow \infty} \sum_{x \leq \nu}^{\leq (1+\delta)x} |b_\nu| \leq \eta(2\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Consequently $\sum b_\nu$ converges, which in view of (8) implies the convergence of $\sum c_\nu$. †

3. A similar reduction and generalization is possible for the case (4). Since the expression $e^{-\lambda_k t} - e^{-\nu t}$ is > 0 and monotonely decreasing in k , we have, by (4a),

$$\left| \sum_{\nu-1 < \lambda_k}^{\leq \nu} c_k (e^{-\lambda_k t} - e^{-\nu t}) \right| \leq (e^{-(\nu-1)t} - e^{-\nu t}) \max_{x \leq \nu} \left| \sum_{\nu-1 < \lambda_k}^{\leq x} c_k \right| \leq (1 - e^{-t}) e^{-(\nu-1)t} o(1) \text{ as } \nu \rightarrow \infty.$$

Hence again

$$F(t) - f(t) \rightarrow 0, \quad f(t) \rightarrow s \text{ as } t \rightarrow 0.$$

Now on putting $\sum_{\nu=1}^n b_\nu = B_n$ we have

$$B_n = \sum_{\nu=1}^n \sum_{\nu-1 < \lambda_k}^{\leq \nu} c_k = \sum_{0 < \lambda_k}^{\leq n} c_k, \quad B_n - B_m = \sum_{n < \lambda_k}^{\leq m} c_k, \quad n < m,$$

and, by (4a),

$$\limsup_{m \rightarrow \infty} \max_{m(1+\delta)^{-1} \leq n \leq m(1+\delta)} |B_n - B_m| = \psi(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

We conclude that $\sum b_\nu$ is convergent and from

$$\lim_{x \rightarrow \infty} \max_{x \leq \nu} \left| \sum_{\nu-1 < \lambda_k}^{\leq x} c_k \right| = 0$$

the convergence of $\sum c_\nu$ follows immediately.

The results of Neder and Landau without the assumptions $\lambda_{n+1}/\lambda_n \rightarrow 1$ and (4b) can also be derived from (7), for

$$\text{minimum}_{x \leq \lambda_\nu \leq (1+\delta)x} \sum c_\nu \geq - \text{maximum}_{x \leq \lambda_\nu \leq (1+\delta)x} \left| \sum c_\nu \right|.$$

Finally, in (6) also the restriction $\lambda_{n+1}/\lambda_n \rightarrow 1$ can be removed. During the

† For a similar argument see Ananda-Rau [3].

writing of this paper there appeared an interesting paper by Iyer,* where a proof of this generalization is given. In the present paper I give a proof which is somewhat simpler, and a further generalization.

4. In what follows it is simpler to use Laplace integrals. We intend to prove two auxiliary theorems which are of interest in themselves.

THEOREM 1. *Assume that*

$$(1') \quad F(t) = t \int_0^\infty A(u)e^{-ut}du \text{ converges for } t > 0,$$

$$(2') \quad F(t) \rightarrow s \text{ as } t \rightarrow 0,$$

$$(9) \quad v(x) \equiv xA(x) - \int_0^x A(u)du \geq -Kx, \quad x > 0,$$

where K is a positive constant. Then

$$(10) \quad \frac{1}{x}A_1(x) \equiv \frac{1}{x} \int_0^x A(u)du \rightarrow s \text{ as } x \rightarrow \infty.$$

For the proof we need three lemmas.

LEMMA 1. *Assumptions (1') and (2') imply that the integral*

$$F_1(t) = t \int_0^\infty \frac{1}{u} A_1(u)e^{-ut}du$$

converges for $t > 0$ and

$$(11) \quad F_1(t) \rightarrow s \text{ as } t \rightarrow 0.$$

On integrating by parts in (1') and using (2') we have

$$F(t) = t^2 \int_0^\infty A_1(u)e^{-ut}du \rightarrow s, \quad t \rightarrow 0,$$

where the integral converges absolutely for $t > 0$. Hence

$$\begin{aligned} \int_t^\infty x^{-2}F(x)dx &= \int_0^\infty A_1(u)du \int_t^\infty e^{-ux}dx \\ &= \int_0^\infty \frac{1}{u} A_1(u)e^{-ut}du = O\left(\frac{1}{t}\right), \quad t \rightarrow 0. \end{aligned}$$

Now

$$F(t) - F_1(t) = t \int_t^\infty [F(t) - F(x)] \frac{dx}{x^2} = t \int_t^1 + t \int_{1/2}^\infty, \quad t < 1,$$

* [5]. In the proof on p. 112 the author refers to his Theorem 4 in the statement of which the "O" is to be replaced by "o," as is seen from the proof.

whence

$$|F(t) - F_1(t)| \leq \max_{t \leq x \leq t^{1/2}} |F(t) - F(x)| + t^{1/2} |F(t)| + O(t^{1/2}) \rightarrow 0,$$

and so, by (2'), $F_1(t) \rightarrow s$ as $t \rightarrow 0$.

LEMMA 2. If $a(x) \rightarrow 0$ as $x \rightarrow \infty$, then

$$(12) \quad t \int_0^\infty a(u) e^{-ut} du \rightarrow 0 \text{ as } t \rightarrow 0.$$

This lemma is well known.

LEMMA 3. If $xb(x) \rightarrow 0$ as $x \rightarrow \infty$, and

$$(13) \quad \int_0^\infty b(u) e^{-ut} du \rightarrow B \text{ as } t \rightarrow 0,$$

then the integral $\int_0^\infty b(u) du$ exists as an improper integral and

$$(14) \quad \int_0^\infty b(u) du = B.$$

We have

$$\begin{aligned} \int_0^x b(u) du - \int_0^\infty b(u) e^{-ut} du &= \int_0^x b(u) (1 - e^{-ut}) du - \int_x^\infty b(u) e^{-ut} du \\ &= H_1 - H_2. \end{aligned}$$

For $t = 1/x$, $x \rightarrow \infty$, it follows that

$$|H_1| \leq \frac{1}{x} \int_0^x u |b(u)| du = o(1),$$

$$|H_2| = o\left(\int_x^\infty u^{-1} e^{-u/x} du\right) = o\left(\frac{1}{x} \int_x^\infty e^{-u/x} du\right) = o(1).$$

We now pass on to the proof of Theorem 1. From (2') and (11) it follows that

$$t \int_0^\infty \frac{v(u)}{u} e^{-ut} du \rightarrow 0 \text{ as } t \rightarrow 0,$$

whence

$$t \int_0^\infty \left(\frac{v(u)}{u} + K\right) e^{-ut} du \rightarrow K \text{ as } t \rightarrow 0.$$

By a well known theorem, this and (9) yield

$$\int_0^x \left(\frac{v(u)}{u} + K\right) du \sim Kx \text{ as } x \rightarrow \infty,$$

or

$$(15) \quad \frac{1}{x} \Phi(x) \equiv \frac{1}{x} \int_0^x \frac{v(u)}{u} du \rightarrow 0 \text{ as } x \rightarrow \infty.$$

Furthermore,

$$v(x) = xA(x) - A_1(x) = x^2 \frac{d}{dx} \frac{A_1(x)}{x}.$$

On assuming, as we may without loss of generality, $A(x) = o(x)$ as $x \rightarrow 0$, we have

$$(10') \quad \frac{A_1(x)}{x} = \int_0^x \frac{v(u)}{u^2} du,$$

and, on integrating by parts,

$$\int_0^x \frac{v(u)}{u^2} du = \frac{1}{x^2} v_1(x) + 2 \int_0^x \frac{v_1(u)}{u^3} du; \quad v_1(x) \equiv \int_0^x v(u) du.$$

Thus (11) becomes

$$(11') \quad F_1(t) = t \int_0^\infty \left(\frac{1}{u^2} v_1(u) + 2 \int_0^u \frac{v_1(\tau)}{\tau^3} d\tau \right) e^{-ut} du \rightarrow s, \quad t \rightarrow 0.$$

But on integrating by parts we have by (15)

$$(16) \quad v_1(x) = x\Phi(x) - \int_0^x \Phi(u) du = o(x^2) \text{ as } x \rightarrow \infty,$$

whence, by Lemma 2,

$$t \int_0^\infty u^{-2} v_1(u) e^{-ut} du \rightarrow 0 \text{ as } t \rightarrow 0.$$

Now from (11') it follows that

$$2t \int_0^\infty \left(\int_0^u \tau^{-3} v_1(\tau) d\tau \right) e^{-ut} du \rightarrow s \text{ as } t \rightarrow 0,$$

or, on integrating by parts again,

$$2 \int_0^\infty u^{-3} v_1(u) e^{-ut} du \rightarrow s \text{ as } t \rightarrow 0.$$

Lemma 3 shows that

$$2 \int_0^\infty u^{-3} v_1(u) du = s,$$

whence, by another integration by parts and by (16),

$$\int_0^{\infty} u^{-2}v(u)du = s.$$

Combined with (10') this proves Theorem 1.

5. In order to apply this result to the series (1) put

$$\begin{aligned} \lambda_0 &= 0, \quad s_0 = 0, \\ A(x) &= s_n \text{ for } \lambda_n \leq x < \lambda_{n+1} \quad (n = 0, 1, 2, \dots). \end{aligned}$$

A summation by parts yields $s_n = o(e^{\lambda_n t})$, $t > 0$, and

$$F(t) = \sum_{\nu=1}^{\infty} s_{\nu}(e^{-\lambda_{\nu}t} - e^{-\lambda_{\nu+1}t}) = t \int_0^{\infty} A(u)e^{-ut}du, \quad t > 0.$$

Furthermore, for $\lambda_n \leq x < \lambda_{n+1}$,

$$\begin{aligned} \int_0^x A(u)du &= \sum_{\nu=1}^n (\lambda_{\nu} - \lambda_{\nu-1})s_{\nu-1} + (x - \lambda_n)s_n \\ &= \sum_{\nu=1}^n (x - \lambda_{\nu})c_{\nu} = \sum_{\lambda_{\nu} < x} (x - \lambda_{\nu})c_{\nu}. \end{aligned}$$

Thus it appears that $(1/x)\int_0^x A(u)du$ is the typical mean (R, λ) of the first order of the series $\sum c_{\nu}$. Again

$$v(x) = \lambda_n s_n - \sum_{\nu=1}^n (\lambda_{\nu} - \lambda_{\nu-1})s_{\nu-1} = \sum_{\nu=1}^n \lambda_{\nu} c_{\nu}, \quad \lambda_n \leq x < \lambda_{n+1},$$

or

$$v(x) = \sum_{\lambda_{\nu} \leq x} \lambda_{\nu} c_{\nu}.$$

Now assume

$$(17) \quad v_n = \sum_{\nu=1}^n \lambda_{\nu} c_{\nu} \geq -K\lambda_n \quad (n = 1, 2, 3, \dots).$$

Then

$$\frac{v(x)}{x} = \frac{v_n}{x} \geq -\frac{K\lambda_n}{x} \geq -K, \quad \lambda_n \leq x < \lambda_{n+1},$$

and Theorem 1 yields

THEOREM 1'. *Conditions (1), (2) and (17) imply*

$$(18) \quad \sum_{\lambda_{\nu} < x} (x - \lambda_{\nu})c_{\nu} = \int_0^x A(u)du \sim sx, \quad x \rightarrow \infty.$$

6. Next we prove

THEOREM 2. *Let*

$$(18) \quad \sum_{\lambda_\nu < x} (x - \lambda_\nu) c_\nu \sim sx, \quad x \rightarrow \infty,$$

and let

$$(6') \quad \sum_{\nu=2}^n |c_\nu| \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n) \text{ as } n \rightarrow \infty, \quad p > 1.$$

Then

$$s_n = A(\lambda_n) \rightarrow s \text{ as } n \rightarrow \infty.$$

We start with the identity

$$\delta \lambda_n s_n = \int_0^{(1+\delta)\lambda_n} A(u) du - \int_0^{\lambda_n} A(u) du - \int_{\lambda_n}^{(1+\delta)\lambda_n} [A(u) - A(\lambda_n)] du,$$

or

$$(19) \quad s_n = \frac{1}{(1+\delta)\lambda_n} \int_0^{(1+\delta)\lambda_n} A(u) du \cdot \frac{1+\delta}{\delta} - \frac{1}{\lambda_n} \int_0^{\lambda_n} A(u) du \cdot \frac{1}{\delta} - \frac{1}{\delta \lambda_n} \int_{\lambda_n}^{(1+\delta)\lambda_n} [A(u) - A(\lambda_n)] du, \quad \delta > 0.$$

Here

$$(20) \quad A(u) - A(\lambda_n) = \begin{cases} \sum_{\nu=1}^k c_{n+\nu}, & \lambda_{n+k} \leq u < \lambda_{n+k+1}, \quad k \geq 1, \\ 0, & \lambda_n \leq u < \lambda_{n+1}, \end{cases}$$

and

$$\left| \sum_{\nu=n+1}^{n+k} c_\nu \right| \leq \sum_{\nu=n+1}^{n+k} |c_\nu| = \sum_{\nu=n+1}^{n+k} |c_\nu| \lambda_\nu (\lambda_\nu - \lambda_{\nu-1})^{-1/p'} \frac{(\lambda_\nu - \lambda_{\nu-1})^{1/p'}}{\lambda_\nu}.$$

By Hölder's inequality and by (6'),

$$\sum_{n+1}^{n+k} |c_\nu| = O[\lambda_{n+k}^{1/p} \lambda_{n+1}^{-1} (\lambda_{n+k} - \lambda_n)^{1/p'}], \quad p' = \frac{p}{p-1}.$$

Since $\lambda_{n+k} \leq u \leq (1+\delta)\lambda_n$, we have

$$\sum_{n+1}^{n+k} |c_\nu| = O[(1+\delta)^{1/p} \delta^{1/p'}] = O(\delta^{1/p'}),$$

and

$$A(u) - A(\lambda_n) = O(\delta^{1/p'}), \lambda_n < u \leq (1 + \delta)\lambda_n, n \rightarrow \infty.$$

For a fixed δ it follows from (18) and (19) that

$$\limsup_{n \rightarrow \infty} s_n \leq s + O(\delta^{1/p'}),$$

and, on allowing $\delta \rightarrow 0$,

$$\limsup_{n \rightarrow \infty} s_n \leq s.$$

A similar argument shows

$$\liminf_{n \rightarrow \infty} s_n \geq s,$$

which completes the proof of Theorem 2.

Theorems 1' and 2 immediately yield

THEOREM 3. *Condition (2) and (6') imply that $\sum_1^\infty c_\nu$ converges to s .*

It is plain that (6') implies the convergence of (1) for $t > 0$. It remains only to observe that (6') implies (17). Indeed

$$\begin{aligned} \sum_{\nu=1}^n \lambda_\nu |c_\nu| &= \sum_1^n \lambda_\nu |c_\nu| (\lambda_\nu - \lambda_{\nu-1})^{-1/p'} (\lambda_\nu - \lambda_{\nu-1})^{1/p'} \\ &\leq \left(\sum_1^n |c_\nu|^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} \right)^{1/p} \left(\sum_1^n (\lambda_\nu - \lambda_{\nu-1}) \right)^{1/p'} \\ &= O(\lambda_n^{1/p+1/p'}) = O(\lambda_n). \end{aligned}$$

7. Another generalization of (6) is

$$(6'') \quad \sum_1^n (|c_\nu| - c_\nu)^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n), \quad p > 1, \quad n \rightarrow \infty,$$

while

$$(21) \quad \liminf_{n \rightarrow \infty} c_n \geq 0.*$$

It can be derived from (7), but we can prove it directly, by an argument used above.

First we have

* If $\lambda_{n+1}/\lambda_n \rightarrow 1$, then (21) is a consequence of (6'') as will be shown later (cf. (21')). For the case $\lambda_n = n$ cf. Szász [11].

$$\begin{aligned}
 - \sum_1^n \lambda_r c_r &\leq \sum_1^n \lambda_r (|c_r| - c_r) \leq \left[\sum_1^n (|c_r| - c_r)^p \lambda_r^p (\lambda_r - \lambda_{r-1})^{1-p} \right]^{1/p} \lambda_n^{1/p'} \\
 &= O(\lambda_n).
 \end{aligned}$$

Hence by Theorem 1', (1), (2) and (6'') imply

$$\sum_{\lambda_r < x} (x - \lambda_r) c_r = \int_0^x A(u) du = A_1(x) \sim sx, \quad x \rightarrow \infty.$$

Next we have

$$\begin{aligned}
 - \sum_{n+1}^{n+k} c_r &\leq \sum_{n+1}^{n+k} (|c_r| - c_r) \\
 &\leq \left[\sum_{n+1}^{n+k} (|c_r| - c_r)^p \lambda_r^p (\lambda_r - \lambda_{r-1})^{1-p} \right]^{1/p} \lambda_{n+1}^{-1} (\lambda_{n+k} - \lambda_n)^{1/p'},
 \end{aligned}$$

hence by (20)

$$- \frac{1}{\delta \lambda_n} \int_{\lambda_n}^{(1+\delta)\lambda_n} [A(u) - A(\lambda_n)] du \leq O(\lambda_{n+k}^{1/p} \lambda_{n+1}^{-1} \lambda_n^{1/p'} \delta^{1/p'}) = O(\delta^{1/p'}),$$

and so by (19)

$$\limsup_{n \rightarrow \infty} s_n \leq s + O(\delta^{1/p'}).$$

On allowing here $\delta \rightarrow 0$ we get

$$\limsup_{n \rightarrow \infty} s_n \leq s.$$

Furthermore, since

$$(22) \quad \frac{\delta}{1 + \delta} \lambda_n s_n = \int_0^{\lambda_n} A(u) du - \int_0^{\lambda_n(1+\delta)^{-1}} A(u) du + \int_{\lambda_n(1+\delta)^{-1}}^{\lambda_n} [s_n - A(u)] du$$

and since $s_n = A(\lambda_n)$,

$$s_n - A(u) = \sum_{r=0}^k c_{n-r} \text{ for } \lambda_{n-k-1} \leq u < \lambda_{n-k}, \quad k \geq 0.$$

Now if $k \geq 1$, and $\lambda_{n-k} > u \geq \lambda_n(1 + \delta)^{-1}$, then

$$\begin{aligned}
 - \sum_{r=0}^{k-1} c_{n-r} &\leq \sum_0^{k-1} (|c_{n-r}| - c_{n-r}) \\
 &\leq \left(\sum_{n-k+1}^n (|c_r| - c_r)^p \lambda_r^p (\lambda_r - \lambda_{r-1})^{1-p} \right)^{1/p} \lambda_{n-k}^{-1} (\lambda_n - \lambda_{n-k})^{1/p'},
 \end{aligned}$$

and

$$-\sum_0^{k-1} c_{n-r} \leq \frac{1 + \delta}{\lambda_n} \lambda_n^{1/p'} \left(\frac{\delta}{1 + \delta} \right)^{1/p'} O(\lambda_n^{1/p}) = O(\delta^{1/p'}).$$

Furthermore, by (6''),

$$(21') \quad -c_{n-k} \leq |c_{n-k}| - c_{n-k} = O\left(\left(\frac{\lambda_{n-k} - \lambda_{n-k-1}}{\lambda_{n-k}}\right)^{1/p'}\right).$$

Hence under either one of the assumptions $\lambda_{n+1}/\lambda_n \rightarrow 1$, or $\liminf_{n \rightarrow \infty} c_n \geq 0$, it follows from (22) that

$$\liminf_{n \rightarrow \infty} s_n \geq s - O(\delta^{1/p'}),$$

and on allowing $\delta \rightarrow 0$,

$$\liminf_{n \rightarrow \infty} s_n \geq s.$$

This yields

THEOREM 4. *If (1), (2) and (6'') hold and if at least one of the additional conditions*

$$(a) \quad \frac{\lambda_{n+1}}{\lambda_n} \rightarrow 1, \quad (b) \quad \liminf_{n \rightarrow \infty} c_n \geq 0$$

is satisfied, then $\sum_1^\infty c_n$ converges to s .

Notice that conditions (1), (2) and (6'') imply

$$\sum_{\lambda_\nu < x} (x - \lambda_\nu) c_\nu \sim s \cdot x, \text{ and } \limsup_{n \rightarrow \infty} s_n \leq s,$$

but not the convergence in general. Even the more restrictive condition

$$\lambda_n c_n > -K(\lambda_n - \lambda_{n-1}) \quad (n = 1, 2, 3, \dots)$$

does not imply the convergence, as is shown by an example of Ananda-Rau [2].

8. We now shall state a theorem which includes as special cases not only the results of Landau and Neder but also condition (3) and even Theorem 3. On putting

$$\psi_n(\delta) = \text{maximum}_{\lambda_{n+k} \leq \lambda_n(1+\delta)} |s_{n+k} - s_n|, \quad \psi_n(\delta) = 0 \text{ if } \lambda_{n+k} > \lambda_n(1 + \delta),$$

let us assume

$$\limsup_{n \rightarrow \infty} \psi_n(\delta) = \psi(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

This can be written in the form

$$(23) \quad \limsup_{n \rightarrow \infty} \max_{\lambda_n \leq x \leq (1+\delta)\lambda_n} |A(x) - A(\lambda_n)| = \psi(\delta) \rightarrow 0, \delta \rightarrow 0,$$

or

$$(23') \quad |A(x) - A(\lambda_n)| < \epsilon \text{ for } \lambda_n \leq x \leq (1 + \delta)\lambda_n, \delta = \delta(\epsilon).$$

This condition is satisfied automatically if we have a series with gaps, that is, if for a constant $\theta > 1$

$$\lambda_{n+1} > \theta \lambda_n \quad (n = 1, 2, 3, \dots).$$

Assume the conditions of Theorem 1', so that

$$\sum_{\lambda_r < x} (x - \lambda_r) c_r = A_1(x) \sim s \cdot x, \quad x \rightarrow \infty.$$

This and (23') hold if we assume (2) and (6').

Now using (19) and (23') we get

$$\limsup_{n \rightarrow \infty} s_n \leq s + \epsilon, \quad \liminf_{n \rightarrow \infty} s_n \geq s - \epsilon.$$

Since ϵ is arbitrary, it follows that

$$\lim_{n \rightarrow \infty} s_n = \lim_{x \rightarrow \infty} A(x) = s.$$

Thus we obtain

THEOREM 5. *Conditions (1), (2), (17) and (23) imply*

$$\sum_{r=1}^{\infty} c_r = s.$$

9. Hardy and Littlewood have proved that from

$$\sum_{r=1}^{\infty} \left(\frac{\lambda_r}{\lambda_r - \lambda_{r-1}} \right)^{\rho} |c_r|^{\rho+1} < \infty, \quad \rho > 0,$$

and from (2) follows the convergence of $\sum c_r$. The following generalization is an easy consequence of Theorem 4.

THEOREM 6. *Conditions (1), (2) and*

$$\sum_{r=1}^{\infty} \left(\frac{\lambda_r}{\lambda_r - \lambda_{r-1}} \right)^{\rho} (|c_r| - c_r)^{\rho+1} < \infty, \quad \rho > 0,$$

imply $\sum_{r=1}^{\infty} c_r = s.$

For now we have

$$|c_\nu| - c_\nu = o\left(\left(\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right)^{\rho/(\rho+1)}\right) = o(1),$$

hence condition (b) is satisfied. Moreover on setting

$$u_n = \sum_1^n \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}}\right)^\rho (|c_\nu| - c_\nu)^{\rho+1},$$

$$\sum_1^n (|c_\nu| - c_\nu)^{\rho+1} \lambda_\nu^{\rho+1} (\lambda_\nu - \lambda_{\nu-1})^{-\rho} = u_n \lambda_n - \sum_1^{n-1} u_\nu (\lambda_{\nu+1} - \lambda_\nu) = o(\lambda_n);$$

(6'') holds a posteriori and Theorem 6 is proved.

Finally we observe that condition

$$(6a) \quad \sum_1^n a_\nu^\rho \lambda_\nu^\rho (\lambda_\nu - \lambda_{\nu-1})^{1-\rho} = O(\lambda_n),$$

where a_ν stands for $|c_\nu|$ or for $|c_\nu| - c_\nu$, is equivalent to the following: there exists a constant $g > 1$ such that

$$(6b) \quad \sum_{x < \lambda_\nu \leq gx} a_\nu^\rho (\lambda_\nu - \lambda_{\nu-1})^{1-\rho} = O(x^{1-\rho}) \text{ as } x \rightarrow \infty.$$

For from (6a) it follows that

$$\begin{aligned} \sum_{x < \lambda_\nu \leq 2x} a_\nu^\rho (\lambda_\nu - \lambda_{\nu-1})^{1-\rho} &\leq x^{-\rho} \sum_{x < \lambda_\nu \leq 2x} a_\nu^\rho \lambda_\nu^\rho (\lambda_\nu - \lambda_{\nu-1})^{1-\rho} \\ &= O(x^{1-\rho}). \end{aligned}$$

Conversely on putting $x_\nu = \lambda_\nu g^{-\nu}$ ($\nu = 0, 1, 2, \dots$) we have

$$\begin{aligned} \sum_1^n a_\nu^\rho \lambda_\nu^\rho (\lambda_\nu - \lambda_{\nu-1})^{1-\rho} &= \sum_\nu \sum_{x_{\nu-1} < \lambda_k \leq x_\nu} a_k^\rho \lambda_k^\rho (\lambda_k - \lambda_{k-1})^{1-\rho} \\ &\leq \sum_\nu x_\nu^\rho \sum_{x_{\nu-1} < \lambda_k \leq x_\nu} a_k^\rho (\lambda_k - \lambda_{k-1})^{1-\rho}, \end{aligned}$$

and by (6b)

$$\sum_1^n a_\nu^\rho \lambda_\nu^\rho (\lambda_\nu - \lambda_{\nu-1})^{1-\rho} = O\left(\sum_\nu x_\nu\right) = O\left(\lambda_n \sum_0^\infty g^{-\nu}\right) = O(\lambda_n).*$$

* After this paper was completed and sent to the editors, the author learned of an interesting paper by G. Ricci, *Sui teoremi Tauberiani*, Annali di Matematica, (4), vol. 13 (1935), pp. 287-308, where bounds for oscillation of $A(x)$ are given, under the assumptions (2') and $A(y) - A(x) > -K$ for $0 \leq x \leq y \leq x(1+H)$.

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