A NEW METHOD FOR WARING THEOREMS WITH POLYNOMIAL SUMMANDS, II*

BY

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1. In a paper† with the same title, I showed how to deduce instantaneously a Waring theorem for an even polynomial $f(x)$ of degree $2n$ from a known Waring theorem for a polynomial $q(x)$ of degree n. Here I extend the method to the new case in which $f(x)$ contains also a term in x.

2. First, let $n=2$ and

$$f(x) = ux^4 + vx^2 - wx + k, \qquad q(x) = ux^2 + vx + 2k. \tag{1}$$

We have the identity in a, b, c, d, u, v, w, k

$$6q(s) \equiv \sum f(z), \qquad s = a^2 + b^2 + c^2 + d^2, \tag{2}$$

in which z takes the following twelve values:

$$b \pm a, \qquad c \pm a, \qquad d \pm a, \qquad \pm b - c, \qquad \pm d - b, \qquad \pm c - d, \tag{3}$$

whose sum is zero. Since some of the numbers (3) are negative, we impose the condition

$$f(x) \text{ is an integer } \geqq 0 \text{ for all integers } x. \tag{4}$$

But when x ranges over all integers (positive, negative, or zero), evidently $f(-x)$ takes the same values as $f(x)$. Without loss of generality we may therefore take $w \geqq 0$. Since $f(-x) = f(x) + 2wx$, $f(-x)$ will be $\geqq 0$ for all integers $x \geqq 0$ if the same is true for $f(x)$. Hence (4) follows from

$$f(x) \text{ is an integer } \geqq 0 \text{ for every integer } x \geqq 0. \tag{5}$$

Since $u > 0$, only a limited number of integers x yield negative values of $f(x) - k$. If one of these values is $-P$, while all the remaining are $\geqq -P$, then (5) holds if and only if $k \geqq P$. In brief, we need only take k sufficiently large in (1).

Consider triangular, pyramidal, and figurate numbers

$$\begin{aligned} T(x) &= (x^2 - x)/2, \qquad P(x) = (x^3 - x)/6, \\ F(x) &= (x + 2)(x + 1)x(x - 1)/24 = (x^4 + 2x^3 - x^2 - 2x)/24. \end{aligned} \tag{6}$$

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† These Transactions, vol. 36 (1934), pp. 731–748. In (36) read $\pm d$ for $+d$. In (38) delete exponent 6.

Any quartic function with rational coefficients can evidently be expressed in the form

$$f = AF + gP + hT + tx + k, \tag{7}$$

where $A, \cdots, k$ are rational numbers. Let (5) hold. Taking $x=0, \cdots, 4$, we see that

$$k, \quad t+k, \quad A+g+h+2t+k, \quad 5A+4g+3h+3t+k,$$
$$15A+10g+6h+4t+k$$

must be integers. Hence k, t, A, g, h are integers. The coefficients of x^3 and x in (7) are

$$A/12+g/6, \qquad -(A/12+g/6)-h/2+t.$$

These must be 0 and $\leqq 0$ respectively if (7) shall be of the form (1) with $w \geqq 0$. Hence

$$A+2g=0, \qquad h \geqq 2t. \tag{8}$$

3. Of special interest are functions f for which

(9) Every integer $\geqq 0$ is a sum of V values of $f(x)$

for integers x. The smaller A is, the more slowly will f increase with x, and the smaller V will be in general. Hence we give to A its minimum (even) value 2. Evidently (9) requires that

$$f(y)=0, \quad f(z)=1 \text{ for certain integers } y, z. \tag{10}$$

The functions (7) satisfying (4), (8), and (10) and having $A=2$, $t \geqq -5$, are found to be those with

t	1	0	0	-1	-1	-2	-2	-2	-2	-3	-3	-3
h	$\geqq 2$	0	1	-1	$\geqq 1$	-2	-1	0	2	-4	1	3
k	0	0	0	2	1	6	4	3	2	16	4	3

t	-4	-4	-4	-4	-4	-5	-5	-5
h	-3	-1	0	2	4	-6	3	5
k	15	9	7	5	4	36	6	5

Hence each of these functions represents 0 and 1, and is an integer $\geqq 0$ for every integer x. The general theory therefore yields a value of V in (9) and hence a universal Waring theorem for summands $f(x)$.

4. Consider, for example, the seventh set $t=-2$, $h=-1$, $k=4$. Then

$$(11)\qquad \begin{aligned} f(x) &= 2F - P - T - 2x + 4 = (x^4 - 7x^2)/12 - \tfrac{4}{3}x + 4, \\ Q &= 6q = H + 42, \qquad H = \tfrac{1}{2}(x-3)(x-4). \end{aligned}$$

Every integer $\geqq 0$ is a sum of three values of the triangular number H for integers $x \geqq 4$. Thus every integer $\geqq 126$ is a sum of three values of Q. Hence by (2), every integer $\geqq 126$ is a sum of 36 values of $f(x)$. We next verify this fact also for positive integers <126 and indicate a probable reduction from 36 to 5. By a table to 1000 of sums of three values of $f(x)$, we find that 415, 734, and 749 are the only positive integers <1000 which are not sums of four values. We find at once that all integers $\leqq 5114$ are sums of five values of $f(x)$ for integers x (positive, negative, or zero).

5. **Waring theorem for sextic polynomials.** Let

$$(12)\qquad \begin{aligned} f(x) &= ux^6 + vx^4 + wx^2 - hx + k, \\ q(x) &= 120ux^3 + 72vx^2 + 60wx + 108k. \end{aligned}$$

Then a like generalization of (38) of the former paper gives

$$(13)\qquad q(s) \equiv \sum f(y) + 8\sum f(z) + f(2a) + f(2b) + f(2c) + f(2d),$$

where z ranges over the twelve values (3), and y ranges over the eight values

$$(14)\qquad \begin{gathered} -a-b-c\pm d, \qquad \pm a-b+c-d, \qquad \mp a+b-c-d, \\ \pm(a-b-c)+d, \end{gathered}$$

whose sum is $-2a-2b-2c-2d$. Hence the sum of the 108 arguments of f in (13) is zero. In case $f(x)$ is an integer $\geqq 0$ for every integer x (which is true when k is sufficiently large), a Waring theorem for q leads instantly to one for f. This condition (4) holds if $f(x)$ is

$$(15)\qquad x^6 + x^2 - x, \qquad \tfrac{1}{2}(x^6 + 3x^2) - x,$$

each of which represents 0 and 1. Hence each yields at once a universal Waring theorem.

6. **Quartics with property (4).** Replacing x by $-x-1$ in (7), we get

$$(16)\qquad AF(x) - gP(x) + (h-g)T(x) + (2h-g-t)x + h - t + k.$$

Hence (7) remains unaltered if and only if

$$(17)\qquad g = 0, \qquad h = t.$$

In this case the values of $f(x)$ for negative integers coincide with its values for integers $x \geqq 0$. Such unfavorable cases are

$$(18)\qquad f = F(x), \qquad F - T - x + 2, \qquad F - 6T - 6x + 56,$$

whose values are $\geqq 0$ for an integer $x \geqq 0$ and hence for all integers. By tables to 1000, all integers from 0 to 3366 inclusive are sums of 7 values of $F(x)$ except only 64, 99, 119, 189, 314, 774. Hence all $\leqq 23841$ are sums of 8.

Since $T(-x)=T(x+1)$, $F+T$ is $\geqq 0$ for all integers. By a table to 1000 of sums by three, it was verified that all integers from 0 to 3900 are sums of four values of $F+T$.

Miss H. Rees found that (7) has property (4) if

$$A = 1, \quad g = -p-2, \quad h = p+1+\tfrac{1}{6}p(2p+1)+m, \quad t=1, \quad k=0,$$
$$p \geqq 0, \qquad m \geqq 0.$$

To obtain an integer h take m to be the sum of an integer $\geqq 0$ and $\frac{1}{2}$ if $p\equiv 1$ or 3 (mod 6); 0 if $p\equiv 0$ or 4; $\frac{1}{3}$ if $p\equiv 2$; $\frac{5}{6}$ if $p\equiv 5$ (mod 6). We may remove the term involving P by the transformation $x=y+p+2$. We get

$$\begin{aligned} &F+(m-r)T+\{1-r+m(p+2)\}y+(p+2)\{1+(p+1)J\}, \\ &r=\tfrac{1}{6}p(p+2), \qquad J=(p+1)(p-12)/24+\tfrac{1}{2}(m+p+1). \end{aligned} \tag{19}$$

When $p=1$, $m=\frac{1}{2}$, (19) is $f=F+2y+5$. By a table of sums of three values from 0 to 3000 and from 9000 to 11000, it was found that every integer $\leqq 16151$, except only 11784, is a sum of four (positive) values of (16) for positive and negative integers x. It follows that all $\leqq 210739$ are sums of five such values.

Another favorable function $f=F+y+2$ is the case $p=m=0$ of (19); it represents 0, 1, 2, 3, 5, 10, 12, 21, etc.

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