NOTES ON LINEAR TRANSFORMATIONS, I*

BY

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Under the above title the author intends to publish some investigations on the properties of linear transformations in abstract spaces. In the present note the space is a suitable subset of the set of all measurable functions defined for $-\infty < x < \infty$, and the transformations are of the form

$$K_\alpha[f] \equiv \alpha \int_{-\infty}^{\infty} K(\alpha t) f(x+t)\,dt, \qquad \alpha > 0. \tag{1}$$

The results, which are somewhat loosely knit together, cluster around four problems. (i) *The originators of zero*, i.e., the solutions of the equation

$$K_\alpha[f] = 0. \tag{2}$$

(ii) *The invariant elements*, i.e., the solutions of the equation

$$K_\alpha[f] = f. \tag{3}$$

(iii) *The functional equations* satisfied by $K_\alpha[f]$ for special choices of the kernel. (iv) *The metric properties* of the transformation $K_\alpha[f]$, including *properties of contraction*, and *degree of approximation* of f by $K_\alpha[f]$ for large values of α. The material is grouped as follows. §1 gives a survey of problems (i), (ii) and (iv) for a general kernel $K(u) \in L_1(-\infty, \infty)$, $K(u) \geqq 0$. It lies in the nature of things that the results for this case are rather incomplete. They probably do not offer much of any novelty to the workers in the field, but serve as background for the discussion in §§3–4. The existence of functional equations obtained by superposition is established in §2, and the equations are given for four particular kernels which may be associated with the names of Dirichlet, Picard, Poisson, and Weierstrass. A closer study of the last two kernels, which satisfy the same functional equation, is given in §3, whereas the kernel of Picard is treated in §4. It turns out that the study of problems (i), (ii) and (iv) for these special kernels is much simplified by the corresponding functional equations. Some results on the Dirichlet kernel occur in §5, but lack the same degree of completeness, sharpness and simplicity.†

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* Presented to the Society, April 20, 1935; received by the editors March 14, 1935.

† The author is indebted to Professor J. D. Tamarkin for helpful criticism.

1. Non-negative kernels in $L_1(-\infty, \infty)$

1.1. We shall be concerned with kernels $K(u)$ satisfying the following conditions:

(K_1) *$K(u)$ is defined as a measurable non-negative function in $(-\infty, \infty)$.*

(K_2) *$\int_{-\infty}^{\infty} K(u)du$ exists and equals unity.*

Let $S = S(K)$ be the set of all functions $f(x)$ satisfying the two conditions

(S_1) *$f(x)$ is defined as a measurable function in $(-\infty, \infty)$, and*

$$K_\alpha[f] \equiv \alpha \int_{-\infty}^{\infty} K(\alpha t) f(x+t) dt \tag{1.11}$$

exists as an ordinary Lebesgue integral for almost all x and all $\alpha > 0$.

(S_2) *$K_\alpha[f] \in S(K)$ for all $\alpha > 0$ whenever $f \in S(K)$.*

It is obvious from these definitions that $S(K)$ is a linear vector space closed under the transformations K_α. We note that if $f(x) \in S(K)$ then all translations of $f(x)$, i.e., the functions $f(x+h)$, also belong to $S(K)$, and that the two operations K_α and translation by h commute.

1.2. Problem (i) calls for the solution of the equation

$$K_\alpha[f] = 0. \tag{1.21}$$

A solution is clearly $f \sim 0$. But is this the only solution? Not always, as we shall see.

Let us denote the Fourier transform of $g(x)$ by $T[x; g]$. Suppose that $K(u) \in L_2(-\infty, \infty)$. It then has a Fourier transform in the same space. Suppose that $f(x)$ is a solution of (1.21) in L_2. Then by a well known formula

$$T[x; f] T[-x/\alpha; K] = 0. \tag{1.22}$$

Here we have two possibilities. (1) $T[-x/\alpha; K]$ vanishes only in a null set. In this case (1.22) implies that $T[x; f] \sim 0$, and consequently also $f \sim 0$, so that $f \sim 0$ is the only solution of (1.21) in L_2. (2) $T[-x/\alpha; K]$ vanishes on a set S of positive measure. We can assume S to be bounded. Let $g(x)$ be a measurable function which is bounded in S and vanishes outside $\overline{S}$, and put $f(x) = T[-x; g]$. This function $f(x)$ is in L_2 and is a solution of (1.21) which is not equivalent to zero. That this case can actually arise is shown by the kernel $K(u) = \pi^{-1} u^{-2}(1 - \cos u)$ whose Fourier transform vanishes for $|x| > 1$.

It is obvious that this method is capable of some extension, but it suffers from the usual limitations due to the severe restrictions which must be imposed upon the function in order that it shall have a Fourier transform. The special kernels considered below in §§3–4 have Fourier transforms nowhere equal to zero, and the particular properties of the kernels will enable us to prove that $f=0$ is the only solution of problem (i) in the corresponding space $S(K)$.

1.3. Problem (ii) calls for the fixed points of $S(K)$, i.e., the solutions of the equation*

$$K_\alpha[f] = f. \tag{1.31}$$

Condition (K_2) shows that $f=1$ is a solution. In many important cases $K(u)$ is an even function of u. If this is so, and $x \in S(K)$, then $f(x)=x$ is a solution of (1.31) for every $\alpha>0$. Consequently every linear function is an invariant. This case is realized for instance for the kernels of Picard and Weierstrass, treated below, but not for that of Poisson, because $f=x$ does not belong to the corresponding space $S(K)$.

The method of Fourier transforms leads to the equation

$$(2\pi)^{1/2}T[x;f]T[-x/\alpha;K] = T[x;f], \tag{1.32}$$

if we assume for the sake of simplicity that $K(u)$ and $f(x)$ are in L_2. We have again two cases. (1) If $T[-x/\alpha;K]=(2\pi)^{-1/2}$ only on a null set, $T[x;f]$ must vanish almost everywhere, i.e., $f\sim 0$ is the only solution of (1.31) in L_2. (2) If, on the other hand, $T[-x/\alpha;K]=(2\pi)^{-1/2}$ on a set of positive measure, a construction similar to that of §1.2 will lead to an invariant manifold in L_2.

1.4. Let us now consider a metric space $M(K)$ which is a sub-set of $S(K)$. We shall suppose that $M(K)$ has the following properties.

(M_1) *It is a normed linear vector space in the sense of Banach, complete with respect to its metric.*

(M_2) $f(x)\in M(K)$ *implies* $K_\alpha[f]\in M(K)$ *for every* $\alpha>0$.

(M_3) $\|K_\alpha[f]\| \leqq \|f\|$.

We shall first consider the possibilities of finding such spaces $M(K)$ in $S(K)$. It is a simple matter to see that every Lebesgue space $L_p(-\infty,\infty)$, $1\leqq p\leqq\infty$, is a sub-space of every $S(K)$, and the same is true of the space $C[-\infty,\infty]$ of the functions which are continuous for $-\infty\leqq x\leqq\infty$. That the customary metrics of these spaces satisfy condition (M_1) is well known, and

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* There are some passing remarks on this problem by N. Wiener and E. Hopf in the introduction to their paper *Ueber eine Klasse singulärer Integralgleichungen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Mathematisch-Physikalische Klasse, 1931, pp. 696–706. They assume that the kernel $K(u)$ vanishes exponentially for large values of $|u|$. In this case the method of bilateral Laplace transforms applies and shows that the solutions are essentially exponential functions. The discussion of the invariant elements of the Weierstrass kernel in §3.4 could have been made somewhat shorter with the aid of this method. [Added in proof, November 2, 1935.]

in order to see that (M_2) and (M_3) are satisfied it is enough to recall the following inequalities:

$$\int_{-\infty}^{\infty} | K_\alpha[f] | \, dx \leqq \alpha \int_{-\infty}^{\infty} K(\alpha t) dt \int_{-\infty}^{\infty} | f(x + t) | \, dx, \tag{1.41}$$

$$\int_{-\infty}^{\infty} | K_\alpha[f] |^p dx \leqq \alpha \int_{-\infty}^{\infty} K(\alpha t) dt \int_{-\infty}^{\infty} | f(x + t) |^p dx, \tag{1.42}$$

$$\text{e.l.u.b.} \, | K_\alpha[f] | \leqq \text{e.l.u.b.} \, | f |. \tag{1.43}$$

The first inequality refers to the case in which $f \epsilon L_1$, and is immediate. The second inequality presupposes $f \epsilon L_p$, $1 < p < \infty$; it follows from Jensen's inequality for convex functions. In (1.43) $f \epsilon L_\infty$, but we have merely to replace the *essential least upper bound* by the *maximum* in order to get the corresponding inequality for $f \epsilon C$.

There is consequently no lack of sub-spaces of $S(K)$ which satisfy our conditions. It is perhaps also possible to find a metric satisfying these conditions which applies to the whole of $S(K)$. Various metrics valid for the space of measurable functions come to mind in this connection, but these metrics normally fail to satisfy the condition $\|\alpha f\| = |\alpha| \, \|f\|$ which is a part of (M_1). This condition is used extensively below, especially in §3.6. But this is actually the only part of our conditions which it seems difficult to impose on $S(K)$; in particular, (M_2) and (M_3) do not cause any trouble.

Condition (M_3) is consequently a natural assumption to make in the study of these kernels. Its geometric significance is that the transformation $K_\alpha[f]$ defines a contraction of the space $M(K)$ for every fixed $\alpha > 0$. In special cases this contraction will be continuous and monotone with respect to α; this is the case with the particular kernels discussed below.

1.5. It is an easy matter to show that (K_1) and (K_2) imply that

$$\lim_{\alpha \to \infty} K_\alpha[f] = f(x) \tag{1.51}$$

at every point of continuity of $f(x)$. Under certain circumstances we can also show convergence of $K_\alpha[f]$ to f in the sense of the metric in $M(K)$.

For this purpose let us introduce the *modulus of continuity* of $f(x)$ defined as

$$\omega(h; f) = \|f(x + h) - f(x)\|, \tag{1.52}$$

where h is fixed, and the norm is taken with respect to x. Further, let $P(u)$ and $Q(u)$ be even continuous functions of u, monotone increasing for $u > 0$, and vanishing for $u = 0$. We have then the following

THEOREM 1.5. *A sufficient condition that*

$$\lim_{\alpha\to\infty} \|K_\alpha[f] - f\| = 0 \tag{1.53}$$

for every $f(x)\epsilon M(K)$ *is that the following assumptions hold*:

(C_1) $f(x)\epsilon M(K)$ *implies* $f(x+h)\epsilon M(K)$, *and* $\|f(x+h)\| = \|f(x)\|$ *for every real* h.

(C_2) *There shall exist two functions* $P(u)$ *and* $Q(u)$ *with the properties stated above such that*

$$\|K_\alpha[f] - f\| \leqq P\{K_\alpha[Q(\omega(t; f))]\}$$

for every $f\epsilon M(K)$.

(C_3) $\lim_{h\to 0} \omega(h; f) = 0$ *for every* $f\epsilon M(K)$.

The proof of this theorem follows standard lines, and can be omitted here. Let us instead consider the justification of imposing such conditions. Our assumptions are satisfied in $L_p(-\infty, \infty)$ for $1 \leqq p < \infty$, but not for $p = \infty$. Indeed, in L_1

$$\int_{-\infty}^{\infty} | K_\alpha[f] - f | \, dx \leqq \alpha \int_{-\infty}^{\infty} K(\alpha t) dt \int_{-\infty}^{\infty} | f(x + t) - f(x) | \, dx$$
$$= K_\alpha[\omega(t; f)],$$

so that (C_2) is satisfied with $P(u) = Q(u) = |u|$. Conditions (C_1) and (C_3) are evidently also satisfied. If $1 < p < \infty$, we have instead

$$\int_{-\infty}^{\infty} | K_\alpha[f] - f |^p dx \leqq \alpha \int_{-\infty}^{\infty} K(\alpha t) dt \int_{-\infty}^{\infty} | f(x + t) - f(x) |^p dx$$
$$= K_\alpha[(\omega(t; f))^p],$$

so that (C_2) is satisfied with $P(u) = |u|^{1/p}$, $Q(u) = |u|^p$. The other conditions are also known to hold. If $p = \infty$, conditions (C_1) and (C_2) still hold, but not (C_3). Moreover, formula (1.53) cannot hold for every $f(x)\epsilon L_\infty$. Indeed, convergence in this space is essentially uniform convergence, and a sequence of continuous functions converges essentially uniformly if and only if it converges uniformly. This of course implies that the limit function is continuous. Since $K_\alpha[f]$ is always a continuous function of x, (1.53) cannot hold when $f(x)$ is discontinuous. In the case of $C[-\infty, \infty]$ we have

$$\max_x | K_\alpha[f] - f | \leqq \alpha \int_{-\infty}^{\infty} K(\alpha t) \max_x | f(x + t) - f(x) | \, dt$$
$$= K_\alpha[\omega(t; f)],$$

so that all three conditions hold.

The assumptions of Theorem 1.5 are clearly not necessary, and various modifications of these assumptions could be given which would preserve their sufficient character. The reader who reconstructs the omitted proof of the theorem will find that the convergence of $K_\alpha[f]$ to f as $\alpha\to\infty$ is uniform in any family of uniformly bounded, equi-continuous functions. He will also get some idea of what degree of approximation is to be expected. In the special cases treated in §§3–4 it is possible to find a best degree of approximation valid for all elements of $M(K)$ which are not invariant.

2. Some functional equations

2.1. For the work of the present paragraph it is convenient to add the following postulate:

$$(K_3) \qquad K(u)\epsilon L_2(-\infty,\infty).$$

We shall also need (K_1), (K_2), (S_1) and (S_2).

For every function $f(x)\epsilon S(K)$ we can form the iterated transformations $K_\alpha[K_\beta[f]]$ and $K_\beta[K_\alpha[f]]$, and they are also elements of $S(K)$. We are particularly interested in those cases in which these superposed transforms are expressible in terms of simple transforms $K_\gamma[f]$, where γ is some function of α and β. Such cases are revealed by the method of Fourier transforms.

Proceeding formally, let us write

$$K_\alpha[K_\beta[f]] = \int_{-\infty}^{\infty} K(u;\alpha,\beta)f(u+x)du, \tag{2.11}$$

where

$$K(u;\alpha,\beta) = \alpha\beta\int_{-\infty}^{\infty} K(\alpha s)K(\beta(u-s))ds. \tag{2.12}$$

Then

$$\begin{aligned} T[x;K(u;\alpha,\beta)] &= \alpha\beta(2\pi)^{1/2}T[x;K(\alpha s)]T[x;K(\beta s)] \\ &= (2\pi)^{1/2}T[x/\alpha;K(u)]T[x/\beta;K(u)], \end{aligned} \tag{2.13}$$

so that

$$K(u;\alpha,\beta) = (2\pi)^{1/2}T^{-1}\{u;T[x/\alpha;K(v)]T[x/\beta;K(v)]\}, \tag{2.14}$$

which can be used for the computation of the composed kernel. This formula is the basis of all the functional equations in the following.

2.2. Let us consider some important special cases.

I. Weierstrass's singular integral. Here

$$K(u) = \pi^{-1/2}e^{-u^2},$$

and

$$T[x; K(u)] = (2\pi)^{-1/2}e^{-x^2/4}.$$

It follows that the Fourier transform of the composed kernel is

$$\exp\left[-\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\frac{x^2}{4}\right],$$

so that the kernel itself becomes

$$\gamma\pi^{-1/2}e^{-\gamma^2u^2}, \qquad \frac{1}{\gamma^2} = \frac{1}{\alpha^2}+\frac{1}{\beta^2}.$$

Hence putting

$$\overline{W}_\alpha[f] = \alpha\pi^{-1/2}\int_{-\infty}^{\infty} e^{-\alpha^2u^2}f(u+x)du, \tag{2.21}$$

we obtain

$$\overline{W}_\alpha[\overline{W}_\beta[f]] = \overline{W}_\gamma[f], \qquad \frac{1}{\gamma^2} = \frac{1}{\alpha^2}+\frac{1}{\beta^2}. \tag{2.22}$$

II. Poisson's integral for the half-plane. Here

$$K(u) = \pi^{-1}(1+u^2)^{-1},$$

and putting

$$\overline{P}_\alpha[f] = \alpha\pi^{-1}\int_{-\infty}^{\infty} \frac{f(u+x)}{1+\alpha^2u^2}du, \tag{2.23}$$

we get

$$\overline{P}_\alpha[\overline{P}_\beta[f]] = \overline{P}_\gamma[f], \qquad \frac{1}{\gamma} = \frac{1}{\alpha}+\frac{1}{\beta}. \tag{2.24}$$

We note that this is essentially the same functional equation as that of the Weierstrass kernel.

III. Picard's singular integral. Here

$$K(u) = \tfrac{1}{2}e^{-|u|},$$

and putting

$$\Pi_\alpha[f] = \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha|u|}f(u+x)du, \tag{2.25}$$

we obtain

$$(\alpha^2 - \beta^2)\Pi_\alpha[\Pi_\beta[f]] = \alpha^2\Pi_\beta[f] - \beta^2\Pi_\alpha[f]. \tag{2.26}$$

IV. Dirichlet's singular integral. Here

$$K(u) = \frac{\sin u}{\pi u}.$$

Putting

$$D_\alpha[f] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin \alpha u}{u} f(u + x)du, \tag{2.27}$$

we get

$$D_\alpha[D_\beta[f]] = D_\gamma[f], \qquad \gamma = \min(\alpha, \beta). \tag{2.28}$$

We note that this kernel does not satisfy either (K_1) or (K_2). This fact makes the investigation of the corresponding transformation much more complicated.

Other examples of simple functional equations could undoubtedly be found in this connection. The importance of these four transformations is such, however, that a special investigation of their properties as revealed by the functional equations is warranted. This will be done below.

3. The Poisson-Weierstrass case

3.1. Equations (2.22) and (2.24) reduce to the common form

$$F_\lambda[F_\mu[f]] = F_{\lambda+\mu}[f] \tag{3.11}$$

by an obvious change of parameters. This equation is consequently satisfied by the two transformations

$$P_\lambda[f] = \frac{\lambda}{\pi}\int_{-\infty}^{\infty}\frac{f(u + x)}{u^2 + \lambda^2}du, \tag{3.12}$$

$$W_\lambda[f] = (\pi\lambda)^{-1/2}\int_{-\infty}^{\infty} e^{-u^2/\lambda}f(u + x)du. \tag{3.13}$$

This fact is undoubtedly well known to mathematical physicists.* $P_\lambda[f]$ is

* Several writers on the theory of the equation of heat conduction have observed such functional equations. P. Appell gave equation (3.11) for $W_\lambda[f]$ in Journal de Mathématiques, (4), vol. 8 (1892), pp. 187–216, p. 201. Cesàro, Académie Royale de Belgique, Bulletin de la Classe des Sciences, 1902, pp. 387–407, p. 392, noted that certain solutions form the elements of an Abelian group. G. Doetsch has produced a number of related transcendental addition theorems; see especially Mathematische Zeitschrift, vol. 25 (1926), pp. 608–626, p. 615. I am not aware of similar considerations having been made for Poisson's integral.

the solution of Dirichlet's problem for the upper half-plane corresponding to the boundary values $f(x)$ on the x-axis, whereas $W_\lambda[f]$ is a solution of the equation of heat conduction in one dimension corresponding to a given initial temperature $f(x)$. These interpretations make equation (3.11) intuitively obvious.

We choose for $S(P)$ and $S(W)$ the classes of measurable functions defined on $(-\infty, \infty)$ for which (3.12) and (3.13) respectively exist as proper Lebesgue integrals for every $\lambda > 0$. This choice is evidently in agreement with (S_1), and a moment's consideration will show that (S_2) is also fulfilled, and that (3.11) holds for any such function $f(x)$. $S(P)$ is simply the class of all $f(x)$ such that $f(x)/(1+x^2)\epsilon L_1(-\infty, \infty)$. $S(W)$ cannot be characterized in such simple terms.

The transforms $P_\lambda[f]$ and $W_\lambda[f]$ are analytic functions of x and of λ. For a fixed real x, $P_\lambda[f]$ defines one analytic function of λ in the right half-plane and another in the left, which are holomorphic in the half-planes in question, whereas $W_\lambda[f]$ is holomorphic in the right half-plane and ordinarily does not exist in the left one. For a fixed positive λ, $P_\lambda[f]$ is an analytic function of x, holomorphic in the strip $-\lambda < \mathfrak{J}(x) < \lambda$,* whereas $W_\lambda[f]$ is an entire function of x.

In the present case formula (1.51) holds in a sharper form, viz.,

$$\lim_{\lambda\to 0} P_\lambda[f] = f(x), \qquad \lim_{\lambda\to 0} W_\lambda[f] = f(x), \tag{3.14}$$

for almost all x whenever $f(x)\epsilon S(P)$ or $S(W)$.

3.2. Problem (i) has a very simple solution in this case:

THEOREM 3.2. *If $P_\alpha[f]=0$ or $W_\alpha[f]=0$ for a fixed α, and $f\epsilon S(P)$ or $S(W)$ respectively, then $f(x)\sim 0$.*

This is pretty well known. A proof is obtained by observing that $F_\alpha[f]=0$ implies $F_{\alpha+\beta}[f]=0$ for every $\beta>0$ by (3.11). $F_\lambda[f]$ being analytic in λ must then vanish identically, and (3.14) shows that this implies $f(x)\sim 0$.

The same argument shows that $F_\alpha[f_1]\neq F_\alpha[f_2]$ unless $f_1(x)\sim f_2(x)$.

3.3. Let us now consider problem (ii). It is required to find whether, for a fixed α, the equation

$$F_\alpha[f] = f \tag{3.31}$$

can have any solution in S other than the trivial one, $f=$constant. If there exists such a solution $f(x)$ then (3.11) shows that the corresponding transform $F_\mu[f]$ satisfies the equation

* $P_\lambda[f]$ also defines two other analytic functions of x, one holomorphic above this strip, the other one below it.

$$F_{\lambda+\alpha}[f] = F_\lambda[f]$$

for every λ. Hence $F_\lambda[f]$ is an analytic function of λ with period α. From this point onwards the two cases must be treated separately.

In the Poisson case we note that if $f(x)\epsilon S(P)$ and x is fixed, then $P_\lambda[f] = o(|\lambda|)$. Hence, if $P_\lambda[f]$ is periodic in λ with period α, it must be a constant with respect to λ, i.e., $P_\lambda[f]=f$ identically in λ. But $P_\lambda[f]$ is a potential function for $\lambda>0$, i.e.,

$$\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\lambda^2}\right\}P_\lambda[f] = 0.$$

Here

$$\frac{\partial^2}{\partial\lambda^2} P_\lambda[f] = 0,$$

since $P_\lambda[f]$ is independent of λ. Hence

$$\frac{\partial^2}{\partial x^2} P_\lambda[f] = 0,$$

so that $P_\lambda[f]=f$ is a linear function of x. But x is clearly not in $S(P)$, hence f is a constant. Thus we have proved

THEOREM 3.3. *The only function in $S(P)$ which is invariant under a Poisson transformation P_α is $f(x)=$*const., *and this function is invariant under all such transformations.*

3.4. The Weierstrass case is rather different. We have seen that $W_\lambda[f]$ must be an analytic function of λ with period α. Being holomorphic in the right half-plane, $W_\lambda[f]$ must then be an entire function of λ as well as of x. We have consequently

$$W_\lambda[f] = \sum_{-\infty}^{\infty} A_n(x)e^{2\pi in\lambda/\alpha}. \tag{3.41}$$

Here the coefficients are entire functions of x which tend to zero faster than $\exp[-B|n|]$, B arbitrary, as $n\to\infty$, x being fixed. But $W_\lambda[f]$ is a solution of the partial differential equation

$$\frac{\partial^2 W}{\partial x^2} = 4\frac{\partial W}{\partial\lambda}, \tag{3.42}$$

and we are clearly entitled to differentiate term by term in (3.41). It follows that

$$A_n''(x) = 8\pi in\alpha^{-1}A_n(x) \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{3.43}$$

Consequently $A_0(x)$ is a linear combination of

$$(3.44)\qquad 1 \text{ and } x,$$

$A_n(x)$ is a linear combination of

$$(3.45)\qquad \exp\,[(2\pi in/\alpha)^{1/2}(1+i)x] \quad \text{and} \exp\,[(2\pi in/\alpha)^{1/2}(-1-i)x]$$

if $n>0$, and of

$$(3.46)\qquad \exp\,[(2\pi i|n|/\alpha)^{1/2}(1-i)x] \text{ and } \exp\,[(2\pi i|n|/\alpha)^{1/2}(-1+i)x]$$

if $n<0$. It follows that the equation (3.31) has a continuum of solutions in the Weierstrass case. These solutions have a denumerable basis, viz., the functions of (3.44)–(3.46). Any linear combination of these functions, the coefficients of which satisfy the restriction of tending to zero faster than any function of n of the form $\exp\,[-B|n|\,]$ as $n\to\infty$, assuming that there are infinitely many terms, is a solution of (3.31).

These solutions are entire functions of x. Their rate of growth is subject to rather interesting limitations. Suppose that x is real and $|f(x)|\leqq Ae^{k|x|}$, where k is a positive constant. A simple calculation shows that for $\lambda=\sigma+i\tau$

$$(3.47)\qquad |W_\lambda[f]|\leqq 2Ae^{k|x|}(\sigma^2+\tau^2)^{1/4}\sigma^{-1/2}\exp\,[k^2(\sigma^2+\tau^2)/(4\sigma)].$$

Suppose now that it is known that $W_\lambda[f]$ has the period α. Then we have the same estimate if we replace λ by $\lambda+n\alpha$. Here we can choose n so as to minimize $((\sigma+n\alpha)^2+\tau^2)/(\sigma+n\alpha)$. This minimum lies arbitrarily close to $2|\tau|$ if $|\tau|$ is large. Hence for an $f(x)$ which produces a periodic solution we can replace (3.47) by

$$(3.48)\qquad |W_\lambda[f]|\leqq 2^{3/2}A\exp\,[k|x|+k^2|\tau|/2].$$

But this estimate implies that $W_\lambda[f]$ is a rational function of $w=\exp\,[2\pi i\lambda/\alpha]$ with singularities only at 0 and ∞, or more precisely, $W_\lambda[f]=w^{-n}P_{2n}(w)$ where $P_{2n}(w)$ is a polynomial in w of degree $\leqq 2n$ and $n=[k^2\alpha/(4\pi)]$. This result gives us additional information about the solutions of (3.31). It follows that any solution which involves infinitely many functions of the basis must occasionally grow faster than any function of the form $e^{k|x|}$ on the real axis. On the other hand, a simple calculation shows that if such a solution is an entire function of order two, it is of the minimal type of that order. Suitably chosen "lacunary series" in terms of the basis functions show that this estimate cannot be essentially improved upon. In the other direction we notice that *the only solution which is at most of the minimal type of order one is $Ax+B$, and this is the only invariant common to all Weierstrass transformations.*

It should be added that the preceding results also permit a complete determination of the solutions of the equation

$$F_\alpha[f] = F_\beta[f]$$

in the two cases under consideration. The reader will have no difficulties in supplying the details.

3.5. We shall now study the character of the deformation defined by $F_\lambda[f]$ in metric sub-spaces of $S(K)$. We consider two sub-sets $M(P)$ and $M(W)$ of $S(P)$ and $S(W)$ respectively which we suppose satisfy postulates (M_1), (M_2) and (M_3). In addition we shall require

(M_4) *$f(x) \epsilon M(K)$ implies $|f(x)| \epsilon M(K)$, and the inequality $|f(x)| \leqq |g(x)|$ for almost all x implies $\|f\| \leqq \|g\|$.*

A particular consequence of (M_4) is that $f(x)$ and $|f(x)|$ have the same norm since $|f(x)| \leqq ||f(x)||$ and vice versa.

An immediate consequence of (M_3) together with the functional equation (3.11) is that

$$\|F_\beta[f]\| \leqq \|F_\alpha[f]\| \quad \text{for } 0 < \alpha < \beta, \tag{3.51}$$

so that the transformation $F_\lambda[f]$ is a steady contraction of the space M, and

$$F_\beta[M] \subset F_\alpha[M] \subset M. \tag{3.52}$$

It follows that $\lim_{\alpha\to\infty} \|F_\alpha[f]\|$ exists and is $\geqq 0$. If $f(x) \epsilon L_p(-\infty, \infty)$, $1 \leqq p < \infty$, or more generally, if

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt = 0, \qquad \frac{1}{2T}\int_{-T}^{T} |f(t)|\, dt \leqq A, \tag{3.53}$$

then

$$\lim_{\alpha\to\infty} F_\alpha[f] = 0 \tag{3.54}$$

for all x by a theorem of N. Wiener.*

$\|F_\lambda[f]\|$ is a functional of $f(x)$ and a function of λ. For a fixed λ it is clearly a continuous functional of $f(x)$ in M by virtue of (M_3). Let us now consider its properties as a function of the real positive variable λ for fixed $f(x)$.

Formula (3.51) expresses that $\|F_\lambda[f]\|$ is a monotone decreasing function of λ. We have for $0<h<\lambda$,

$$\begin{aligned} 0 &\leqq \|F_\lambda[f]\| - \|F_{\lambda+h}[f]\| \leqq \|F_{\lambda+h}[f] - F_\lambda[f]\| \\ &= \|F_\lambda[F_h[f] - f]\| \leqq \|F_h[f] - f\|, \end{aligned}$$

and

* See S. Bochner, *Fouriersche Integrale*, p. 30.

$$0 \leqq \|F_{\lambda-h}[f]\| - \|F_\lambda[f]\| \leqq \|F_\lambda[f] - F_{\lambda-h}[f]\|$$
$$= \|F_{\lambda-h}[F_h[f] - f]\| \leqq \|F_h[f] - f\|.$$

Hence if

$$\lim_{h\to 0} \|F_h[f] - f\| = 0, \tag{3.55}$$

then $\|F_\lambda[f]\|$ is continuous for every $\lambda \geqq 0$, and the elements $F_\lambda[f]$ form a continuous curve in M having f as one of its end points. On the other hand, if $\|F_\lambda[f]\|$ is not continuous at $\lambda = \lambda_0$, but has a jump j at this point, then

$$\lim_{h\to 0} \|F_{\lambda+h}[f] - F_{\lambda-h}[f]\| \geqq j$$

for every λ, $0 < \lambda \leqq \lambda_0$, and the distance from $F_\alpha[f]$ to $F_\beta[f]$ would be at least j if either α or β belongs to the range $[0, \lambda_0]$. In particular, the distance between $f(x)$ and any one of its transforms must be at least j. It seems difficult to exclude this possibility a priori, but we shall show that it cannot occur if F_λ be interpreted as P_λ or W_λ.

3.6. We have

$$P_{\lambda+h}[f] - P_\lambda[f] = \frac{h}{\lambda} P_\lambda[f]$$
$$- h(\lambda + h)(2\lambda + h)\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u + x)du}{(u^2 + \lambda^2)(u^2 + (\lambda + h)^2)}.$$

Here we take norms on both sides, noting that the norm of a sum is not greater than the sum of the norms. In the second term we note that $u^2 + (\lambda + h)^2 \geqq (\lambda + h)^2$, and apply hypothesis (M_4). Combining these steps we get for $h > 0$

$$\|P_{\lambda+h}[f] - P_\lambda[f]\| \leqq \frac{h}{\lambda}\left\{\|P_\lambda[f]\| + \frac{2\lambda + h}{\lambda + h}\|P_\lambda[\,|f|\,]\|\right\}$$
$$< \frac{h}{\lambda}\{\|P_\lambda[f]\| + 2\|P_\lambda[\,|f|\,]\|\},$$

or

$$\|P_{\lambda+h}[f] - P_\lambda[f]\| < 3\frac{h}{\lambda}\|P_\lambda[\,|f|\,]\| \leqq 3\frac{h}{\lambda}\|f\|. \tag{3.61}$$

If $-\lambda < h < 0$ we get instead

$$\|P_{\lambda+h}[f] - P_\lambda[f]\| < 3\frac{|h|}{\lambda}\|P_{\lambda+h}[\,|f|\,]\| \leqq 3\frac{|h|}{\lambda}\|f\|. \tag{3.62}$$

In the case of $W_\lambda[f]$ we have

$$W_{\lambda+h}[f] - W_\lambda[f] = \frac{h}{(\lambda+h)^{1/2}[\lambda^{1/2}+(\lambda+h)^{1/2}]} W_\lambda[f] + \frac{1}{(\pi(\lambda+h))^{1/2}} \int_{-\infty}^{\infty} [e^{-u^2/(\lambda+h)} - e^{-u^2/\lambda}] f(u+x) du.$$

A simple calculation shows that

$$\left| e^{-u^2/(\lambda+h)} - e^{-u^2/\lambda} \right| \leqq \begin{cases} \dfrac{h}{e\lambda} e^{-u^2/(2(\lambda+h))}, & 0 < h, \\ \dfrac{|h|}{e(\lambda+h)} e^{-u^2/(2\lambda)}, & -\lambda < h < 0. \end{cases}$$

Hence we get for $h>0$

$$\|W_{\lambda+h}[f] - W_\lambda[f]\| \leqq \frac{h}{\lambda}\left\{\tfrac{1}{2}\|W_\lambda[f]\| + \frac{2^{1/2}}{e}\|W_{2(\lambda+h)}[|f|]\|\right\},$$

or

$$(3.63)\qquad \|W_{\lambda+h}[f] - W_\lambda[f]\| \leqq 2\frac{h}{\lambda}\|W_\lambda[|f|]\| \leqq \frac{2h}{\lambda}\|f\|,$$

where we have used formula (3.51) in addition to hypothesis (M_4). For $-\lambda<h<0$ we get instead

$$(3.64)\qquad \|W_{\lambda+h}[f] - W_\lambda[f]\| < \frac{2|h|}{\lambda+h}\|W_{\lambda+h}[|f|]\| \leqq \frac{2|h|}{\lambda+h}\|f\|.$$

These formulas show that $\|P_\lambda[f]\|$ and $\|W_\lambda[f]\|$ are continuous families of continuous transformations in $M(P)$ and $M(W)$ respectively for $\lambda>0$. If $\lambda \geqq \lambda_0 > 0$, $\|f\| \leqq B$, these families satisfy a Lipschitz condition of order one with respect to λ, uniformly in λ and in f. It follows in particular that the monotone decreasing functions $\|P_\lambda[f]\|$ and $\|W_\lambda[f]\|$ are continuous for $\lambda>0$.

It is not possible to prove continuity at $\lambda=0$ by these considerations. As a matter of fact we recall from the result of the discussion in §1.5 that (3.55) is not true for all metric sub-spaces of S. In particular, it was shown to be false in $L_\infty(-\infty, \infty)$.

3.7. Let us now consider the transformation

$$(3.71)\qquad E_\lambda = I - F_\lambda,$$

where I is the identity. As a consequence of (3.11) we get

$$E_{\lambda+\mu}[f] = E_\lambda[f] + E_\mu[f] - E_\lambda[E_\mu[f]]. \tag{3.72}$$

It is easy to see that the operations E_λ and F_μ commute. Formula (3.72) is less useful to us than the mixed equation

$$E_{\lambda+\mu}[f] = E_\lambda[f] + E_\mu[F_\lambda[f]] = E_\lambda[f] + F_\lambda[E_\mu[f]]. \tag{3.73}$$

Using the first and the last member of this equation we get

$$\|E_{\lambda+\mu}[f]\| \leqq \|E_\lambda[f]\| + \|F_\lambda[E_\mu[f]]\|,$$

whence, by virtue of (M_3),

$$\|E_{\lambda+\mu}[f]\| \leqq \|E_\lambda[f]\| + \|E_\mu[f]\|. \tag{3.74}$$

A particular consequence of this relation is that

$$\|E_{\alpha 2^{-n}}[f]\| \geqq 2^{-n}\|E_\alpha[f]\|,$$

and this leads to the important conclusion that

$$\limsup_{h\to 0} \frac{1}{h}\|E_h[f]\| \geqq \frac{1}{\alpha}\|E_\alpha[f]\| \tag{3.75}$$

for every fixed positive α. It follows that the degree of approximation of a function $f(x)$ by its Poisson or Weierstrass transform is definitely limited to be of the first order at best. Indeed, if the limit on the left-hand side is zero, then $\|E_\alpha[f]\| = 0$ for every α, i.e., $f(x)$ is an invariant element of the space M under all transformations F_α. These were determined in §3.3 for the Poisson case and in §3.4 for that of Weierstrass. We have consequently proved

THEOREM 3.7. *If $f(x)\epsilon M(P)$ and*

$$\lim_{h\to 0} \frac{1}{h}\|P_h[f] - f\| = 0, \tag{3.76}$$

then $f(x) = \text{const}$. If $f(x)\epsilon M(W)$ and

$$\lim_{h\to 0} \frac{1}{h}\|W_h[f] - f\| = 0, \tag{3.77}$$

then $f(x) = Ax + B$.

It follows in particular that if $M(P)$ or $M(W)$ coincides with $L_p(-\infty, \infty)$, $1 \leqq p < \infty$, then (3.76) or (3.77) implies $f(x) = 0$. The theorem shows that an inequality of the form

$$\|F_h[f] - f\| > Ch \tag{3.78}$$

holds for every $f \epsilon M$ and for infinitely many values of $h \to 0$. Here C is a non-negative constant depending only upon f which equals zero if and only if f is invariant under all transformations F_α. The estimates of §3.6 show on the other hand that the inequality (3.78) can be reversed for all those functions of the space M which are themselves transforms, i.e., which can be written as $f = F_\alpha[g]$ with $g \epsilon M$. It follows that in a space M whose metric satisfies the conditions stated in §3.5 the degree of approximation of a function $f(x)$ by its Poisson or Weierstrass transform is at best of the first order with respect to α, except for the fixed elements, and that this order is actually reached for an infinite subclass of the space, namely by all the transforms.

4. The Picard case

4.1. We shall now take up for discussion Picard's equation

$$(4.11) \qquad (\alpha^2 - \beta^2)\Pi_\alpha[\Pi_\beta[f]] = \alpha^2\Pi_\beta[f] - \beta^2\Pi_\alpha[f], \qquad \alpha \neq \beta.$$

$S(\Pi)$ is the class of all measurable functions $f(x)$ such that (2.25) exists as a proper Lebesgue integral for every $\alpha > 0$. This assumption means that (S_1) is satisfied, and it is easy to see that (S_2) is then also satisfied, and that (4.11) holds for any such function.

The transform $\Pi_\alpha[f]$ is an analytic function of α, regular in the right half-plane. It can be shown that $\Pi_\alpha[f]$ is absolutely continuous and possesses a second-order partial derivative with respect to x for almost all x, and satisfies the differential equation

$$(4.12) \qquad \frac{\partial^2}{\partial x^2}\Pi_\alpha[f] = \alpha^2\{\Pi_\alpha[f] - f\}$$

almost everywhere.

It is well known that

$$(4.13) \qquad \lim_{\alpha \to \infty} \Pi_\alpha[f] = f(x)$$

for almost all x when $f(x) \epsilon S(\Pi)$.

4.2. Formula (4.12) gives us the following complete solution of problem (i).

Theorem 4.2. *If* $\Pi_\alpha[f] = 0$ *for some* $\alpha > 0$, *then* $f(x) \sim 0$.

The same conclusion can be drawn from (4.11) combined with (4.13). The same argument shows that $\Pi_\alpha[f_1] = \Pi_\alpha[f_2]$ implies $f_1(x) \sim f_2(x)$.

4.3. The question of invariant elements is also easily answered. Suppose that for some $\alpha > 0$

$$(4.31) \qquad \Pi_\alpha[f] = f.$$

Formula (4.32) then shows that

$$\frac{\partial^2}{\partial x^2}\Pi_\alpha[f] = 0.$$

Hence we have proved

THEOREM 4.3. *The only functions in $S(\Pi)$ which are invariant under a Picard transformation Π_α are the linear functions, $Ax+B$, and these functions are invariant under all such transformations.*

We recall that it was shown in §1.3 that the linear functions are left invariant by every transformation $K_\alpha[f]$ whose kernel is an even function and which satisfies (K_1) and (K_2). The Picard transformation has consequently no other invariant elements than those common to this class of transformations.

The equation

$$\Pi_\alpha[f] = \Pi_\beta[f] \tag{4.32}$$

can be treated in the same manner. Together with (4.11) it implies

$$\Pi_\alpha[\Pi_\beta[f]-f]=0,$$

whence $\Pi_\beta[f]-f=0$, and $f(x)=Ax+B$.

4.4. Let us now consider a linear sub-space $M(\Pi)$ of $S(\Pi)$ in which we introduce a metric subject to postulates (M_1), (M_2) and (M_3). Note that (M_4) is not assumed. We shall show that (M_3), i.e., the assumption

$$\|\Pi_\alpha[f]\| \leqq \|f\| \tag{4.41}$$

for every $\alpha>0$, implies that

$$\|\Pi_\alpha[f]\| \leqq \|\Pi_\beta[f]\|, \qquad \alpha < \beta, \tag{4.42}$$

i.e., the analogue of (3.51). We can write (4.11)

$$\Pi_\alpha[f] = \frac{1}{\beta^2}\{\alpha^2\Pi_\beta[f] + (\beta^2-\alpha^2)\Pi_\alpha[\Pi_\beta[f]]\}.$$

This gives

$$\begin{aligned}\|\Pi_\alpha[f]\| &\leqq \frac{1}{\beta^2}\{\alpha^2\|\Pi_\beta[f]\| + (\beta^2-\alpha^2)\|\Pi_\alpha[\Pi_\beta[f]]\|\}\\ &\leqq \frac{1}{\beta^2}(\alpha^2+\beta^2-\alpha^2)\|\Pi_\beta[f]\| = \|\Pi_\beta[f]\|.\end{aligned}$$

It follows that $\|\Pi_\alpha[f]\|$ is a monotone increasing function of α, and

(4.43) $$\Pi_\alpha[M] \subset \Pi_\beta[M] \subset M.$$

In particular, $\|\Pi_\alpha[f]\|$ tends to a finite limit $\geqq 0$ as $\alpha \to 0$. The transformation $\Pi_\alpha[f]$ is ordinarily not defined for $\alpha = 0$, and need not tend to any finite limit as $\alpha \to 0$, as is shown by the simple example $\Pi_\alpha[x^2] = x^2 + 2\alpha^{-2}$. On the other hand, if the mean value of $f(x)$ over the range $(-T, T)$ is uniformly bounded with respect to T, and tends to a finite limit $\mathfrak{M}[f]$ as $T \to \infty$, then by Wiener's theorem

(4.44) $$\lim_{\alpha \to 0} \Pi_\alpha[f] = \mathfrak{M}[f]$$

for all x, uniformly over any fixed finite interval. But it is obvious a priori that this result does not enable us to draw any conclusion regarding the numerical value of $\lim_{\alpha \to 0} \|\Pi_\alpha[f]\|$.

4.5. The continuity properties of the Picard transform are on the whole simpler than in the Poisson-Weierstrass case. We can rewrite (4.11) in the form

(4.51) $$\Pi_\alpha[f] - \Pi_\beta[f] = \frac{\alpha^2 - \beta^2}{\beta^2} \Pi_\beta[f - \Pi_\alpha[f]].$$

Putting

(4.52) $$H_\alpha[f] = f - \Pi_\alpha[f],$$

we get

(4.53) $$\Pi_\alpha[f] - \Pi_\beta[f] = \frac{\alpha^2 - \beta^2}{\beta^2} \Pi_\beta[H_\alpha[f]].$$

This relation leads to the inequalities

(4.54) $$\begin{aligned} \|\Pi_\alpha[f] - \Pi_\beta[f]\| &\leqq \left| \left(\frac{\alpha}{\beta}\right)^2 - 1 \right| \|\Pi_\beta[H_\alpha[f]]\| \\ &\leqq \left| \left(\frac{\alpha}{\beta}\right)^2 - 1 \right| \|H_\alpha[f]\| \\ &\leqq 2\left| \left(\frac{\alpha}{\beta}\right)^2 - 1 \right| \|f\|, \end{aligned}$$

since obviously

(4.55) $$\|H_\alpha[f]\| \leqq 2\|f\|.$$

Since Π_β and H_α commute, we have also

$$\|\Pi_\alpha[f] - \Pi_\beta[f]\| \leqq 2\left|\left(\frac{\alpha}{\beta}\right)^2 - 1\right| \|\Pi_\beta[f]\|. \tag{4.56}$$

It follows from these inequalities that $\Pi_\alpha[f]$ regarded an an element of $M(\Pi)$ is continuous with respect to α, $0<\alpha<\infty$, and that $\|\Pi_\alpha[f]\|$ is a continuous function of α in the same range. We do not have continuity at either zero or infinity except in special cases. Formula (4.42) expresses the fact that $\|\Pi_\alpha[f]\|$ is an increasing function of α. The rate of growth is limited by the inequality

$$\|\Pi_\beta[f]\| \leqq \left[2\left(\frac{\beta}{\alpha}\right)^2 - 1\right] \|\Pi_\alpha[f]\|, \qquad \alpha < \beta, \tag{4.57}$$

which is a consequence of (4.54).

4.6. Let us now consider the transformation $H_\alpha[f]$ in more detail. It satisfies the functional equation

$$(\alpha^2 - \beta^2)H_\alpha[H_\beta[f]] = \alpha^2 H_\alpha[f] - \beta^2 H_\beta[f], \tag{4.61}$$

and the mixed equations

$$\alpha^2(H_\alpha[f] - H_\beta[f]) = (\beta^2 - \alpha^2)\Pi_\alpha[H_\beta[f]], \tag{4.62}$$

$$\alpha^2 \Pi_\beta[H_\alpha[f]] = \beta^2 \Pi_\alpha[H_\beta[f]]. \tag{4.63}$$

Suppose that $\alpha<\beta$. Then (4.62) gives

$$\begin{aligned} \alpha^2\|H_\alpha[f]\| &\leqq \alpha^2\|H_\beta[f]\| + (\beta^2 - \alpha^2)\|\Pi_\alpha[H_\beta[f]]\| \\ &\leqq (\alpha^2 + \beta^2 - \alpha^2)\|H_\beta[f]\|, \end{aligned}$$

so that

$$\alpha^2\|H_\alpha[f]\| \leqq \beta^2\|H_\beta[f]\|, \qquad \alpha < \beta. \tag{4.64}$$

This inequality states that

$$\alpha^2\|H_\alpha[f]\| \equiv \alpha^2\|f - \Pi_\alpha[f]\|$$

is an increasing function of α. Hence it can tend to zero as $\alpha\to\infty$ if and only if it is identically zero, i.e., if and only if $f(x)$ is invariant under all Picard transformations. We have consequently proved

THEOREM 4.6. *If $f(x)\epsilon M(\Pi)$, and*

$$\lim_{\alpha\to\infty} \alpha^2\|f - \Pi_\alpha[f]\| = 0, \tag{4.65}$$

then $f(x) = Ax+B$.

It follows that there exists a non-negative constant C for every $f(x)\epsilon M(\Pi)$ such that

$$\alpha^2\|f - \Pi_\alpha[f]\| \geqq C \tag{4.66}$$

for infinitely many values of $\alpha \to \infty$, and $C=0$ if and only if $f(x)=Ax+B$. Hence in a space $M(\Pi)$ whose metric is subject to the restrictions stated above, the degree of approximation of $f(x)$ by its Picard transform is of the second order with respect to $1/\alpha$ at the best. This order is actually reached, however, namely by all elements which are themselves transforms of elements of M, i.e., for every $g=\Pi_\beta[f]$. Indeed, formula (4.63) tells us that

$$\alpha^2\|\Pi_\alpha[\Pi_\beta[f]]\| = \beta^2\|\Pi_\beta[\Pi_\alpha[f]]\|. \tag{4.67}$$

The right-hand side does not exceed $2\beta^2\|f\|$ independently of α. Hence the left-hand side remains bounded as $\alpha \to \infty$, i.e.,

$$\limsup_{\alpha\to\infty} \alpha^2\|g - \Pi_\alpha[g]\| \leqq 2\beta^2\|f\|, \qquad g = \Pi_\beta[f]. \tag{4.68}$$

This proves the assertion.

5. The Dirichlet case

5.1. The kernel in the Dirichlet case differs fundamentally in some respects from the kernels in the cases which we have discussed so far. Thus it satisfies neither (K_1) nor (K_2). One is constantly hampered by these defects when trying to extend the preceding theory to the Dirichlet case. The difficulties start right at the beginning, viz., with the determination of $S(D)$. It is by no means sufficient that (S_1) is satisfied in order that (S_2) be also satisfied as well as the functional equation

$$D_\alpha[D_\beta[f]] = D_\gamma[f], \qquad \gamma = \min(\alpha, \beta). \tag{5.11}$$

Both the originators of the zero element and the invariant elements form linear manifolds which are difficult to characterize. Finally if we come to the question of metric sub-spaces $M(D)$, it turns out that (M_3), which was basic in the previous discussion, is no longer valid in the cases of main interest. The only instance to which our methods obviously apply is the space $L_2(-\infty, \infty)$. Here the transforms exist, belong to the same space, and satisfy (5.11). Problems (i) and (ii) can be completely solved. The space is metric and (M_3) holds. It is not possible to extend all of what we are doing to the case $L_p(-\infty, \infty)$, $p\neq 2$, but we shall note below what results are valid in the more general case. In view of this situation the space will be taken to be $L_2(-\infty, \infty)$ unless otherwise stated.

5.2. The solutions of problem (i) can be obtained by the method of Fourier transforms along the lines given in §1.2. We have

$$T[x; K(u)] = \begin{cases} 0 & \text{for } |x| > 1, \\ (2\pi)^{-1/2} & \text{for } |x| < 1. \end{cases} \tag{5.21}$$

Hence we are confronted with case (2) in the notation of §1.2. It follows that a necessary and sufficient condition that

$$D_\alpha[f] = 0, \qquad f\epsilon L_2, \tag{5.22}$$

is that

$$f(x) = (2\pi)^{-1/2} \operatorname*{l.i.m.}_{a\to\infty} \left\{ \int_{-a}^{-\alpha} + \int_{\alpha}^{a} \right\} e^{ixu}F(u)du, \tag{5.23}$$

where $F(u)$ is an arbitrary function in L_2. The set $\mathfrak{M}$ of all such functions $f(x)$ is obviously a linear manifold in L_2.

5.3. The same method applies to problem (ii). Suppose

$$D_\alpha[g] = g. \tag{5.31}$$

We have again case (2) of §1.3. It follows that a necessary and sufficient condition in order that $g(x)$ shall satisfy (5.31) is that

$$g(x) = (2\pi)^{-1/2} \int_{-\alpha}^{\alpha} e^{ixv}G(u)du, \tag{5.32}$$

where $G(u)$ is an arbitrary function of L_2. The set of all such functions forms a linear manifold $\mathfrak{F}$. We note that $\mathfrak{F}$ and $\mathfrak{M}$ are orthogonal complements of each other in L_2, since

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} T[x; f]\overline{T[x; g]}dx = 0.$$

The discussion and results of §§5.2 and 5.3 extend without difficulty to the case in which we replace L_2 by L_p, $1<p<2$. The case $p=1$ is not accessible because $D_\alpha[f]$ need not be in L_1 when $f\epsilon L_1$.* In case $p>2$ the method breaks down because the method of Fourier transforms fails.

5.4. Supposing $f(x)\epsilon L_2$, let us put $T[x; f]=F(x)$. A simple calculation shows that

$$D_\alpha[f] = (2\pi)^{-1/2} \int_{-\alpha}^{\alpha} e^{ixu}F(u)du. \tag{5.41}$$

We have consequently

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* For the properties of $D_\alpha[f]$ in $L_1(-\infty, \infty)$, $1 \leqq p \leqq \infty$, see E. Hille and J. D. Tamarkin, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 768–774.

$$\|D_\alpha[f]\|^2 = \int_{-\alpha}^{\alpha} |F(u)|^2 du \leqq \|F\|^2 = \|f\|^2, \tag{5.42}$$

or

$$\|D_\alpha[f]\| \leqq \|f\|, \tag{5.43}$$

so that (M_3) holds. We then get from (5.41) that

$$\|D_\alpha[f]\| \leqq \|D_\beta[f]\|, \qquad \alpha < \beta. \tag{5.44}$$

It is obvious that $\|D_\alpha[f]\|$ is continuous, and

$$\lim_{\alpha \to 0} \|D_\alpha[f]\| = 0, \qquad \lim_{\alpha \to \infty} \|D_\alpha[f]\| = \|f\|. \tag{5.45}$$

Further, $\{D_\alpha[f]\}$ is a continuous family of continuous transformations defined over L_2. Let us put

$$E_\alpha[f] = f - D_\alpha[f]. \tag{5.46}$$

$E_\alpha[f]$ is also in L_2. Its Fourier transform equals $F(x)$ for $|x| > \alpha$, and zero for $|x| < \alpha$. Hence

$$\|E_\alpha[f]\|^2 = \left\{ \int_{-\infty}^{-\alpha} + \int_{\alpha}^{\infty} \right\} |F(u)|^2 du. \tag{5.47}$$

It follows that

$$\|E_\alpha[f]\| \geqq \|E_\beta[f]\|, \qquad \alpha < \beta, \tag{5.48}$$

$$\lim_{\alpha \to \infty} \|E_\alpha[f]\| = 0. \tag{5.49}$$

It is clear that (5.49) does not hold uniformly for all $f(x)$ having norms under a fixed bound, nor is it possible to assign any limits one way or the other to the degree of approximation with respect to $1/\alpha$.

Formula (5.41) remains valid for $f(x) \epsilon L_p$, $1 < p < 2$, but while it is true that $D_\alpha[f]$ is a bounded transformation in L_p, it does not seem likely that the bound should be equal to unity. It follows that (5.44) is likely to be false for $p \neq 2$.

5.5. Let us define

$$D_\alpha[f] = 0, \qquad \alpha < 0. \tag{5.51}$$

The family of transformations D_α is then defined for $-\infty < \alpha < \infty$, D_α is zero for $\alpha < 0$, tends to the identity as $\alpha \to \infty$, and is continuous for all values of α. These properties together with formula (5.11) express the fact that $\{D_\alpha[f]\}$ is a *family of projection operators forming the resolution of the identity* *of a self-adjoint transformation H*, in the terminology of J. von Neumann and M. H. Stone. We shall show that

$$H[f] = \tilde{f}'(x) \sim \tilde{f}'(x), \tag{5.52}$$

where

$$\tilde{F}(x) = -\frac{1}{\pi}\text{P.V.}\int_{-\infty}^{\infty} F(u+x)\frac{du}{u}, \tag{5.53}$$

and P.V. denotes that the Cauchy principal value of the integral is to be taken at $u=0$. We recall that $\tilde{F}(x)$ exists for almost all x and is in L_2 if $F(x)$ is in L_2. In the following $f(x)$ is an absolutely continuous function in L_2 whose derivative, $f'(x)$, is also in L_2. We have

$$\begin{aligned} T[\alpha;\tilde{f}'] &= -i\,\text{sgn}\,\alpha T[\alpha; f'] = |\alpha|\, T[\alpha; f], \\ T[\alpha;\tilde{f}'] &= i\alpha T[\alpha;\tilde{f}] \qquad\qquad = |\alpha|\, T[\alpha; f]. \end{aligned} \tag{5.54}$$

These relations also prove the equivalence of the conjugate of the derivative and the derivative of the conjugate function. With the usual notation for the inner product, and assuming $g(x)\epsilon L$,

$$\begin{aligned} (\tilde{f}', g) &= (|\alpha|\, T[\alpha; f],\ T[\alpha; g]) = \int_{-\infty}^{\infty} |\alpha|\, F(\alpha)\overline{G(\alpha)}d\alpha \\ &= \int_0^{\infty} \alpha d_\alpha \int_{-\alpha}^{\alpha} F(u)\overline{G(u)}du = \int_0^{\infty} \alpha d_\alpha (D_\alpha[f], g) \end{aligned}$$

by formula (5.41). It follows that

$$(\tilde{f}', g) = \int_{-\infty}^{\infty} \alpha d_\alpha (D_\alpha[f], g). \tag{5.55}$$

This relation proves formula (5.52).

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