STEREOGRAPHIC PARAMETERS AND PSEUDO-MINIMAL HYPERSURFACES*

BY

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Introduction

This paper consists of two parts: in the first we discuss a general method of representation of a hypersurface by means of a special system of parameters, $x_1, \cdots, x_n$, which we shall call stereographic parameters. In §1 we show how a hypersurface (n-dimensional in E_{r+1}) is expressed in terms of a single function of the parameters which is closely related to a quantity introduced by Painvin† whose value gives the distance of the tangent E_n from the origin of the coordinate system. Next the coefficients of the first and second fundamental forms (g_{ik} and l_{ik}) are obtained and shown to be related in the following manner:

$$g_{ik} = \lambda^2 l_{i\rho} l_{k\rho},$$

where $2\lambda = 1 + x_\rho x_\rho$. This relation suggests that it would be especially simple to determine the surface by giving the l's. In fact (see §2) this problem, which in general requires the integration of a complicated system of partial differential equations, here reduces to quadratures (integral formulas (2.2)). The l tensor must obey certain integrability conditions, the Codazzi equations, which take a surprisingly simple form since they are free from any g_{ik}. In the case of a two-dimensional minimal surface these equations are the Cauchy-Riemann equations and the integral formulas reduce to the Weierstrass representation of a minimal surface in terms of an analytic function. It becomes clear that the ideas underlying the Weierstrass representation are not limited to surfaces of special curvature properties; in particular we show that the real reason of the Weierstrassian theorem about algebraic minimal surfaces is a trivial one, for it lies in the choice of parameters; therefore it is obvious how to generalize it to any hypersurface (§3). Since the stereographic parameters depend upon the coordinate system in the E_{n+1} our next task (§4) is to investigate the resulting arbitrariness. We find that the situation is governed by a certain $n(n+1)/2$-parameter group Ω which is a subgroup of the conformal group in the space $(x_1, \cdots, x_n)$, and which is induced by the rotation group in the E_{n+1}. Because of application in §10 the infinitesimal

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† Journal de Mathématiques, (2), vol. 17 (1872), pp. 219–248.

operators arising from Ω are derived. Our special choice of stereographic representation results in a restricted form of tensor analysis corresponding to Ω (§5). The criterion that an expression or relation involving our representation have geometrical meaning is invariance under Ω. As the simplest invariant relation appears

$$l_{\rho\rho} = 0,$$

whose geometric meaning is that the sum of the n principal radii of curvature vanishes (§6).

The second part of the paper is devoted to the study of the hypersurfaces thus characterized, which we shall refer to as pseudo-minimal hypersurfaces. In view of the connection between these surfaces and analytic functions established by the Weierstrass formulas, the following developments may be considered as a generalization of the theory of analytic functions to higher dimensions. The requirement $l_{\rho\rho}=0$ results in a differential equation for the scalar function ϕ:

$$\lambda \frac{\partial^2 \phi}{\partial x_\rho \partial x_\rho} - n x_\rho \frac{\partial \phi}{\partial x_\rho} + n\phi = 0.$$

Any pseudo-minimal hypersurface can be obtained from a solution of this equation in the following way:

$$X_i = \frac{x_i}{n} \frac{\partial^2 \phi}{\partial x_\rho \partial x_\rho} + \frac{\partial \phi}{\partial x_i}, \qquad X_0 = \frac{1}{n} \frac{\partial^2 \phi}{\partial x_\rho \partial x_\rho}.$$

For $n=2$ this gives a representation of minimal surfaces different from that of Weierstrass. The l_{ik} are also given by very simple formulas in terms of ϕ (§7). We proceed to the integration of the above differential equation using a method analogous to that of separation of variables in mathematical physics by making the formal "Ansatz"

$$\phi = f_l(r) H_l(x_1, \cdots, x_n).$$

Here H_l is a homogeneous polynomial of degree l; f a function of r. It then turns out that H_l must satisfy the n-dimensional Laplace equation, and f_l a hypergeometric differential equation. In the next section (§8) the solutions of this equation are studied in detail. Due to the appearance of integer and half-integer arguments α, β, γ, exceptional cases occur, and the solutions turn out to be Jacobi polynomials in many instances.

Interesting and important are perhaps the cases of centrally symmetric solutions which give rise to axially symmetric hypersurfaces (§9). For $n=2, 3$, and 5 these are actually obtained. We note here the unexpected result that for $n=3$ the hypersurface is generated by "rotating" two-dimensionally a parabola around an axis.

The conclusion briefly touches upon the following topics: development of a general solution; explicit connection with harmonic functions for even n; obtaining particular solutions by applying infinitesimal operators to the central symmetric solution; hypersurfaces of constant sum of radii.

The literature on hypersurfaces can be found in a book by Struik.* The integral representation of §2 was indicated first by one of us.† The representation in terms of the potential was found during the course of the investigation. Although related potentials in three dimensions had been proposed before (Painvin,‡ Minkowski,§ Blaschke||), their use in connection with stereographic parameters, which results in the simple representation of §1, seems not to occur in the literature.

It appears likewise that the special class of hypersurfaces here considered never has been treated in the literature and that no examples of them are known, although the invariant whose vanishing characterizes them can be found expressed by means of other parameters in Forsyth.¶

Part I

1. Stereographic parameters

It is convenient to begin the discussion by considering a surface in ordinary space and then generalize, introducing index notation.

Denote by X, Y, Z the coordinates of a point of a surface, by ξ, η, ζ the components of a unit normal vector at that point, and by x, y the coordinates of the stereographic projection of the point ξ, η, ζ, which moves on a unit sphere when X, Y, Z move on the surface. If the surface is not developable, X, Y, Z will in general be functions of x, y which we shall call "stereographic parameters." From now on we shall consider X, Y, Z and ξ, η, ζ as functions of x, y. We shall write

$$(1.1)\quad r^2 = x^2 + y^2; \quad \lambda = \frac{1 + r^2}{2}; \quad \mu = \frac{1 - r^2}{2}; \quad x^2 + y^2 + \mu^2 = \lambda^2.$$

We have then

$$(1.2)\quad \xi = \frac{x}{\lambda}; \quad \eta = \frac{y}{\lambda}; \quad \zeta = \frac{1 - \lambda}{\lambda} = \frac{\mu}{\lambda}; \quad x = \frac{\xi}{1 + \zeta}; \quad y = \frac{\eta}{1 + \zeta}.$$

* D. J. Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie*, Berlin, 1922. Also A. R. Forsyth, *Geometry of Four Dimensions*, Cambridge, 1930.

† G. Y. Rainich, Comptes Rendus, vol. 180 (1925), p. 801.

‡ Loc. cit.

§ Mathematische Annalen, vol. 57 (1903), pp. 447–496.

|| *Vorlesungen über Differentialgeometrie*, 2d edition, vol. I, 1924, pp. 85, 135.

¶ Loc. cit., p. 42 of vol. II.

We have the following relations, which express the fact that ξ, η, ζ is normal to the surface:

$$(1.3)\qquad \frac{\partial X}{\partial x}\xi+\frac{\partial Y}{\partial x}\eta+\frac{\partial Z}{\partial x}\zeta=0;\qquad \frac{\partial X}{\partial y}\xi+\frac{\partial Y}{\partial y}\eta+\frac{\partial Z}{\partial y}\zeta=0.$$

We now introduce an auxiliary quantity ϕ which for reasons which will be clear later we shall call the potential, viz.,

$$(1.4)\qquad \phi = Xx+Yy+Z\mu = \lambda(X\xi+Y\eta+Z\zeta).$$

The quantity $p=X\xi+Y\eta+Z\zeta$ seems to have been introduced by Painvin.* Minkowski's Stützfunktion is also closely related to it.† Geometrically p means the distance from the origin of coordinates to the tangent plane at the point considered. We note here for future reference the relations

$$(1.5)\qquad \phi=\lambda p;\qquad p=\frac{1}{\lambda}\phi.$$

Returning to the potential ϕ we shall show that X, Y, Z may be expressed in terms of it and its derivatives with respect to x and y. Denoting differentiation with an index, we have from (1.4)

$$\phi_x = X_x x + Y_x y + Z_x\mu + X - Z_x.$$

The first three terms vanish because of (1.3) and we are left with

$$(1.6)\qquad \phi_x = X - Zx,$$

and in the same way we obtain

$$(1.6')\qquad \phi_y = Y - Zy.$$

We substitute X and Y from these relations into (1.5) and obtain, taking into account (1.1),

$$(1.7)\qquad Z=\frac{1}{\lambda}(\phi - x\phi_x - y\phi_y).$$

Using this value in (1.6) and (1.6′) we have

$$(1.7')\qquad \begin{aligned} X &= \frac{1}{\lambda}[x\phi+(\lambda-x^2)\phi_x - xy\phi_y],\\ Y &= \frac{1}{\lambda}[y\phi - xy\phi_x+(\lambda-y^2)\phi_y]. \end{aligned}$$

* Loc. cit.

† Loc. cit.

The formulas (1.7), (1.7′) give a representation of any non-developable surface. The formulas we have obtained generalize without difficulty to an n-dimensional hypersurface in an $(n+1)$-dimensional euclidean space. We use index notations, replacing $n+1$ as an index by 0. The indices a, b, c will range from 0 to n, and all other indices i, j, k, etc., from 1 to n. Greek indices α, β indicate summation from 0 to n, and other Greek letters summation from 1 to n. Although we use index notation and omit summation signs, we do not use orthodox tensor analysis. Quantities such as λ, r are not scalars, in the sense that they are affected by transformation of coordinates in E_{n+1}; but they *are* invariant under rotations in n-space, and therefore as long as we keep the same n-space, distinction between covariant and contravariant quantities is irrelevant and we make all our indices subscripts. We shall discuss these questions in detail in §5.

We write now the extensions of the above formulas giving to the new formulas the same numbers with stars:

$$r^2 = x_\rho x_\rho, \qquad 2\lambda = 1 + x_\rho x_\rho, \qquad x_\rho x_\rho + \mu = \lambda; \tag{1.1*}$$

$$\xi_i = \frac{x_i}{\lambda}, \qquad \xi_0 = \frac{\mu}{\lambda}; \qquad x_i = \frac{\xi_i}{1 + \xi_0}; \tag{1.2*}$$

$$\frac{\partial X_\alpha}{\partial x_i} \xi_\alpha = 0; \tag{1.3*}$$

$$\phi = X_\rho x_\rho + X_0 \mu; \tag{1.4*}$$

$$X_0 = \frac{1}{\lambda}(\phi - x_\rho \phi_\rho); \qquad X_i = \phi_i + \frac{x_i}{\lambda}(\phi - x_\rho \phi_\rho). \tag{1.7*}$$

In these formulas differentiation of ϕ with respect to x_i is indicated by affixing the index i.

Our next task is to compute in our parameters the coefficients of the fundamental forms. We begin by calculating the derivatives

$$\frac{\partial X_i}{\partial x_k} = \phi_{ik} + \frac{\delta_{ik}}{\lambda}(\phi - \phi_\sigma x_\sigma) - \frac{x_i x_\rho}{\lambda}\left[\phi_{\rho k} + \frac{\delta_{\rho k}}{\lambda}(\phi - \phi_\sigma x_\sigma)\right].$$

It is convenient to introduce the notation

$$a_{ik} = \phi_{ik} + \frac{\delta_{ik}}{\lambda}(\phi - \phi_\sigma x_\sigma). \tag{1.8}$$

We can write then

$$\frac{\partial X_i}{\partial x_k} = a_{\rho k}\left(\delta_{\rho i} - \frac{x_\rho x_i}{\lambda}\right). \tag{1.9}$$

We next calculate

$$\frac{\partial X_0}{\partial x_k} = -\frac{x_\rho}{\lambda}\left[\phi_{\rho k} + \frac{\delta_{\rho k}}{\lambda}(\phi - \phi_\sigma x_\sigma)\right]. \tag{1.10}$$

Thus we have for the metric tensor

$$g_{ik} = \frac{\partial X_\alpha}{\partial x_i}\frac{\partial X_\alpha}{\partial x_k} = a_{\rho i}\left(\delta_{\rho\sigma} - \frac{x_\rho x_\sigma}{\lambda}\right)a_{\tau k}\left(\delta_{\tau\sigma} - \frac{x_\tau x_\sigma}{\lambda}\right) + x_\rho a_{\rho i} x_\tau a_{\tau k},$$

or, using the definition of λ,

$$g_{ik} = a_{\sigma i}a_{\sigma k}. \tag{1.11}$$

For the calculation of the coefficients of the second differential form we use the formula (see, e.g., Forsyth, vol. 2, p. 348)

$$l_{jk} = \frac{\partial^2 X_\alpha}{\partial x_j \partial x_k}\xi_\alpha = -\frac{\partial X_\alpha}{\partial x_j}\frac{\partial \xi_\alpha}{\partial x_k}.$$

In our case it gives

$$l_{jk} = -\frac{\partial X_\rho}{\partial x_j}\frac{\partial \xi_\rho}{\partial x_k} - \frac{\partial X_0}{\partial x_j}\frac{\partial \xi_0}{\partial x_k}.$$

But differentiating (1.2*) we have

$$\frac{\partial \xi_i}{\partial x_k} = \frac{\delta_{ik}}{\lambda} - \frac{x_i x_k}{\lambda^2}; \qquad \frac{\partial \xi_0}{\partial x_k} = -\frac{x_k}{\lambda^2}.$$

Together with (1.10) and (1.9) this gives

$$l_{jk} = -\frac{a_{jk}}{\lambda}. \tag{1.12}$$

It should be noted that in our system of representation the l's are simpler than the g's. Combining the formulas (1.11) and (1.12) we obtain

$$g_{ik} = \lambda^2 l_{i\rho} l_{\rho k}. \tag{1.13}$$

2. Generalized Weierstrass formulas

In the classical theory of surfaces it has been known for a long time that a surface may be entirely determined, except for its position in space, by giving the E, F, G, L, M, N as functions of the parameters (O. Bonnet). With our choice of parameters the situation simplifies considerably, because if the

l_{jk} are given the g_{jk} may be considered known as well (1.13), so that the giving of the l_{jk} determines the hypersurface (it is not surprising that we need here fewer functions because our system of parameters does not involve arbitrary functions). In the classical theory the surface can be obtained from the fundamental quantities by integrating a system of differential equations. Here we can reconstruct our hypersurface from the l_{jk} (or the a_{jk}) by quadratures. The coordinates X_α of the hypersurface may namely be obtained from (1.9) and (1.10) by curvilinear integration. We thus obtain the formulas

$$(2.1) \qquad X_i = \int a_{\rho\sigma}\left(\delta_{\rho i} - \frac{x_\rho x_\sigma}{\lambda}\right) dx_\sigma; \qquad X_0 = -\int \frac{a_{\rho\sigma}}{\lambda} x_\rho dx_\sigma.$$

If we introduce l_{ik} we have formulas which give everything directly in terms of the coefficients of the second differential form:

$$(2.2) \qquad X_i = \int l_{\rho\sigma}(x_i x_\rho - \lambda\delta_{i\rho}) dx_\sigma; \quad X_0 = \int l_{\rho\sigma} x_\rho dx_\sigma.$$

As we shall see a little later, these formulas may be considered as generalizations of the Weierstrass formulas for a minimal surface.

Of course, the a's or l's in these formulas cannot be given arbitrarily; they must satisfy certain differential equations which may be obtained as integrability conditions for (1.8), i.e., as conditions on the a's that it should be possible to determine ϕ from (1.8). Introducing the notation

$$(2.3) \qquad \frac{\phi - \phi_\sigma x_\sigma}{\lambda} = F,$$

we have from (1.8)

$$(2.4) \qquad a_{ik} = \phi_{ik} + \delta_{ik}F, \quad \text{or} \quad a_{ik} - \delta_{ik}F = \phi_{ik}.$$

Since the right-hand sides are second derivatives, the a_{ik} and F must satisfy the relations

$$(2.4') \qquad \frac{\partial a_{ik}}{\partial x_j} - \delta_{ik}\frac{\partial F}{\partial x_j} = \frac{\partial a_{ij}}{\partial x_k} - \delta_{ij}\frac{\partial F}{\partial x_k},$$

which give, for i, j, k all different,

$$(2.5) \qquad \frac{\partial a_{ik}}{\partial x_j} = \frac{\partial a_{jk}}{\partial x_i}.$$

Since j and k must be distinct (otherwise (2.4′) are satisfied identically) we may assume now $i=j\neq k$ and have

$$\frac{\partial a_{ik}}{\partial x_i} = \frac{\partial a_{ii}}{\partial x_k} - \frac{\partial F}{\partial x_k}.$$

From this it follows that, for i, k, l all different,

$$\frac{\partial a_{ii}}{\partial x_k} - \frac{\partial a_{ik}}{\partial x_i} = \frac{\partial a_{ll}}{\partial x_k} - \frac{\partial a_{lk}}{\partial x_l}, \tag{2.6}$$

and if (2.5) and (2.6) hold it is easy to show that an F exists which together with ϕ satisfies (2.4).

So far we have not taken into account the relation (2.3). Writing it in the form

$$\lambda F + \phi_\sigma x_\sigma - \phi = 0$$

and differentiating, we obtain

$$\lambda \frac{\partial F}{\partial x_i} + x_i F + \phi_{i\sigma} x_\sigma + \phi_\sigma \delta_{\sigma i} - \phi_i = 0,$$

where the last two terms cancel. On the other hand, from (2.4) we have

$$a_{i\sigma} x_\sigma = \phi_{i\sigma} x_\sigma + F x_i.$$

Together with the last relation this gives

$$\lambda \frac{\partial F}{\partial x_i} + a_{i\sigma} x_\sigma = 0.$$

Differentiating (2.4), contracting in two different ways, and subtracting, we have

$$\frac{\partial a_{\sigma\sigma}}{\partial x_i} - \frac{\partial a_{i\sigma}}{\partial x_\sigma} = (n-1)\frac{\partial F}{\partial x_i}.$$

Now we can eliminate F and obtain

$$\lambda\left(\frac{\partial a_{\sigma\sigma}}{\partial x_i} - \frac{\partial a_{i\sigma}}{\partial x_\sigma}\right) + (n-1) a_{i\sigma} x_\sigma = 0,$$

which may also be written as

$$\lambda^{1-n} \frac{\partial a_{\sigma\sigma}}{\partial x_i} = \frac{\partial \lambda^{1-n} a_{i\sigma}}{\partial x_\sigma}. \tag{2.7}$$

We omit the proof that (2.5), (2.6) and (2.7) constitute also sufficient conditions for (1.8). This system can also be obtained from, and regarded as integrability conditions for, the line integrals (2.1), and was so obtained originally (contributions to this work by Mr. J. L. Coe and Dr. B. C. Getchell in connection with a seminar in differential geometry in 1934 are acknowledged here), but it seems that an early introduction of the potential ϕ permits an easier approach.

In terms of the $l_{ij}=-\lambda^{-1}a_{ij}$ we obtain in place of (2.5), (2.6), and (2.7)

$$\text{(2.8)}\quad \frac{\partial\lambda l_{ij}}{\partial x_k}=\frac{\partial\lambda l_{ik}}{\partial x_j};\qquad \frac{\partial\lambda l_{ii}}{\partial x_k}-\frac{\partial\lambda l_{ik}}{\partial x_i}=\frac{\partial\lambda l_{ll}}{\partial x_k}-\frac{\partial\lambda l_{lk}}{\partial x_l},\qquad \frac{\partial\lambda^{2-n}l_{i\sigma}}{\partial x_\sigma}=\lambda^{1-n}\frac{\partial\lambda l_{\sigma\sigma}}{\partial x_i}.$$

These relations constitute the Codazzi equations in stereographic parameters. In the classical theory the Codazzi equations involve the l's and the g's. Here, since the g's are functions of the l's, the latter alone appear in the equations. Any hypersurface may be given by l's which satisfy these equations. It might also be mentioned that the Gauss equations are satisfied identically.

For $n=2$ only the last set of (2.8) remains, because in the other equations all three indices must be distinct. We have thus

$$\lambda\frac{\partial l_{i\sigma}}{\partial x_\sigma}=\frac{\partial\lambda l_{\sigma\sigma}}{\partial x_i},$$

or, going back to ordinary notations,

$$\lambda\left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}\right)=\frac{\partial\lambda(L+N)}{\partial x},\qquad \lambda\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)=\frac{\partial\lambda(L+N)}{\partial y},$$

or

$$\text{(2.9)}\quad \lambda\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=x(L+N),\qquad \lambda\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)=y(L+N).$$

These equations were given in the paper referred to on page 156.

For a minimal surface (as we shall see in §6) we have the relation

$$L+N=0.$$

Denoting M by u and $L=-N$ by v, (2.9) become in this case the Cauchy-Riemann equations, so that $u+iv=w$ is an analytic function of $z=x+iy$. An easy calculation shows that the formulas (2.2) become in this case the Weierstrass formulas.

3. Representation of algebraic hypersurfaces

Weierstrass* proved that his representation of minimal surfaces possesses the property that every *algebraic* minimal surface is given by an algebraic analytic function, and conversely that to every algebraic analytic function there corresponds an algebraic minimal surface. Now our representation is a

* See Bianchi, *Lezioni di Geometria Differenziale*, 3d edition, vol. I, 1922, p. 540, chapter 12, §204.

generalization of Weierstrass's in two respects: it does not presuppose anything about the curvature as is the case for minimal surfaces, and it holds for any hypersurface of n dimensions in an $(n+1)$-space. It is therefore natural to inquire to what extent the Weierstrass theorem can be generalized. The result of this consideration is the following

THEOREM. *For a hypersurface to be an algebraic hypersurface it is necessary and sufficient that the potential* ϕ *in*

$$X_i = \phi_i - \frac{x_i}{\lambda}(x_\rho\phi_\rho - \phi), \qquad X_0 = \frac{1}{\lambda}(x_\rho\phi_\rho - \phi),$$

be an algebraic function of $x_1, \cdots, x_n$.

That this is a sufficient condition is immediately seen from the above formulas, for differentiation cannot introduce a transcendentality and thus X_i and X_0 will be algebraic functions of the x_i and of each other if ϕ is an algebraic function of x_i. The important part of the theorem is that conversely we obtain *all* algebraic hypersurfaces from algebraic functions ϕ. We therefore proceed to find the function ϕ belonging to any algebraic surface and to show that it is also algebraic.

If the surface $X_0 = X_0(X_1, \cdots, X_n)$ is algebraic then certainly the direction cosines of the normal ξ_j are algebraic functions of the X_0 and X_j. But since for stereographic parameters

$$\xi_0 = \frac{\mu}{\lambda}, \qquad \xi_j = \frac{x_j}{\lambda},$$

the relation between the ξ_0 and ξ_j and the X_0 and X_j is also algebraic. We can therefore express X_0 and X_j as algebraic functions of x_i. Introducing these into

$$\phi = x_\rho X_\rho + (\lambda - 1)X_0$$

we obtain ϕ as an algebraic function of x_i. Thus the theorem is proved.

4. Group properties

The formulas we have obtained are based on the use of a rectangular cartesian coordinate system in E_{n+1}. We wish to see now how they are affected by a transformation of these coordinates. Consider first a translation; for $n=2$ we have

$$X' = X + h, \qquad Y' = Y + k, \qquad Z' = Z + l.$$

Substituting in formula (1.4), since x, y, μ are not affected we obtain for the new ϕ

$$\phi' = \phi + hx + ky + l\mu.$$

For a general n this becomes

$$\phi' = \phi + h_\rho x_\rho + h_0\mu. \tag{4.1}$$

Now consider a change of the axes with the origin preserved. ξ, η, ζ and X, Y, Z will be changed, but the scalar product of the vectors of which they are components will not be affected. The change in ϕ will therefore come from λ. Let the transformation formulas be

$$\xi'_a = s_{a\beta}\xi_\beta \tag{4.2}$$

with the orthogonality condition

$$s_{a\beta}s_{c\beta} = \delta_{ac}.$$

This comprises both rotations (determinant $+1$) and rotation-reflexions (determinant -1). Denoting by x_i' the quantities which in the new system correspond to x_i we have

$$x'_j = \frac{\xi'_j}{1 + \xi_0'} = \frac{s_{j\rho}\xi_\rho + s_{j0}\xi_0}{1 + s_{0\rho}\xi_0 + s_{00}\xi_0} = \frac{N_j}{D}, \tag{4.3}$$

where

$$N_j = s_{j\rho}x_\rho + s_{j0}\mu, \qquad D = \lambda + s_{0\rho}x_\rho + s_{00}\mu. \tag{4.4}$$

These formulas include inversions (transformations by reciprocal radii) for

$$s_{ik} = \delta_{ik}, \qquad s_{00} = -1.$$

The substitutions (4.3) constitute a group which we will call Ω. It should be noted that in spite of the linear appearance of these formulas they are essentially quadratic. For $n=2$ using complex numbers x_1+ix_2 we obtain a subgroup of the group of fractional linear transformations where the essentially quadratic character of the transformations is masked by the use of complex division.

We shall have to use in what follows an expression for

$$\lambda' = \tfrac{1}{2}(1 + x_\rho' x_\rho')$$

which it is easy to calculate from the above formulas. Since we already have

$$\lambda = 1/(1 + \xi_0)$$

we have now

$$\lambda' = \frac{1}{1 + \xi_0'} = \frac{1}{1 + s_{0\rho}\dfrac{x_\rho}{\lambda} + s_{00}\dfrac{\mu}{\lambda}} = \frac{\lambda}{D},$$

or

$$\lambda' = \frac{\lambda}{D}. \tag{4.5}$$

Of course, we have also

$$\lambda = \frac{\lambda'}{D'}, \tag{4.5'}$$

where D' is the denominator of the inverse transformation, or

$$D' = \lambda' + s_{\rho 0}x_\rho' + s_{00}\mu'. \tag{4.5''}$$

Now that we know how λ is affected by our transformations, we can find the law of transformation for ϕ. If we denote the potential in the new coordinate system by $\phi'(x_1', \cdots, x_n') = \phi'(x')$ we have, on account of (1.4),

$$\phi'(x') = \frac{1}{\lambda}\, p[x(x')],$$

where we have to consider λ as a function of the x', i.e.,

$$\phi'(x') = \frac{2}{1 + x_\rho(x')x_\rho(x')}\, p[x(x')].$$

Using again (1.4) and (4.5') this may be written as*

$$\phi'(x') = \frac{1 + x_\rho' x_\rho'}{1 + x_\sigma x_\sigma}\,\phi[x(x')] = D'\phi[x(x')],$$

or introducing a symbol T corresponding to a transformation of coordinates (4.2),

$$T\phi = (s_{0\rho}x_\rho + s_{00}\mu + \lambda)\cdot\phi\left[\frac{s_{i\rho}x_\rho + s_{i0}\mu}{s_{0\rho}x_\rho + s_{00}\mu + \lambda}\right]. \tag{4.6}$$

The transformation formulas for the quantities l_{ik} (or a_{ik}) and g_{ik} are easily obtained, but they will not be used in what follows.

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* It is interesting to compare this situation with that of the harmonic functions (let us say, to fix the ideas, in three dimensions). It is well known that the substitution $x/r^2, \cdots$ for $x, \cdots$ does not carry a harmonic into a harmonic function, but that

$$\frac{1}{r}H\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$$

is harmonic. We get a clearer insight into the situation if we remark that the class of functions $\lambda^{1/2}H(x, y, z)$ is invariant under the reciprocal radii transformations as well as under Ω of which the former are special cases. For a general n the exponent of λ is $n/2-1$.

Although our situation is slightly different from that usually considered in the Lie theory, in that ϕ is transformed not only by substitution but also by multiplication, the idea of infinitesimal transformations and the rules for obtaining them are essentially the same. We consider a one-parameter group $T(h)$ (with canonical parameter h) contained in Ω. This makes the parameters of the group, in our case the s_{ik}, functions of h, so that we can write

$$T(h) = T[s_{ik}(h)].$$

The corresponding infinitesimal transformation is given by

$$(4.7)\qquad \left(\frac{\partial T\phi}{\partial h}\right)_{h=0} = \sum_{a,b}\left(\frac{\partial s_{ab}}{\partial h}\right)_{h=0}\left(\frac{\partial T\phi}{\partial s_{ab}}\right)_{h=0}.$$

In applying this general rule to (4.6) we must note that, for $h=0$, T becomes the identical transformation, so that $(s_{ab})_{h=0}=\delta_{ab}$ and that the $(\partial s_{ab}/\partial h)_0$ are the components of an antisymmetric tensor. Therefore (4.7) may be rewritten as

$$(4.8)\qquad \left(\frac{\partial T\phi}{\partial h}\right)_0 = \sum_{a<b}\left(\frac{\partial s_{ab}}{\partial h}\right)_0\left(\frac{\partial T\phi}{\partial s_{ab}} - \frac{\partial T\phi}{\partial s_{ba}}\right)_0 = \sum_{a<b}\left(\frac{\partial s_{ab}}{\partial h}\right)_0 M_{ab}\phi,$$

where the first factors are independent quantities, $n(n+1)/2$ in number, so that the M_{ab} constitute the infinitesimal transformations. Computing first the case where none of the indices is zero, we have

$$(4.9)\qquad M_{ik}\phi = \left(\frac{\partial T\phi}{\partial s_{ik}} - \frac{\partial T\phi}{\partial s_{ki}}\right)_0 = \left(x_k\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_k}\right)\phi.$$

These are, as was to be expected, the familiar infinitesimal rotations in the x-space; they correspond to rotations which leave the X_0 axis invariant in the X-space. More interesting are the

$$(4.10)\qquad M_{k0}\phi = (\mu\delta_{kr} + x_kx_r)\frac{\partial\phi}{\partial x_r} - x_k\phi = Q_k\phi.$$

The first part of these operators involving the derivatives corresponds to rotations of $(n+1)$-space in the n coordinate planes containing the X_0 axis. In the vicinity of the origin they approximate displacements along the $x_1, \cdots, x_n$ axes.* It is therefore natural to compare the group Ω generated by the Q's and M's with the group of rigid motions in n-space.

Since Ω is isomorphic (in the narrow sense) to the group of rotations in

* The term $-x_k\phi$ in (4.10) is due to the fact that ϕ differs according to (1.5) by a factor λ from the true scalar p.

$(n+1)$-space, the commutation relations must be the same as those of the latter group. They may be written as

$$(M_{ab}, M_{bc}) = M_{ac}.$$

All other M's exchange. In our notation, giving preference to the X_0 axis, this becomes

$$(4.11)\qquad \begin{aligned} (M_{ij}, M_{jk}) &= M_{ik}, \\ (M_{ij}, Q_j) &= Q_i, \\ (Q_i, Q_j) &= -M_{ij}. \end{aligned}$$

These formulas will be used in §10.

5. Tensor-analytical aspects

The quantities x_i may be considered as parameters of the hypersurface or as Gaussian coordinates on it; the quantities l_{ij} are the components of a tensor and so are the quantities g_{ij}. In dealing with these tensors we do not employ general coordinates, but only the special systems obtained by stereographic projection. When we change our coordinates we always pass from one special coordinate system to another. The general rules of tensor calculus apply, of course, in all cases. However, due to the special character of the coordinates used it did not seem advantageous to employ the entire apparatus of tensor analysis.

The most striking peculiarity is that for a tensor f_{ik} the equation

$$(5.1)\qquad f_{\alpha\alpha} = 0$$

is invariant under our transformations. In order to explain the situation we have to make some remarks which are almost trivial.

When we speak of tensors we have in mind quantities which obey known laws of transformations; here we are interested only in transformations corresponding to the coordinate changes just mentioned. As to formation of invariants we must remark that we can use any tensor as a "metric tensor," define raising and lowering of indices with respect to it, and apply this operation in order to obtain invariants of other tensors by means of contraction. A better way of expressing this is to say that covariant tensors have no invariants, but that there exist simultaneous invariants of two tensors, and contracting f_{ij} using the metric tensor g_{ij} is simply a method of building these simultaneous invariants. Ordinarily a definite tensor g_{ij} is singled out by the nature of the problem, and it is used exclusively for the formation of invariants with other tensors. However, if we apply the same transformation of coordinates to two spaces, the metric tensor of each may serve to form invariants with the tensors arising in the study of the other; and in practice we often would use the simplest of these.

In our case the simplest seems to be the metric tensor of the unit sphere which we denote by G_{ij}. It may be calculated by use of the formulas

$$G_{ij} = \frac{\partial \xi_\alpha}{\partial x_i} \frac{\partial \xi_\alpha}{\partial x_j}$$

and (1.2*), from which we obtain

$$\frac{\partial \xi_i}{\partial x_j} = \frac{\delta_{ij}}{\lambda} - \frac{x_i x_j}{\lambda^2}, \qquad \frac{\partial \xi_0}{\partial x_j} = -\frac{x_j}{\lambda},$$

so that†

$$G_{ik} = \frac{\delta_{ik}}{\lambda^2}. \tag{5.2}$$

We find without difficulty

$$G^{ik} = \lambda^2 \delta_{ik}, \tag{5.3}$$

and we may use this to form invariants according to what was said above of any tensor referring to any hypersurface. We shall avoid raising indices, and the only place where we use superscripts will be in metric tensors where upper indices simply denote the inverse of a matrix; i.e., m^{ij} denotes a matrix such that

$$m^{ip} m_{pj} = \delta_{ij}.$$

Given any tensor t_{ij} and any other tensor m_{ij}, we know that

$$m^{\rho\sigma} t_{\rho\sigma}$$

† This formula (5.2) permits us to use the tensor character of G to derive rapidly the conformal character of our transformations. The transformation formula

$$G'_{ij} = G_{\rho\sigma} \frac{\partial x_\rho}{\partial x_i'} \frac{\partial x_\sigma}{\partial x_j'}$$

becomes, using (5.2) and an analogous formula in the other coordinate system,

$$\frac{\delta_{ij}}{\lambda'^2} = \frac{\delta_{\rho\sigma}}{\lambda^2} \frac{\partial x_\rho}{\partial x_i'} \frac{\partial x_\sigma}{\partial x_j'},$$

or, using (4.5),

$$D^2 \delta_{ij} = \frac{\partial x_\rho}{\partial x_i'} \frac{\partial x_\rho}{\partial x_j'},$$

which shows that the matrix

$$\frac{\partial x_k}{\partial x_i'}$$

differs from an orthogonal matrix by a numerical factor D; this is another way of saying that the transformation we are considering is conformal.

is an invariant. In particular

$$G^{\rho\sigma}t_{\rho\sigma}$$

is an invariant, and using the above expression for G^{ik} we find that

$$\lambda^2 t_{\rho\rho} \tag{5.4}$$

is an invariant.

Furthermore, if two tensors t_{ij} and s_{pq} are given, of course $t_{ij}s_{pq}$ is a tensor of rank four, and from it we may obtain a tensor of rank two by contracting with respect to any third tensor m_{ik} so that

$$m^{\rho\sigma}t_{i\rho}s_{\sigma q}$$

is a tensor. Using G for m we have that

$$\lambda^2 t_{i\rho}s_{\rho k}$$

is a tensor. As an example we may mention formula (1.13) which gives the metric tensor g of a hypersurface in terms of the l's of that hypersurface. It becomes clear now why $t_{\rho\rho}=0$ is an invariant equation. It is because $\lambda^2 t_{\rho\rho}$ is an invariant. We have

$$\lambda'^2 t'_{\rho\rho} = \lambda^2 t_{\rho\rho},$$

or, according to (4.5),

$$t'_{\rho\rho} = Dt_{\rho\rho}.$$

The result of summing with respect to two lower indices is therefore in our case a *relative* invariant. Equating a relative invariant to zero we obtain, of course, an invariant equation. The situation we have here is intermediate between that of general tensor analysis and the special case when we consider only rectangular cartesian coordinates. In the latter case we may, of course, use δ_{ij} as a metric tensor, and we may form invariants (they will be invariants only under orthogonal transformations) by summing lower indices, and, in general, we do not have to bother about the level on which the indices are.

Of course it is permissible in considering a hypersurface to use in the orthodox way the g's of *that* hypersurface for raising indices and contracting. For that purpose we need the g^{ik}. In our notation l^{ik} is simply a matrix such that

$$l^{i\rho}l_{\rho k} = \delta_{ik}.$$

The elements of that matrix are the $(n-1)$-row minors of the matrix l_{ik} divided by the determinant l. It is easy to see because of (1.12) that

$$g^{ik} = \frac{1}{\lambda^2} l^{i\rho} l^{\rho i}.$$

Now we can contract any tensor; applying this in particular to l_{ik} we obtain

$$g^{\sigma\tau} l_{\sigma\tau} = \frac{1}{\lambda^2} l^{\sigma\rho} l^{\rho\tau} l_{\sigma\tau} = \frac{1}{\lambda^2} l^{\sigma\sigma},$$

which means the sum of the diagonal minors of l_{ik} divided by the determinant of the l's and by λ^2. This is an invariant which will appear again in the next section. On the other hand we may obtain an invariant by contracting l^{ik} by means of g_{ik}. We obtain

$$g_{\rho\sigma} l^{\rho\sigma} = \lambda^2 l_{\rho\tau} l_{\tau\sigma} l^{\rho\sigma} = \lambda^2 l_{\rho\rho},$$

which is the invariant (5.4) obtained above.

We come now to differential invariants. Here, as before, it seems important to emphasize the point that all such invariants must be considered as *simultaneous* invariants of the given tensor and of the fundamental tensor, and that we may use any tensor as the fundamental tensor. It is natural again to use G_{ik}, the metric tensor of the unit sphere. We first calculate the corresponding three-index symbols

$$\Gamma^i{}_{jk} = \frac{1}{2} G^{i\rho}\left(\frac{\partial G_{\rho k}}{\partial x_j} + \frac{\partial G_{\rho j}}{\partial x_k} - \frac{\partial G_{jk}}{\partial x_\rho}\right)$$
$$= \frac{1}{\lambda}(\delta_{jk} x_i - \delta_{ik} x_j - \delta_{ij} x_k).$$

We can now form differential invariants and covariants. Taking the Painvin function p for instance, which is a scalar, we can form its second covariant derivatives

$$p_{,jk} = \frac{\partial}{\partial x_k}\left(\frac{\partial p}{\partial x_j}\right) - \Gamma^\rho{}_{jk}\frac{\partial p}{\partial x_\rho}$$
$$= \frac{\partial^2 p}{\partial x_j \partial x_k} + \frac{1}{\lambda}\left(\frac{\partial p}{\partial x_j} x_k + \frac{\partial p}{\partial x_k} x_j\right) - \frac{\delta_{ij}}{\lambda}\frac{\partial p}{\partial x_\rho} x_\rho.$$

If we express this in terms of $\phi = \lambda p$ and take into account (1.8) and (1.12) we obtain

$$p_{,jk} = l_{jk} - \frac{\delta_{jk}}{\lambda^2} p,$$

or

$$l_{jk} = p_{,jk} + G_{jk} p.$$

This formula is of a more general character than its derivation by means of stereographic parameters would seem to indicate.

The tensor character of the l's is put here in evidence by expressing it tensor-analytically in terms of a scalar, using the fundamental form of the unit sphere.

We thus see that theoretically the Painvin function is the simplest. But in practice when considering special surfaces it is necessary to use explicit expressions rather than symbolic formulas, and then the potential ϕ in stereographic parameters furnishes the simpler expression for l_{ik}.

We may remark in conclusion that the last formula seems to be related to Minkowski's developments concerning the Stützfunktion; see Blaschke, *Vorlesungen über Differentialgeometrie*, I, §78.

6. CURVATURE INVARIANTS

The theory of hypersurfaces may be regarded as the theory of differential invariants of ϕ under the group of transformations which we have been considering in the last sections. Instead of developing this theory we may, of course, use the classical theory of surfaces, and its generalization, the tensor analysis.

From the general theory of hypersurfaces we know (see e.g., Forsyth, vol. 2, p. 39) that the curvature properties of a hypersurface at a point may be expressed in terms of the roots of the equation

$$|\, l_{ik} - \xi g_{ik} \,| = 0, \tag{6.1}$$

where the vertical bars indicate the determinant of the matrix. These roots, which are obviously invariants, are called the principal curvatures of the hypersurface, and their reciprocals the principal radii of curvature. Very often instead of these irrational invariants their symmetric functions ($I_1 = \xi_1 + \xi_2 + \cdots + \xi_n, \cdots, I_n = \xi_1\xi_2 \cdots \xi_n$) are considered. In the case $n=2$, for instance, the product of these roots is the total curvature, and the sum the mean curvature. All curvature properties at a point may be expressed in terms of these invariants, which are rational functions in the g's and l's. The denominator of I_p is the determinant of the g's, and the numerator is the sum of determinants obtained from this determinant by replacing in all possible ways p rows of the g's by the corresponding l's. The vanishing of these various invariants characterizes important classes of hypersurfaces (for $n=2$, $I_1=0$ gives minimal and $I_2=0$ developable surfaces). In n dimensions there will be n such types; those corresponding to $I_1=0$ have been considered in the literature, and are known as minimal hypersurfaces. The problem of determining such types of hypersurfaces, in other words of integrating the differential equations

(6.2) $$I_p = 0,$$

is a difficult one, because of their non-linear character. The only case ($n>2$) known to us where the hypersurface has been determined is that of the axially symmetric minimal hypersurface in the case $n=3$. (The general case $n>3$ does not present additional difficulties.) They have been determined by A. R. Forsyth (loc. cit., vol. 2, p. 328), and independently by M. Born, in whose theory they serve to describe the field of an electron.*

Using stereographic parameters, equation (6.1) may be simplified considerably if we use for the g's their expressions (1.11) in terms of the l's. The equation becomes

(6.3) $$| \delta_{ij} - \xi\lambda^2 l_{ij} | = 0,$$

and the expressions for the invariants I_p become much simpler. In particular I_{n-1} turns out to be

(6.4) $$I_{n-1} = \lambda^2 l_{\rho\rho}.$$

Its vanishing is equivalent to the differential equation

(6.5) $$l_{\rho\rho} = 0.$$

The corresponding hypersurfaces have the property that the sum of the principal radii of curvature is zero. For $n=2$ they happen to coincide with minimal surfaces. But for $n>2$ they do not seem to be derivable from a variation principle as are the hypersurfaces corresponding to $I_1=0$. These latter may properly be called minimal. We introduce for hypersurfaces which satisfy (6.5) the term pseudo-minimal. Part II is devoted to the investigation of these.

PART II

7. DIFFERENTIAL EQUATION OF PSEUDO-MINIMAL HYPERSURFACES AND ITS REDUCTION

We have seen that the hypersurfaces in question are characterized by the equation $l_{\rho\rho}=0$, or, according to (1.12),

$$a_{\rho\rho} = 0.$$

Using the relations (1.8) which define the a's, we find

(7.1) $$\lambda\phi_{\rho\rho} - nx_\rho\phi_\rho + n\phi = 0,$$

or

$$\lambda\Delta\phi - nx_\rho \frac{\partial\phi}{\partial x_\rho} + n\phi = 0.$$

* Proceedings of the Royal Society, (A), vol. 143 (1934), p. 410, and vol. 144 (1934), p. 425.

This equation is fundamental in the study of our hypersurfaces.

If ϕ is a solution of this equation, formulas (1.7) may be written as

$$X_0 = -\frac{1}{n}\Delta\phi; \qquad X_i = \phi_i - \frac{x_i}{n}\Delta\phi.$$

Using this in (1.8) we find for a_{ik}

$$-\lambda l_{ik} = a_{ik} = \frac{\partial^2\phi}{\partial x_i \partial x_k} - \frac{\delta_{ik}}{n}\frac{\partial^2\phi}{\partial x_\rho \partial x_\rho}. \tag{7.2}$$

Therefore every pseudo-minimal hypersurface may be given by the integral formulas (2.1) in which the a's have the above values. We call ϕ the potential because it appears as an auxiliary quantity in terms of whose derivatives the quasi-tensor a_{ik} (and the tensor l_{ik}) may be expressed.

The determining partial differential equation (7.1) is of such form that it is possible to reduce its complete integration to that of an ordinary differential equation and of the n-dimensional Laplace equation. This can be brought about in several ways. We may, for example, introduce n-dimensional polar coordinates $r, \theta_1, \theta_2, \cdots, \theta_{n-1}$. Then the familiar process of separation of variables allows us immediately to split off the radial differential equation, while a partial differential equation in the angles $\theta_1, \cdots, \theta_{n-1}$ remains which is readily identified as the differential equation of the surface harmonics on an $(n-1)$-dimensional hypersphere.

The method adopted here is not essentially but only formally different from the one just described. Since polar coordinates of n dimensions are clumsy, it is preferable to establish connection with the *solid* harmonics rather than the surface ones.

We therefore make the "Ansatz"

$$\phi_l = H_l(x_1, \cdots, x_n)\cdot f_l, \tag{7.3}$$

where H_l is a homogeneous function of degree $l>0$ and where f_l is a function of the radius-vector or (which is equivalent) of λ. Substituting (7.3) in (7.1) and using repeatedly Euler's theorem $x_\rho \partial H_l/\partial x_\rho = lH_l$, we obtain

$$\lambda f \Delta H + PH = 0, \tag{7.4}$$

where

$$P = \lambda(2\lambda - 1)f'' + (n - n\lambda + 2l\lambda)f' - n(l - 1)f. \tag{7.5}$$

Here primes denote differentiation with respect to λ. In order to separate variables we write (7.4) in the form

$$\frac{r^2\Delta H}{H} = -\frac{Pr^2}{\lambda f}. \tag{7.6}$$

Now the right-hand side being a function of r alone has a constant value on a hypersphere, while the left-hand side being a homogeneous function of $x_1, \cdots, x_n$ of degree zero has constant values along a radius vector. Therefore the common value of the two sides must be a constant, say k. We have then

$$r^2 \Delta H_l = kH_l, \tag{7.7}$$

$$r^2 P = k\lambda f. \tag{7.8}$$

We shall now show that we may always and without loss of generality put k equal to zero. For we can always, without changing the form of the "Ansatz" (7.3), multiply f_l by r^s and H_l by r^{-s}, and we may assume that H_l is regular at $r=0$ but can no longer be divided by a power of r without introducing a singularity at that point. If this is the case, then equation (7.7) implies $k=0$ and we have

$$\Delta H = 0, \tag{7.9}$$

$$P \equiv \lambda(2\lambda - 1)f'' + (n - n\lambda + 2l\lambda)f' - n(l - 1)f = 0. \tag{7.10}$$

Homogeneous solutions of (7.9), which is the n-dimensional Laplace equation, are well known; they are the hyperspherical harmonics.

There exist for every degree l and dimension number n

$$\frac{(l+1)(l+2)\cdots(l+n-3)}{(n-2)!}(2l+n-2)$$

independent solid spherical harmonics. In particular, in three dimensions there are $2l+1$ of them which are connected in the familiar way with the Legendre polynomials. For higher dimensions the reader may look up the corresponding Gegenbauer functions in the Encyclopaedia article by Appell.*

8. Discussion of solution of the radial differential equation

The differential equation (7.10) is immediately recognized to be of hypergeometric type. Introducing

$$\zeta = 1 - 2\lambda = -r^2 \tag{8.1}$$

as independent variable, we obtain the equation in standard form:

$$\zeta(1-\zeta)\frac{d^2f}{d\zeta^2} + \left\{l + \frac{n}{2}\left(l - \frac{n}{2}\right)\zeta\right\}\frac{df}{d\zeta} + \frac{n}{2}(l-1)f = 0, \tag{8.2}$$

with the Gaussian elements

* P. Appell and A. Lambert, tome II, vol. 5, fascicule 2 (II, 28a).

$$\alpha = l - 1; \qquad \beta = -\frac{n}{2}; \qquad \gamma = l + \frac{n}{2}. \tag{8.3}$$

The well known theory of the hypergeometric function* furnishes us with integrals in the vicinity of the three singular points 0, 1, and ∞; it suffices here to consider those around $\zeta = 0$. We have then as a first regular integral

$$f^{(1)} = F\left(l - 1, -\frac{n}{2}, l + \frac{n}{2}; \zeta\right). \tag{8.4}$$

Here $F(\alpha, \beta, \gamma; \zeta)$ is the well known Gaussian series:

$$F(\alpha, \beta, \gamma; \zeta) = 1 + \frac{\alpha\beta}{\gamma}\frac{\zeta}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{\zeta^2}{2!} + \cdots. \tag{8.5}$$

The second integral may be written in either of the following forms:

$$f^{(2)} = \zeta^{1-l-n/2}F\left(1 - l - n, -\frac{n}{2}, 2 - l - \frac{n}{2}; \zeta\right), \tag{8.6a}$$

$$= \zeta^{1-l-n/2}(1-\zeta)^{n+1}F\left(2 - l, 1 + \frac{n}{2}, 2 - l - \frac{n}{2}; \zeta\right). \tag{8.6b}$$

Now it is evident that when the first or second argument α, β of a Gaussian hypergeometric series $F(\alpha, \beta, \gamma; \zeta)$ is a negative integer, the series degenerates into a polynomial, the Jacobi polynomial; and that, when the third argument γ is a negative integer, the series $F(\alpha, \beta, \gamma; \zeta)$ loses its meaning. Thus since different kinds of integers may appear according to whether the number of dimensions n is even or odd, we shall separate the discussion of these two cases.

(1) n is an odd integer. Inspecting (8.4) and (8.6) we see that, due to the appearance of $n/2$ in the third argument, we are safe from any breakdown of solutions. As for appearance of polynomials, we see that $f^{(1)}$ will be polynomial if $l-1$ is negative or zero, i.e., for $l=0$ and $l=1$; in which cases we have

$$f_0^{(1)} = 1 + \zeta = 1 - r^2 = 2\mu, \tag{8.7}$$

$$f_1^{(1)} = 1. \tag{8.8}$$

For $l \geqq 2$ the series no longer breaks off. We mention, however, that even then the series may be summed and written in a closed form, involving polynomials and logarithms.

* See, e.g., Whittaker and Watson, *Modern Analysis*, 3d edition, 1920, chapter 14; Courant-Hilbert, *Methoden der Mathematishen Physik*, vol. 1, chapter II, paragraph 10, p. 74.

Looking at the other solution we see that it always is of the following form: negative power of ζ times a Jacobi polynomial. It therefore behaves like $1/\zeta^{l+n/2-1}$, i.e., like $1/r^{2l+n-2}$ in the vicinity of the origin. Closed expressions are obtained easily since the Nth Jacobi polynomial can be written as Nth derivative of a simple generating function with respect to ζ. We have, using (8.6a),

$$f_l^{(2)} = \lambda^{n+1}\left(\frac{d}{d\lambda}\right)^{l+n-1}(r^n\lambda^{l-2}), \tag{8.9a}$$

or using (8.6b),

$$f_l^{(2)} = \left(\frac{d}{d\lambda}\right)^{l-2}\left(\frac{\lambda^{l+n-1}}{r^{n+2}}\right). \tag{8.9b}$$

(2) n is an even integer. Putting $n=2m$ we see that now the first integral *always* (i.e., not only for $l=0$, and 1) degenerates into a Jacobi polynomial by virtue of its second argument being $-m$. For $l=0$ and $l=1$, (8.7) and (8.8) still hold, but for $l\geqq 2$ the following expression may be used:

$$f_l^{(1)} = \frac{\lambda^{2m+1}}{r^{2m+2l-2}}\left(\frac{d}{d\lambda}\right)^{m}\left(\frac{r^{2l+4m-2}}{\lambda^{m+1}}\right). \tag{8.10}$$

We pass on to the discussion of $f^{(2)}$ for even n. Here we have to expect a breakdown of the solutions (8.6a) and (8.6b), for the third argument $2-l-m$ is always a negative integer. It may however happen, and these are the "exceptions of the second order," especially treated by Klein,* that a negative integer as first or second argument causes the series to break off before it has a chance to acquire infinite terms on account of the third argument being a negative integer. In this case, therefore, one of the first two arguments must be a larger negative integer than the third; or, since both members are negative, it is better to say that the absolute magnitude of one of the first two arguments must be *smaller* than that of the third. Inspecting (8.6a) we see that $l+2m-1$ will always be larger than $l+m-2$ but that (from the second argument)

$$0 \leqq m \leqq l+m-2, \qquad \text{as soon as } l \geqq 2. \tag{8.11a}$$

Comparing the first and third arguments in (8.6b) we see, similarly, that

$$0 \leqq l-2 \leqq l+m-2, \text{ as soon as } l \geqq 2. \tag{8.11b}$$

The exceptions of second order are therefore the rule here; and we obtain from (8.6a)

* Felix Klein, *Vorlesungen über die Hypergeometrische Funktion*, p. 35.

$$f_l^{(2)} = \lambda^{-l+2m-1}\left(\frac{d}{d\lambda}\right)^m \left(\frac{\lambda^{l-m+1}}{r^{2l-2}}\right) \qquad (l \geqq 2), \tag{8.12a}$$

and from (8.6b)

$$f_l^{(2)} = \left(\frac{d}{d\lambda}\right)^{l-2} \left(\frac{\lambda^{l+2m-1}}{r^{2m+2}}\right) \qquad (l \geqq 2). \tag{8.12b}$$

(8.12b) is of course the same as (8.9b), but now it would be no longer correct to apply it for $l<2$.

Solutions $f_0^{(2)}$ and $f_1^{(2)}$ are easily obtained, since we know in each case the other solution, viz. (8.7) and (8.8); we have, using a well known formula,

$$f_0^{(2)} = (1+\zeta)\int \frac{(1-\zeta)^{2m}}{(1+\zeta)^2\zeta^m}\, d\zeta, \tag{8.13}$$

$$f_1^{(2)} = \int \frac{(1-\zeta)^{2m}}{\zeta^{m+1}}\, d\zeta. \tag{8.14}$$

Expanding and integrating by terms, we find expressions containing logarithms and power series beginning with $1/\zeta^{m-1}$ and $1/\zeta^m$ respectively. Of course closed expressions can also be obtained easily by direct integration.

In the following table the above results are summarized. The two columns of the table give the solutions for n even or n odd; the rows for various values of l: 0, 1, and from 2 on. Each row is subdivided and gives in its upper part $f^{(1)}$ and in its lower part $f^{(2)}$.

l	n odd	n even
0	$1-r^2$	$1-r^2$
	$\lambda^{n+1}\left(\frac{d}{d\lambda}\right)^{n-1}\left(\frac{r^n}{\lambda^2}\right)$	$(1-r^2)\log r+\frac{a}{r^{n-2}}+\cdots+br^n$
1	1	1
	$\lambda^{n+1}\left(\frac{d}{d\lambda}\right)^{n}\left(\frac{r^n}{\lambda}\right)$	$\log r+\frac{a}{r^n}+\cdots+br^n$
$l\geqq 2$	$F\left(l-1, -\frac{n}{2}, l+\frac{n}{2}; \zeta\right)=1+ar^2+\cdots$ ad inf.	$\frac{\lambda^{2m+1}}{r^{2m+2l-2}}\left(\frac{d}{d\lambda}\right)^m\frac{r^{2l+4m-2}}{\lambda^{m+1}}$
	$\left(\frac{d}{d\lambda}\right)^{l-2}\left(\frac{\lambda^{l+n-1}}{r^{n+2}}\right)$	$\left(\frac{d}{d\lambda}\right)^{l-2}\left(\frac{\lambda^{l+2m-1}}{r^{2m+2}}\right)$

One sees that always *one* solution is algebraic; but a curious complementary character is exhibited in the occurrence of the case where *both* solutions are algebraic functions, i.e., where the *general* integral of (8.2) is algebraic, a case studied by H. A. Schwarz.† For n odd this case occurs for $l=0$ and $l=1$; for n even only for $l\geqq 2$. Needless to say, in these cases the conditions given by Schwarz are fulfilled. Incidentally the Schwarzian conditions show that for $l\geqq 2$ and n odd, the first solution is indeed a transcendental function of ζ.

The appearance of algebraic solutions has been discussed above somewhat more in detail in view of the theorem of §3.

Until now we have considered only positive values of l. If we wished to consider also negative values of l we should find that the following relation holds:

$$f^{(1)}_{-l-n+2} = \zeta^{l+n/2-1} f^{(2)}_{l}, \tag{8.15}$$

as can be proved either by direct computation or by applying the general theory of the Riemann P-function. This is in agreement with the fact that one can obtain from every homogeneous harmonic function of positive degree l by multiplying it by $r^{-2l-n+2}$ one of negative degree $-l-n+2$, and therefore if we admit negative degrees in (7.3) every potential may be split up in two ways:

$$\phi = H_l f_l = H_{-l-n+2} f_{-l-n+2},$$

furnishing thus two solutions of radial differential equations. One of them is regular and the other singular, in keeping with what is indicated by the superscripts in formula (8.15).

We see thus that we have obtained a large class of potentials which may be presented as products of a harmonic function and a hypergeometric function of the radius vector. In case of a singularity at the origin we have a choice between two such representations, in one of which the hypergeometric function and in the other the harmonic function has that singularity.

9. Pseudo-minimal hypersurfaces of revolution

As in the case of the Laplace equation the spherically symmetric solutions of the differential equation (7.1) are of special interest. Considering in formulas (1.7*) ϕ as a function of r alone we obtain a hypersurface of revolution with the X_0 axis as axis of symmetry. In other words, the intersection of the hypersurface with a hyperplane $X_0=$constant will be an $(n-1)$-dimensional hypersphere. We see from the theorem of §3 and from the table of the last

† H. A. Schwarz, Gesammelte Abhandlungen, vol. 2, pp. 211 ff.

section, that for n odd the surface† will be algebraic, for n even transcendental. We shall discuss a few typical cases.

$n=3$. Using (8.6a) we have, suppressing a multiplicative constant,

$$\phi = \frac{1}{r} - 6r + r^3.$$

Substituting this into (1.7*), we obtain

$$X_i = -\frac{x_i}{r}\left(r + \frac{1}{r}\right)^2,$$

$$X_0 = 4\left(r - \frac{1}{r}\right).$$

Introducing the n-dimensional radius vector $R^2 = X_\rho X_\rho$, we see that the "meridian curve" is the parabola which, adjusting a scale factor, may be written

$$X_0^2 = 2R - 1.$$

$$n = 5. \qquad \phi = \frac{1}{r^3} + 20\frac{1}{r} - 90r + 20r^3 + r^5,$$

and the surface is

$$X_i = -\frac{x_i}{r}\left(r + \frac{1}{r}\right), \qquad X_0 = \frac{8}{3}\left(r^3 - \frac{1}{r^3} + 9\left(r - \frac{1}{r}\right)\right).$$

As meridian curve we obtain

$$54(X_0^2 - 2R + 1)^2 = R^3.$$

As an example of even n we treat the case of $n=2$. Equation (8.13) gives

$$\phi = (1 - r^2) \log (-r^2) - 4(1 - r^2) + 4,$$

or, since we are free to add a multiple of the other (trivial) solution $1-r^2$, we may write, adjusting conveniently a multiplicative constant,

$$\phi = \tfrac{1}{2}(1 - r^2) \log r + \tfrac{1}{2}(1 + r^2).$$

This gives as surface

$$X_i = \frac{x_i}{r}\,\frac{1}{2}\left(r + \frac{1}{r}\right), \qquad X_0 = -\log r,$$

which is the familiar catenoid.

† The solution $\phi = 1 - r^2$ gives only a point.

10. Concluding remarks

The material presented may be developed in several different directions. In this conclusion we shall indicate briefly some of these developments.

(α) **Alternative, Maxwellian, method of obtaining solutions of** (7.1). Both the Laplace equation and our equation (7.1) involve a type of operator considered by Casimir† in connection with the general semi-simple group. In the general case, the operator is of the form

$$K = g^{\lambda\mu} D_\lambda D_\mu, \qquad g_{lm} = c^{\rho}_{l\nu} c^{\nu}_{\rho m},$$

where the D_l are the infinitesimal operators of the group and c^r_{lm} the structure constants of Lie's third fundamental theorem; and it is readily seen that any D commutes with K. It follows from this that from any solution ϕ of a differential equation

$$(K + \alpha I)\phi = 0$$

(where I is the identical operator and α a constant), other solutions may be obtained by applying any one of, or a succession of, the operators D_l to ϕ.

In the case of the Laplace equation, the group in question is the group of translation and $\alpha = 0$. The operators D_k are here $\partial/\partial x_k$, and we are led to Maxwell's method of obtaining solutions of the Laplace equation, which in case $n = 3$ and using $\phi = 1/r$ happens to give essentially all solutions.

Turning to our differential equation (7.1), a straightforward calculation shows that it can be written in terms of the infinitesimal operators (4.9) and (4.10) of the group Ω in the form

$$\left(\sum_k Q_k^2 + \sum_{i<k} M_{ik}^2 + n \right) \phi = 0.$$

This leads to an alternative, Maxwellian, method of constructing solutions. We may, by elementary methods, obtain a central symmetric solution ϕ_0. From it an infinity of other solutions may be derived by acting upon it with Q or M or a combination of them. We thus obtain linear combinations of the particular solutions of §§7 and 8. The relationship between the former point of view and the present is best exhibited by the following formula:

$$Q\phi_l = a\phi_{l-1} + b\phi_{l+1},$$

where a and b depend only upon l. This formula is readily derived, using a recursion formula.

† Proceedings of the Royal Academy of Amsterdam, vol. 34 (1931), p. 144. See also *Rotation of a Rigid Body in Quantum Mechanics*, by the same author, Leiden dissertation, 1931, chapter 4.

(β) **Series expansion.** The potentials ϕ_l introduced in §§7 and 8 may be used for expanding a general potential into a series. Assume that ϕ is a pseudo-minimal potential regular within a sphere of radius 1. On the surface of a sphere of radius $\bar{r}<1$, ϕ may be expanded into a series of *surface* harmonics

$$\phi = \sum a_l \overline{H}_l,$$

and the series $\sum a_l H_l$, where the H_l are the corresponding *solid* harmonics, converges in and on that sphere. Consider now the series

$$\sum a_l \frac{f_l}{\bar{f}_l} H_l.$$

If we can prove that the quantities $f_l/\bar{f}_l$ are bounded in the sphere, we shall know that this series converges uniformly, and, since each term satisfies our differential equation, we shall have (taking into account the analytic character of solutions of an equation of elliptic type) a pseudo-minimal potential. Since on the surface of the sphere f_l reduces to $\bar{f}_l$, the series reduces to that for ϕ, so that we have a potential which must coincide with our original potential, because it coincides with it on the surface of the sphere.

The proof of the statement that the quantities $f_l/\bar{f}_l$ are bounded is based on the remark that, according to (8.4), for sufficiently large values of l the coefficients of the expansion for $f^{(1)}$ can be made as close as desired to the coefficients of the binomial series for $(1-\zeta)^{n/2}$.

(γ) **Explicit formulas connecting harmonic functions with pseudo-minimal potentials.** The preceding section contains a method by which we may obtain from a harmonic function, by developing it into a series and multiplying the terms by appropriate functions of r, a pseudo-minimal potential. In the case of an even n we have succeeded in expressing this connection in an explicit form; for $n=2$ and $n=4$, for instance, we have respectively the following formulas:

$$\phi = \mu\mathcal{H} + \lambda\mathcal{H}_\rho x_\rho,$$

and

$$\phi = 3\mu\mathcal{H} + 3\lambda\mathcal{H}_\rho x_\rho + \lambda^2\mathcal{H}_{\rho\sigma} x_\rho x_\sigma.$$

For every harmonic function $\mathcal{H}$ these formulas give a pseudo-minimal potential, as may be verified by a straightforward calculation.

(δ) **Hypersurfaces for which the sum of the radii of curvature is constant.** These may be included in the present theory without any effort. This sum is equal to the invariant I_{n-1} for which we found the formula (6.4). Equating I_{n-1} to a constant c, using (1.12) and (1.8) we obtain

$$\lambda\phi_{\rho\rho} - n(\phi_\rho x_\rho - \phi) + c = 0,$$

which shows that $x = \phi + c/n$ is a solution of the homogeneous equation (7.1).

We see thus that in this fashion, by adding a constant to a pseudo-minimal potential, we may obtain a representation of a hypersurface $I_{n-1} = \text{const}$.

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