A CLASSIFICATION OF GENERATING FUNCTIONS*

BY
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Introduction. By a generating function we mean a function $f(x)$ which can be represented by a Laplace-Stieltjes integral

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \tag{1}$$

which converges for some value of x. Here $\alpha(t)$ is a function of bounded variation in the interval $(0, R)$ for every positive R, and by convergence we mean that

$$\lim_{R\to\infty} \int_0^R e^{-xt} d\alpha(t)$$

exists. For most purposes it is convenient to assume that $\alpha(t)$ is "normalized"; that is,

$$\alpha(t) = \frac{\alpha(t+) + \alpha(t-)}{2} \qquad (t > 0),$$

$$\alpha(0) = 0.$$

Such normalization has no effect on the function $f(x)$. The representation of a function $f(x)$ in the form (1), with $\alpha(t)$ normalized, is unique. The function $\alpha(t)$ is called the "determining" function corresponding to the generating function $f(x)$. In particular, if

$$\alpha(t) = \int_0^t \phi(u) du,$$

the integral (1) becomes

$$f(x) = \int_0^\infty e^{-xt} \phi(t) dt. \tag{2}$$

Or, if $\alpha(t)$ is a step-function with its jumps at the points λ_k,

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \qquad \lim_{k\to\infty} \lambda_k = \infty,$$

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* Presented to the Society, December 29, 1934; received by the editors February 1, 1935, and, in revised form, June 27, 1935.

the integral (1) reduces to the Dirichlet series

$$f(x) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n x} \tag{3}$$

which may, in particular, be a power series in e^{-x}. Thus one sees the appropriateness of the term generating function.*

It is the purpose of the present paper to obtain characterizations of generating functions $f(x)$ which will insure that their corresponding determining functions shall have certain prescribed properties, such as continuity, absolute continuity, etc. We find that these characterizations are most conveniently expressed in terms of an inversion operator of E. L. Post which was discussed in detail in an earlier paper.† For this operator we use the notation

$$L_{k,t}[f(x)] = \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1} \qquad (k = 1, 2, \cdots).$$

We have proved that for (2)

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \phi(t)$$

for almost all positive values of t. In the present paper we also show that for (1)

$$\alpha(t) - \alpha(0+) = \lim_{k\to\infty} \int_0^t L_{k,u}[f(x)]du.$$

We introduce here one further operator,

$$l_{k,t}[f(x)] = t(2\pi/k)^{1/2} L_{k,t}[f(x)],$$

and we are able to show for the integral (1) that

$$l_{k,t}[f(x)] = \alpha(t+) - \alpha(t-) \qquad (t > 0),$$

and for the series (3) that

$$\lim_{k\to\infty} l_{k,\lambda_n}[f(x)] = a_n \qquad (n = 1, 2, \cdots).$$

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* Originally, the generating function of a sequence $\{a_n\}$ was the function whose Maclaurin development had the constants a_n for its coefficients.

† In this and subsequent foot-notes we shall use the following abbreviations: I for D. V. Widder, *A generalization of Dirichlet's series and of Laplace's integrals by means of a Stieltjes integral*, these Transactions, vol. 31 (1929), pp. 694–743.

II for D. V. Widder, *Necessary and sufficient conditions for the representation of a function as a Laplace integral*, these Transactions, vol. 33 (1931), pp. 851–892.

III for D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, these Transactions, vol. 36 (1934), pp. 107–200.

The present reference is to III, where a reference to E. L. Post will be found on p. 108.

This is a determination of the coefficients of a Dirichlet series which is analogous to the determination of the coefficients of a power series in terms of the derivatives of the sum of the series.

We summarize the more important results of the paper in the following table:

(I) $\alpha(t)$ *is of bounded variation in* $(0, \infty)$;

$$\int_0^\infty |L_{k,t}[f(x)]|\,dt \leqq M \qquad (k = 1, 2, \cdots).$$

(II) $\alpha(t)$ *is non-decreasing in* $(0, \infty)$;

$$L_{k,t}[f(x)] \geqq 0 \qquad (t > 0;\ k = 1, 2, \cdots),$$

$$f(x) \geqq 0 \qquad (x > 0).$$

(III) $|\phi(t)| \leqq M$ *in* $(0, \infty)$;

$$|L_{k,t}[f(x)]| \leqq M \qquad (t > 0;\ k = 1, 2, \cdots),$$

$$f(\infty) = 0.$$

(IV) $\phi(t)$ *is of class* L^p $(p > 1)$ *in* $(0, \infty)$;

$$\int_0^\infty |L_{k,t}[f(x)]|^p dt \leqq M \qquad (k = 1, 2, \cdots),$$

$$f(\infty) = 0.$$

(V) $\phi(t)$ *is of class L in* $(0, \infty)$;

$$\text{l.i.m. } L_{k,t}[f(x)] \textit{ exists},$$

$$f(\infty) = 0.$$

(VI) $\phi(t)$ *is of bounded variation in* $(0, \infty)$;

$$\int_0^\infty |L'_{k,t}[f(x)]|\,dt \leqq M \qquad (k = 1, 2, \cdots), \quad f(\infty) = 0.$$

(VII) $\phi^{(n)}(t)$ *is of bounded variation in* $(0, \infty)$;

$$\int_0^\infty |L_{k,t}^{(n+1)}[f(x)]|\,dt \leqq M \qquad (k = 0, 1, 2, \cdots), f(\infty) = 0.$$

(VIII) $|\alpha(t)| \leqq M$ *in* $(0, \infty)$;

$$\left|\int_0^R L_{k,t}[f(x)]dt\right| \leqq N \qquad (R > 0;\ k = 1, 2, \cdots),$$

$$\int_0^R |L_{k,t}[f(x)]|\,dt \leqq N(R) \qquad (R > 0;\ k = 1, 2, \cdots).$$

(IX) *$\alpha(t)$ is such that* (1) *converges for* $x>0$;

$$\left|\int_0^R L_{k,t}[f(x+\epsilon)]dt\right| \leqq N_\epsilon,$$

$$\int_0^R |L_{k,t}[f(x+\epsilon)]|\,dt \leqq N_\epsilon(R) \qquad (R>0;\ k=1,2,\cdots),$$

for every $\epsilon>0$.

(X) *$\alpha(t)$ is of bounded variation and continuous in* $(0,\infty)$;

$$\int_0^\infty |L_{k,t}[f(x)]|\,dt \leqq M \qquad (k=1,2,\cdots),$$

$$f(\infty)=0$$

$$\lim_{k\to\infty} l_{k,t}[f(x)]=0 \qquad (t>0).$$

(XI) *$\phi(t)$ is completely monotonic in* $(0,\infty)$;

$$(-1)^k[x^n f(x)]^{(n+k)} \geqq 0 \qquad (x>0;\ n,\ k=0,1,2,\cdots).$$

(XII) *$\alpha(t)$ is a step-function making* (1) *converge*;
same as (IX) *and*

$$\lim_{k\to\infty} L_{k,t}[f(x)]=0$$

uniformly in every sub-interval of an interval in which $\alpha(t)$ is to be constant.

In each of these cases the condition on $f(x)$ is merely abbreviated. For the exact statement the reader is referred to the corresponding theorem in the text. In each case the condition is necessary and sufficient. In (VI) and (VII) the upper index on the operator L indicates differentiation with respect to t. In (V), l.i.m. means limit in the mean of order one,

$$\lim_{\substack{k\to\infty\\ l\to\infty}} \int_0^\infty |L_{k,t}[f(x)] - L_{l,t}[f(x)]|\,dt = 0.$$

We note that the result (II) is due to S. Bernstein, the present statement being new only in form. The proof given in the present paper is new. The results (I) and (III) were proved earlier in a different form and with a different method by the author. All remaining results are new.

We call special attention to the results (IX), (XI), (XII). In (IX) we have for the first time a necessary and sufficient condition for the representation of a function $f(x)$ in the most general convergent Laplace-Stieltjes integral.

In (XII) we have a characterization of functions which can be represented in the most general convergent Dirichlet series.*

The result (XI) really has to do with the Stieltjes integral equation of the type

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

and gives for the first time a necessary and sufficient condition that it shall have a bounded non-decreasing solution $\alpha(t)$.

Throughout the paper the determining function is thought of as real. It should be pointed out here, however, that most of the results of the paper also hold for complex determining functions. Thus all the results cited in the foregoing table except those under (II), (XI), and (XII) hold without any change whatever in the complex case. The proof in each case is made by breaking the complex function in question into real and imaginary parts. Thus if the function $f(x+iy)$ is to have the form

$$f(x+iy) = \int_0^\infty e^{-xt-iyt}d[\alpha_1(t) + i\alpha_2(t)]$$

where $\alpha_1(t)+i\alpha_2(t)$ is to be of bounded variation in $(0, \infty)$ it is necessary and sufficient that

$$\int_0^\infty |L_{k,t}[f(x)]|\, dt \leqq M \qquad (k = 1, 2, \cdots).$$

To obtain this result from the corresponding real case one has only to use the relations

$$f(x) = f_1(x) + if_2(x),$$
$$|L_{k,t}[f_1(x)]| \leqq |L_{k,t}[f(x)]|,$$
$$|L_{k,t}[f_2(x)]| \leqq |L_{k,t}[f(x)]|,$$
$$|L_{k,t}[f(x)]| \leqq |L_{k,t}[f_1(x)]| + |L_{k,t}[f_2(x)]|.$$

The author wishes to express his indebtedness to Professor J. D. Tamarkin, who read the original manuscript, for many valuable suggestions. In particular, the proof of Lemma 1 of §12 is his. The author's original proof was longer, less ingenious.

1. **Some preliminary results.** In this section we will establish certain results of a general nature which will be of fundamental importance for later parts of the paper.

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* Compare Th. Kaluza, *Entwickelbarkeit von Funktionen in Dirichletsche Reihen*, Mathematische Zeitschrift, vol. 28 (1928), p. 203, where Dirichlet series with positive coefficients are treated. Compare also II, p. 876.

THEOREM 1.1. *If*

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t),$$

where $\alpha(t)$ is a normalized function of bounded variation in $0 \leqq t \leqq R$ for every positive R and the integral converges for some value of x, then

$$\alpha(t) = f(\infty) + \lim_{k\to\infty} \int_0^t L_{k,u}[f(x)]du \qquad (0 < t < \infty).$$

To prove this we note first that the change of variable $k = uv$ gives us

$$\int_0^t L_{k,u}[f(x)]du = \frac{(-1)^k}{k!}\int_0^t f^{(k)}\left(\frac{k}{u}\right)\left(\frac{k}{u}\right)^{k+1} du$$

$$= (-1)^k \int_{k/t}^\infty \frac{f^{(k)}(v)v^{k-1}dv}{(k-1)!}.$$

The operator $S_{k,t}[f(x)]$ defined earlier* is closely related to this latter integral. In fact

$$S_{k,t}[f(x)] = f(\infty) + \frac{(-1)^{k+1}}{k!}\int_{k/t}^\infty f^{(k+1)}(v)v^k dv.$$

An integration by parts gives

$$S_{k,t}[f(x)] = f(\infty) + \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k + \frac{(-1)^k}{(k-1)!}\int_{k/t}^\infty f^{(k)}(v)v^{k-1}dv.$$

Hence

$$f(\infty) + \int_0^t L_{k,u}[f(x)]du = S_{k,t}[f(x)] + \frac{(-1)^{k+1}}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k.$$

But we proved in III that

$$\lim_{k\to\infty} S_{k,t}[f(x)] = \alpha(t) \qquad (0 < t < \infty).$$

Consequently our result will be established if we can prove that

$$\lim_{k\to\infty} \frac{1}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k = 0 \qquad (0 < t < \infty).$$

Set

$$F(x) = f(x)/x.$$

* III, p. 116.

Then

$$\frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k = \frac{(-1)^k}{k!} F^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1} - \frac{(-1)^{k-1}}{(k-1)!} F^{(k-1)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k,$$

$$\frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k = L_{k,t}[F(x)] - \frac{(-1)^{k-1}}{(k-1)!} F^{(k-1)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k. \tag{1.1}$$

We proved earlier* that

$$\lim_{k\to\infty} L_{k,t}[F(x)] = \alpha(t).$$

It will now be sufficient to show that second term on the right-hand side of (1.1) approaches $-\alpha(t)$. Simple computation gives

$$\frac{(-1)^{k-1}}{(k-1)!} F^{(k-1)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k = \left(\frac{k}{t}\right)^k \int_0^\infty e^{-ku/t} \frac{u^{k-1}}{(k-1)!} \alpha(u)du$$
$$= \frac{k^k}{(k-1)!} \int_0^\infty e^{-kv} v^{k-1} \alpha(tv)dv.$$

On the other hand

$$\lim_{k\to\infty} L_{k-1,t}[F(x)] = \lim_{k\to\infty} \frac{(k-1)^k}{(k-1)!} \int_0^\infty e^{-(k-1)v} v^{k-1} \alpha(tv)dv = \alpha(t).$$

If we replace $\alpha(v)$ by $\alpha(v)e^{-v/t}$ in this equation we have

$$\lim_{k\to\infty} \frac{(k-1)^k}{(k-1)!} \int_0^\infty e^{-(k-1)v} v^{k-1} e^{-v} \alpha(tv)dv = \alpha(t)e^{-1}.$$

Hence

$$\lim_{k\to\infty} \frac{(-1)^k}{(k-1)!} F^{(k-1)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k$$
$$= \lim_{k\to\infty} \left(\frac{k}{k-1}\right)^k \frac{(k-1)^k}{(k-1)!} \int_0^\infty e^{-kv} v^{k-1} \alpha(tv)dv = \alpha(t) \qquad (t > 0).$$

The theorem is then completely established.

We turn next to

* III, p. 119.

THEOREM 1.2. *Let*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a normalized function of bounded variation in (0, *R*) *for every $R>0$, the integral converging for some value of x. If $V(x)$ is the total variation of $\alpha(t)$ in the interval $0 \leqq t \leqq x$, then*

$$V(R) = |f(\infty)| + \lim_{k\to\infty} \int_0^R |L_{k,t}[f(x)]| \, dt \qquad (R > 0).$$

Let S be an arbitrary number greater than R. Set

$$f_1(x) = \int_0^S e^{-xt} d\alpha(t),$$

$$f_2(x) = \int_S^\infty e^{-xt} d\alpha(t).$$

Then

$$\int_0^R |L_{k,t}[f(x)]| \, dt \leqq \int_0^R |L_{k,t}[f_1(x)]| \, dt + \int_0^R |L_{k,t}[f_2(x)]| \, dt.$$

Simple computation shows that

$$|L_{k,t}[f_1(x)]| \leqq L_{k,t}\left[\int_0^S e^{-xu} dV(u)\right],$$

$$\int_0^R |L_{k,t}[f_1(x)]| \, dt \leqq \int_0^R L_{k,t}\left[\int_0^S e^{-xu} dV(u)\right] dt.$$

By Theorem 1.1 we have

$$\lim_{k\to\infty} \int_0^R L_{k,t}\left[\int_0^S e^{-xu} dV(u)\right] dt = V(R) - V(0+).$$

On the other hand it may be seen that

$$\lim_{k\to\infty} L_{k,t}[f_2(x)] = 0 \tag{1.2}$$

uniformly in $0 \leqq t \leqq R$. For

$$f_2(x) = \int_0^\infty e^{-xt} d\beta(t),$$

$$\beta(t) = \alpha(t) \qquad (S \leqq t < \infty),$$

$$\beta(t) = \alpha(S) \qquad (0 \leqq t \leqq S).$$

Hence by an earlier result* (1.2) is established.

Consequently

$$\limsup_{k\to\infty}\int_0^\infty |L_{k,t}[f(x)]|\,dt \leqq V(R) - V(0+).$$

But

$$V(0+) = |\alpha(0+)| = |f(\infty)|,$$

so that

$$(1.3)\qquad |f(\infty)| + \limsup_{k\to\infty}\int_0^R |L_{k,t}[f(x)]|\,dt \leqq V(R).$$

Next set

$$\alpha_k(t) = \int_0^t L_{k,u}[f(x)]du \qquad (0 \leqq t < \infty).$$

Then by Theorem 1.1

$$\lim_{k\to\infty}\alpha_k(t) = \alpha(t) - f(\infty) = \alpha(t) - \alpha(0+) \qquad (0 < t < \infty).$$

Let

$$0 = t_0 < t_1 < \cdots < t_n = R.$$

Then

$$\sum_{i=0}^{n-1} |\alpha_k(t_{i+1}) - \alpha_k(t_i)| \leqq \int_0^R |L_{k,t}[f(x)]|\,dt.$$

Let k become infinite in this inequality:

$$|\alpha(t_1) - \alpha(0+)| + \sum_{i=1}^{n-1}|\alpha(t_{i+1}) - \alpha(t_i)| \leqq \liminf_{k\to\infty}\int_0^R |L_{k,t}[f(x)]|\,dt.$$

The left-hand side of this inequality can be brought as close as desired to $V(R)-V(0+)$ by a suitable choice of the number and position of the points t_i. Hence

$$V(R) - V(0+) \leqq \liminf_{k\to\infty}\int_0^R |L_{k,t}[f(x)]|\,dt,$$

$$V(R) \leqq \liminf_{k\to\infty}\int_0^R |L_{k,t}[f(x)]|\,dt + |f(\infty)|.$$

* III, Theorem 13, p. 137. An examination of the proof of Theorem 13 will show that the convergence is uniform since $\alpha(t)$ is constant in $0\leqq t\leqq R$.

Combining this last inequality with (1.3) the theorem is proved.

We now introduce the following

DEFINITION. *The operator $l_{k,t}[f(x)]$ is defined by the equation*

$$l_{k,t}[f(x)] = t\left(\frac{2\pi}{k}\right)^{1/2} L_{k,t}[f(x)] = \frac{(-1)^k}{k!}(2\pi k)^{1/2} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^k \qquad (t > 0,\ k > 0).$$

By means of this operator we shall be able to discuss the saltus of the determining function $\alpha(t)$. In fact we shall prove

THEOREM 1.3. *If the function $f(x)$ has the representation*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t), \tag{1.4}$$

where $\alpha(t)$ is of bounded variation in $(0, R)$ for every positive R, the integral converging for some value of x, then

$$\lim_{k\to\infty} l_{k,t}[f(x)] = \alpha(t+) - \alpha(t-) \qquad (t > 0).$$

Let us suppose that the integral (1.4) converges for $x > \sigma_c$. Simple computations give

$$l_{k,t}[f(x)] = \frac{(2\pi k)^{1/2}}{k!}\left(\frac{k}{t}\right)^k \int_0^\infty e^{-ku/t} u^k d\alpha(u) \qquad (k > t\sigma_c).$$

We first discuss the case $t = 1$,

$$l_{k,1}[f(x)] = \frac{(2\pi k)^{1/2}}{k!} k^k \int_0^\infty e^{-ku} u^k d\alpha(u) \qquad (k > \sigma_c).$$

Integration by parts gives

$$l_{k,1}[f(x)] = \frac{(2\pi k)^{1/2}}{k!} k^{k+1} \int_0^\infty e^{-ku} u^{k-1}(u-1)\alpha(u) du,$$

or, by Stirling's formula,

$$l_{k,1}[f(x)] \sim ke^k \int_0^\infty e^{-ku} u^{k-1}(u-1)\alpha(u) du \qquad (k \to \infty).$$

Set

$$\begin{aligned} \beta(x) &= \alpha(1-) && (0 \leqq x < 1), \\ \beta(1) &= \alpha(1), \\ \beta(x) &= \alpha(1+) && (1 < x < \infty), \\ \phi(x) &= \alpha(x) - \beta(x) && (0 \leqq x < \infty). \end{aligned}$$

Then

$$l_{k,1}[f(x)] \sim ke^k \int_0^\infty e^{-ku}u^{k-1}(u-1)\beta(u)du + ke^k \int_0^\infty e^{-ku}u^{k-1}(u-1)\phi(u)du \qquad (k \to \infty).$$

Since

$$x \int_1^\infty e^{-xu}du = e^{-x} \qquad (x > 0),$$

we have by successive differentiation

$$e^{-x} = \int_1^\infty e^{-xu}u^{k-1}[xu - k]du \qquad (x > 0),$$

$$e^{-k} = k \int_1^\infty e^{-ku}u^{k-1}[u-1]du,$$

$$-e^{-k} = k \int_0^1 e^{-ku}u^{k-1}[u-1]du.$$

Hence it is clear that

$$ke^k \int_0^\infty e^{-ku}u^{k-1}(u-1)\beta(u)du = \alpha(1+) - \alpha(1-).$$

To prove that

$$\lim_{k\to\infty} l_{k,1}[f(x)] = \alpha(1+) - \alpha(1-) \tag{1.5}$$

it will be sufficient to show that

$$\lim_{k\to\infty} ke^k \int_0^\infty e^{-ku}u^{k-1}(u-1)\phi(u)du = 0.$$

But $\phi(u)$ is continuous and has the value zero at $u=1$. Hence to an arbitrary positive ϵ there corresponds a positive δ such that

$$|\phi(u)| < \epsilon \qquad (|1-u| < \delta).$$

By writing the integral (1.5) as the sum of three integrals over the intervals $(0, 1-\delta)$, $(1-\delta, 1+\delta)$, $(1+\delta, \infty)$ we easily arrive at the following inequality:

$$\left| ke^k \int_0^\infty e^{-ku}u^{k-1}(u-1)\phi(u)du \right|$$
$$\leq Mke[e^\delta(1-\delta)]^{k-1} + 2\epsilon + ke^k[e^{-(1+\delta)}(1+\delta)]^{k-n}N.$$

Here

$$|\phi(u)| \leqq M \qquad (0 \leqq u \leqq 1),$$

n is a positive integer greater than σ_c, and

$$N = \int_1^\infty e^{-nu}u^n |\phi(u)| du.$$

Since

$$e^\delta(1-\delta) < 1 \qquad (\delta > 0),$$

$$e^{-\delta}(1+\delta) < 1 \qquad (\delta > 0),$$

the first and third terms on the right-hand side of the above inequality approach zero as k becomes infinite, so that (1.5) is established.

Returning to the general case we have by an obvious change of variable

$$l_{k,t}[f(x)] = \frac{(2\pi k)^{1/2}}{k!} k^k \int_0^\infty e^{-ku}u^k d\alpha(tu) = l_{k,1}[g(x)],$$

where

$$g(x) = \int_0^\infty e^{-xu}d\beta(u), \qquad \beta(u) = \alpha(tu).$$

Applying the previous result, we obtain

$$\lim_{k\to\infty} l_{k,t}[f(x)] = \lim_{k\to\infty} l_{k,1}[g(x)] = \beta(1+) - \beta(1-) = \alpha(t+) - \alpha(t-) \quad (t > 0).$$

The theorem is thus completely established.

COROLLARY 1. *Under the conditions of the theorem*

$$\lim_{k\to\infty}\left(\frac{-e}{t}\right)^k f^{(k)}\left(\frac{k}{t}\right) = \alpha(t+) - \alpha(t-) \qquad (t > 0).$$

This follows by an application of Stirling's formula.

COROLLARY 2. *If $f(x)$ can be expressed in terms of a Dirichlet series*

$$f(x) = \sum_{n=0}^\infty a_n e^{-\lambda_n x},$$

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \qquad \left(\lim_{n\to\infty}\lambda_n = \infty\right),$$

convergent for some value of x, then

$$\lim_{k\to\infty} l_{k,\lambda_n}[f(x)] = a_n \ (n = 1, 2, \cdots), \quad \lim_{x\to\infty} f(x) = a_0.$$

2. Determining function of bounded variation. In this section we shall discuss generating functions $f(x)$ for which the corresponding determining function is of bounded variation in the interval $(0, \infty)$. We introduce

CONDITION A. *A function $f(x)$ satisfies Condition* A *in the interval* $(0, \infty)$ *if*
(a) *$f(x)$ is of class C^∞ in the interval* $0<x<\infty$,
(b) *a constant M exists such that*

$$\int_0^\infty |L_{k,t}[f(x)]|\, dt \leqq M \qquad (k = 1, 2, \cdots).$$

By use of this condition we may state the fundamental result concerning functions of the class under discussion in

THEOREM 2.1. *Condition* A *is necessary and sufficient that $f(x)$ can be expressed in the form*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \qquad (x > 0)$$

where $\alpha(t)$ is of bounded variation in $(0, \infty)$.

This result was proved earlier by the author* in a slightly different form. In the earlier form Condition A (b) was replaced by

$$\int_0^\infty \frac{t^k}{k!} |f^{(k+1)}(t)|\, dt \leqq M \qquad (k = 0, 1, 2, \cdots).$$

The equivalence of these two forms follows immediately from an obvious change of the variable of integration. In a later section of the present paper we shall give a new proof of the theorem and a new form of the condition.

We turn next to

THEOREM 2.2. *If*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a normalized function of bounded variation V in the interval $(0, \infty)$, *then*

$$\begin{aligned} V &= |f(\infty)| + \lim_{k\to\infty} \int_0^\infty |L_{k,t}[f(x)]|\, dt \\ &= |f(\infty)| + \lim_{k\to\infty} \int_0^\infty \frac{t^k}{k!} |f^{(k+1)}(t)|\, dt. \end{aligned} \tag{2.1}$$

* II, p. 866.

Moreover, the integral

$$\int_0^\infty |L_{k,t}[f(x)]|\,dt$$

is a non-decreasing function of the integer k.

This result is an extension of Theorem 1.2 to the case of an infinite interval and can be easily derived from that theorem. First suppose that $f(\infty)=\alpha(0+)=0$.

By Theorem 2.1 we have

$$\int_0^\infty |L_{k,t}f(x)|\,dt \leqq M \qquad (k = 1, 2, \cdots). \tag{2.2}$$

Since

$$\int_0^R |L_{k,t}[f(x)]|\,dt \leqq \int_0^\infty |L_{k,t}[f(x)]|\,dt \leqq M,$$

we have by Theorem 1.2

$$V(R) \leqq \liminf_{k\to\infty} \int_0^\infty |L_{k,t}[f(x)]|\,dt \leqq M,$$

whence

$$V \leqq \liminf_{k\to\infty} \int_0^\infty |L_{k,t}[f(x)]|\,dt.$$

On the other hand,

$$\int_0^\infty |L_{k,t}[f(x)]|\,dt \leqq \int_0^\infty \left(\frac{k}{t}\right)^{k+1} dt \int_0^\infty e^{-ku/t}\frac{u^k}{k!}\,dV(u) = V,$$

$$\limsup_{k\to\infty} \int_0^\infty |L_{k,t}[f(x)]|\,dt \leqq V.$$

Hence

$$\lim_{k\to\infty} \int_0^\infty |L_{k,t}[f(x)]|\,dt = V.$$

If $\alpha(0+)$ is different from zero, we apply the result just established to the function $f(x)-f(\infty)$ whose normalized determining function will have total variation equal to that of $\alpha(t)$ increased by $|f(\infty)|$. Hence (2.1) is completely established.

Finally, to prove the last assertion of the theorem we note that

$$f^{(k)}(t) = -\int_t^\infty f^{(k+1)}(u)du,$$

$$|f^{(k)}(t)| \leqq \int_t^\infty |f^{(k+1)}(u)|\, du.$$

These integrals clearly converge by virtue of (2.2). It follows that

$$\int_0^\infty \frac{t^{k-1}}{(k-1)!}|f^{(k)}(t)|\, dt \leqq \int_0^\infty \frac{t^{k-1}}{(k-1)!}dt \int_t^\infty |f^{(k+1)}(u)|\, du$$

$$= \int_0^\infty |f^{(k+1)}(t)|\, dt \int_0^t \frac{u^{k-1}}{(k-1)!}du = \int_0^\infty |f^{(k+1)}(t)|\, \frac{t^k}{k!}dt.$$

This concludes the proof.

This result enables us to prove at once

THEOREM 2.3. *Let*

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t), \tag{2.3}$$

where $\alpha(t)$ is a normalized function of bounded variation $V(R)$ in the interval $0 \leqq t \leqq R$ for every positive R, the integral converging absolutely for $x > \sigma_a$. Then

$$\int_0^\infty e^{-xt}dV(t) = |f(\infty)| + \lim_{k\to\infty} \int_x^\infty \frac{(u-x)^k}{k!}|f^{(k+1)}(u)|\, du \qquad (x > \sigma_a).$$

For $x > \sigma_a$ we have

$$f(u+x) = \int_0^\infty e^{-ut}e^{-xt}d\alpha(t) \qquad (u > 0)$$

$$= \int_0^\infty e^{-ut}d\beta(t),$$

where

$$\beta(t) = \int_0^t e^{-xu}d\alpha(u) \qquad (t > 0), \qquad \beta(0) = 0.$$

The total variation of $\beta(t)$ in the interval $0 \leqq t < \infty$ is

$$\int_0^\infty e^{-xu}dV(u),$$

an integral that converges for $x > \sigma_a$ since (2.3) converges absolutely there. Now applying Theorem 2.2 to $f(x+u)$ considered as a function of u we have

$$\lim_{k\to\infty}\int_0^\infty \frac{u^k}{k!}\left| f^{(k+1)}(x+u)\right| du = \lim_{k\to\infty}\int_x^\infty \frac{(u-x)^k}{k!}\left| f^{(k+1)}(u)\right| du$$
$$= \int_0^\infty e^{-xt}dV(t) - \left| f(\infty)\right|,$$

and the theorem is established.

3. **Determining function monotonic.** The results of the previous section may be applied to completely monotonic functions. We recall the following

Definition. *A function $f(x)$ is completely monotonic in the interval $c<x<\infty$ if*

$$(-1)^k f^{(k)}(x) \geqq 0 \qquad (c < x < \infty;\ k = 0, 1, 2, \cdots);$$

$f(x)$ is completely monotonic in the interval $c\leqq x<\infty$ if, in addition, it approaches a limit as x approaches c from the right.

We can easily show that if $f(x)$ is completely monotonic in $0\leqq x<\infty$ then Condition A is satisfied. It is a familiar fact that such a function is analytic in the interval $0<x<\infty$ so that A(a) is satisfied. Furthermore,

$$\int_0^\infty \left| L_{k,t}[f(x)]\right| dt = \int_0^\infty \frac{t^{k-1}}{(k-1)!}(-1)^k f^{(k)}(t)dt = f(\infty) - f(0+).$$

Hence A(b) is also fulfilled. By Theorem 2.1

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t) \qquad (0 \leqq x < \infty), \tag{3.1}$$

where $\alpha(t)$ is of bounded variation in the interval $(0, \infty)$. Appealing to Theorem 1.1 we have

$$\alpha(t) = f(\infty) + \lim_{k\to\infty}\int_0^t L_{k,u}[f(x)]du \qquad (0 < t < \infty).$$

Since $L_{k,u}[f(x)]$ is non-negative, the function $\alpha(t)$ is non-decreasing.

Conversely, any function $f(x)$ of the form (3.1) with $\alpha(t)$ non-decreasing and bounded is clearly completely monotonic in $0\leqq x<\infty$. Hence we have proved

Theorem 3.1. *A necessary and sufficient condition that $f(x)$ can be expressed in the form*

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t),$$

where $\alpha(t)$ is non-decreasing and bounded, is that $f(x)$ should be completely monotonic in $0\leqq x<\infty$.

4. Determining function the integral of a function of class L^p, $p>1$. In this section we discuss generating functions $f(x)$ of the form

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt, \tag{4.1}$$

where the function $\phi(t)$ is of class L^p in the interval $(0, \infty)$; that is, the integral

$$\int_0^\infty |\phi(t)|^p dt \qquad (p > 1)$$

is finite. We first introduce

CONDITION B. *A function $f(x)$ satisfies Condition* B, *if and only if*

(a) *it is of class C^∞ in $0<x<\infty$,*

(b) *a constant M exists such that*

$$\int_0^\infty |L_{k,t}[f(x)]|^p dt \leqq M \qquad (k = 1, 2, \cdots),$$

(c)
$$\lim_{x\to\infty} f(x) = 0.$$

We now prove

THEOREM 4.1. *A necessary and sufficient condition that $f(x)$ can be represented in the form*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt$$

with $\phi(t)$ of class L^p in the interval $(0, \infty)$ is that Condition B *should hold.*

We note first that Condition B (b) is meaningless for $k=0$ since $\phi(t)$ need not belong to L. That is, $f(0)$ need not exist, as the following example shows:

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt = \int_1^\infty e^{-xt}t^{-1}dt.$$

Here the function $\phi(t)$ belongs to the class L^p for any $p>1$. But $f(0)=\infty$.

We prove first the necessity of Condition B. If $f(x)$ has the representation (4.1) with $\phi(t)$ of class $L^p(p>1)$, then that integral converges absolutely for $x>0$. For, by use of the Hölder inequality,

$$\int_0^\infty e^{-xt}|\phi(t)|\,dt \leqq \left[\int_0^\infty |\phi(t)|^p dt\right]^{1/p}[xq]^{-1/q} \qquad \left(\frac{1}{p}+\frac{1}{q}=1, x>0\right),$$

and the dominant integral converges by hypothesis.

Another application of the Hölder inequality gives

$$|L_{k,t}[f(x)]|^p = \left|\int_0^\infty e^{-ku/t}\frac{u^k}{k!}\left(\frac{k}{t}\right)^{k+1}\phi(u)du\right|^p$$
$$\leqq \int_0^\infty e^{-ku/t}\frac{u^k}{k!}\left(\frac{k}{t}\right)^{k+1}|\phi(u)|^p du\left[\int_0^\infty e^{-ku/t}\frac{u^k}{k!}\left(\frac{k}{t}\right)^{k+1}du\right]^{p/q}.$$

This last factor is equal to unity, so that

$$\int_0^\infty |L_{k,t}[f(x)]|^p dt \leqq \int_0^\infty dt\int_0^\infty e^{-ku/t}\frac{u^k}{k!}\left(\frac{k}{t}\right)^{k+1}|\phi(u)|^p du.$$

By use of the Fubini theorem we may interchange the order of integration in the iterated integral, since the resulting iterated integral,

$$\int_0^\infty |\phi(u)|^p u^k du\int_0^\infty e^{-ku/t}\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1}dt,$$

converges and has the value

$$\int_0^\infty |\phi(u)|^p du.$$

If we denote the value of this integral by M we have Condition B (b). Conditions B (a) and B (c) are known consequences* of the representation (4.1) regardless of the class of $\phi(t)$.

We turn next to the sufficiency of the condition. Let ϵ be an arbitrary positive constant. Then

$$\int_0^\infty |L_{k,t}[f(x+\epsilon)]|\,dt = \frac{1}{k!}\int_0^\infty \left|f^{(k)}\left(\frac{k}{t}+\epsilon\right)\right|\left(\frac{k}{t}\right)^{k+1}dt. \tag{4.2}$$

That the integral converges follows as a result of subsequent operations. First set

$$\frac{k}{t}+\epsilon = \frac{k}{u}.$$

Then (4.2) becomes

$$\frac{1}{k!}\int_0^{k/\epsilon}\left|f^{(k)}\left(\frac{k}{u}\right)\right|\left(\frac{k}{u}-\epsilon\right)^{k-1}\frac{k^2}{u^2}du. \tag{4.3}$$

Again applying Hölder's inequality we see that the integrals (4.2) and (4.3) are not greater than

* I, p. 702, and II, p. 864.

$$\left[\int_0^{k/\epsilon} |L_{k,t}[f(x)]|^p dt\right]^{1/p} \left[\int_0^{k/\epsilon} \left(1 - \frac{\epsilon u}{k}\right)^{q(k-1)} du\right]^{1/q}.$$

The second integral of the product can be evaluated and has the value

$$\frac{k}{\epsilon(qk - q + 1)}.$$

But by B (b)

$$\left[\int_0^{k/\epsilon} |L_{k,t}[f(x)]|^p dt\right]^{1/p} \leqq \left[\int_0^{\infty} |L_{k,t}[f(x)]|^p dt\right]^{1/p} \leqq M^{1/p},$$

so that (4.2) certainly converges and

$$\int_0^{\infty} | L_{k,t}[f(x + \epsilon)] | \, dt \leqq \frac{M^{1/p}k}{\epsilon(qk - q + 1)} \leqq \frac{2M^{1/p}}{\epsilon q} \qquad (k = 2, 3, \cdots).$$

Hence the function $f(x+\epsilon)$ satisfies Condition A, and by Theorem 2.1

$$f(x + \epsilon) = \int_0^{\infty} e^{-xt} d\beta_\epsilon(t),$$

the integral converging absolutely for $x=0$. By the uniqueness theorem† we see that

$$f(x) = \int_0^{\infty} e^{-xt} e^{\epsilon t} d\beta_\epsilon(t) = \int_0^{\infty} e^{-xt} d\alpha^*(t),$$

the integral converging absolutely for $x>0$. Here

$$\alpha^*(t) = \int_0^t e^{\epsilon u} d\beta_\epsilon(u).$$

If we set

$$\alpha(0) = 0, \qquad \alpha(t) = \frac{\alpha^*(t+) + \alpha^*(t-)}{2} \qquad (t > 0),$$

we have finally

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t),$$

the determining function being now normalized.

It remains to show that $\alpha(t)$ is the integral of a function of class L^p. Set

† I, p. 705.

$$\alpha_k(t) = \int_0^t L_{k,u}[f(x)]du.$$

Then by Theorem 1.1

$$\lim_{k\to\infty} \alpha_k(t) = \alpha(t) \qquad (0 \leqq t < \infty).$$

By Condition B (b) we see that there exists a subsequence of functions $L_{k,u}[f(x)]$ ($k = k_1, k_2, \cdots$) which converges weakly to a function $\phi(u)$ of class L^p in $(0, \infty)$. That is,

$$\lim_{j\to\infty} \int_0^\infty L_{k_j,u}[f(x)]\psi(u)du = \int_0^\infty \phi(u)\psi(u)du$$

for every function $\psi(u)$ of class L^q in $(0, \infty)$. By choosing $\psi(u)$ as a step-function we have

$$\alpha(t) = \lim_{j\to\infty} \alpha_{k_j}(t) = \int_0^t \phi(u)du,$$

so that $\alpha(t)$ is the integral of a function which is of class L^p in $(0, \infty)$. Our theorem is completely established.

Our next result is

THEOREM 4.2. *If $f(x)$ can be represented in the form*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt \tag{4.4}$$

with $\phi(t)$ of class $L^p (p > 1)$ in the interval $0 \leqq t < \infty$, then

$$\lim_{k\to\infty} \int_0^\infty | L_{k,t}[f(x)] |^p dt = \int_0^\infty | \phi(t) |^p dt,$$

the integral on the left-hand side constantly increasing with k.

We prove this last statement first. Make the change of variable $xt = k$ in the above integral. Then

$$\begin{aligned} \int_0^\infty | L_{k,t}[f(x)] |^p dt &= \int_0^\infty \left| f^{(k)}\left(\frac{k}{t}\right)\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \right|^p dt \\ &= \int_0^\infty \left| f^{(k)}(x)\frac{x^{k+1}}{k!} \right|^p \frac{k}{x^2}\, dx. \end{aligned} \tag{4.5}$$

Since† $f^{(k)}(\infty) = 0$, we have

† II, p. 864.

$$f^{(k)}(x) = -\int_x^\infty f^{(k+1)}(u)du,$$

or

$$(k+1)x^{k+1}f^{(k)}(x) = -\int_x^\infty \frac{(k+1)x^{k+1}}{u^{k+2}}[u^{k+2}f^{(k+1)}(u)]du.$$

Since

$$x^{k+1}\int_x^\infty (k+1)u^{-k-2}du = 1,$$

we have by Hölder's inequality

$$(4.6) \qquad |(k+1)x^{k+1}f^{(k)}(x)|^p \leqq \int_x^\infty \frac{(k+1)x^{k+1}}{u^{k+2}}|u^{k+2}f^{(k+1)}(u)|^p du.$$

By Theorem 4.1 we know that $f(x)$ satisfies Condition B. This fact enables us to conclude that the integral (4.6) converges. For,

$$\int_x^\infty \frac{1}{u^{k+2}}|f^{(k+1)}(u)u^{k+2}|^p du \leqq \frac{1}{x^k}\int_x^\infty \frac{1}{u^2}|f^{(k+1)}(u)u^{k+2}|^p du,$$

the dominant integral converging by (4.5) and Condition B.

Consequently we have

$$\int_0^\infty |L_{k,t}[f(x)]|^p dt \leqq \int_0^\infty \frac{k}{[k!]^p}x^{kp+p-2}g(x)dx,$$

$$g(x) = \frac{1}{(k+1)^p x^{kp+p}}\int_x^\infty (k+1)x^{k+1}u^{kp+2p-k-2}|f^{(k+1)}(u)|^p du.$$

If it is permissible to interchange the order of integration, this iterated integral becomes

$$\int_0^\infty \frac{k}{[k!]^p}|f^{(k+1)}(u)|^p \frac{u^{kp+2p-k-2}}{(k+1)^{p-1}}du\int_0^u x^{k-1}dx$$

$$= \int_0^\infty \frac{k+1}{[(k+1)!]^p}|f^{(k+1)}(u)|^p u^{kp+2p-2}du = \int_0^\infty \left|\frac{f^{(k+1)}(u)u^{k+2}}{(k+1)!}\right|^p \frac{(k+1)}{u^2}du.$$

The change of variable $ut=k+1$ in this last integral gives us the inequality

$$\int_0^\infty |L_{k,t}[f(x)]|^p dt \leqq \int_0^\infty |L_{k+1,t}[f(x)]|^p dt.$$

The above interchange in the order of integration was permissible since the integrand was positive and since the resulting iterated integral is seen to converge.

We now prove the first statement of the theorem. Since $f(x)$ has the representation (4.4) we know* that

$$\lim_{k\to\infty} |L_{k,t}[f(x)]|^p = |\phi(t)|^p$$

for almost all values of t in $(0, \infty)$. It follows by Fatou's Lemma that

$$\int_0^\infty |\phi(u)|^p du \leqq \liminf_{k\to\infty} \int_0^\infty |L_{k,u}[f(x)]|^p du.$$

This inequality, combined with the inequality

$$\int_0^\infty |L_{k,u}[f(x)]|^p du \leqq \int_0^\infty |\phi(u)|^p du$$

proved in Theorem 4.1, establishes our result.

5. Determining function the integral of a bounded function. By letting p become infinite in the results of the previous section we are led to introduce

CONDITION C. *A function $f(x)$ satisfies Condition* C *if and only if*

(a) *it is of class C^∞ in $0<x<\infty$,*

(b) *a constant M exists such that*

$$|L_{k,t}[f(x)]| \leqq M \qquad (k = 1, 2, \cdots; 0 \leqq t < \infty),$$

(c)

$$\lim_{x\to\infty} f(x) = 0.$$

We can now prove

THEOREM 5.1. *A necessary and sufficient condition that $f(x)$ can be represented in the form*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt,$$

where

$$|\phi(t)| \leqq M \qquad (0 \leqq t < \infty),$$

is that it should satisfy Condition C.

If $f(x)$ has the above representation, it is clear that it is of class C^∞ in $0<x<\infty$ and that $f(\infty)=0$. Condition C (b) follows from the inequality

* III, p. 122.

$$|L_{k,t}[f(x)]| \leqq ML_{k,t}[1/x] = M \qquad (k = 1, 2, \cdots).$$

For the sufficiency let ϵ be an arbitrary positive constant. Then by use of the same transformation as that applied to (4.2) we have

$$\int_0^\infty |L_{k,t}[f(x+\epsilon)]|\,dt = \int_0^{k/\epsilon} |L_{k,u}[f(x)]| \left(1 - \frac{u\epsilon}{k}\right)^{k-1} du$$

$$\leqq M \int_0^{k/\epsilon} \left(1 - \frac{u\epsilon}{k}\right)^{k-1} du = \frac{M}{\epsilon}.$$

Hence, as in §4, we have

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

where $\alpha(t)$ is a normalized function of bounded variation in $(0, \infty)$. Now defining $\alpha_k(t)$ as in §4 we have by Theorem 1.1

$$\lim_{k\to\infty} \alpha_k(t) = \alpha(t) \qquad (0 \leqq t < \infty).$$

To show that $\alpha(t)$ is the integral of a bounded function it is sufficient to show that its difference quotient is bounded.*

But

$$\frac{\alpha(t_2) - \alpha(t_1)}{t_2 - t_1} = \lim_{k\to\infty} \left| \frac{\alpha_k(t_2) - \alpha_k(t_1)}{t_2 - t_1} \right| \leqq \lim_{k\to\infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |L_{k,u}[f(x)]|\,du \leqq M$$

$$(0 \leqq t_1 < t_2).$$

Hence Theorem II is completely established.

6. **Determining function the integral of a function of class L.** Theorem 4.1 was proved only for the case in which p is greater than unity. We can easily see that it is no longer valid if p is equal to unity. For, in that case Condition B reduces to Condition A. But a generating function satisfying the latter condition gives rise to a determining function which is merely of bounded variation and hence not necessarily the integral of any function. For the discussion of the case when p is equal to unity we introduce

CONDITION D. *A function $f(x)$ satisfies Condition* D *if and only if*

(a) *it is of class C^∞ in $0<x<\infty$,*

(b) *the function $L_{k,t}[f(x)]$ converges in the mean of order unity as k becomes infinite:*

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* E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, 2d edition, vol. 1, 1921, p. 549.

$$\lim_{\substack{k\to\infty \\ l\to\infty}} \int_0^\infty | L_{k,t}[f(x)] - L_{l,t}[f(x)] | \, dt = 0,$$

(c)

$$\lim_{x\to\infty} f(x) = 0.$$

The proof of the main result of this section depends on the following

LEMMA. *If the function* $\phi(t)$ *is of class L in the interval* $(0, \infty)$, *then the function*

$$g(u) = \int_0^\infty | \phi(ut) - \phi(t) | \, dt \tag{6.1}$$

is continuous for $u=1$ *and a constant A exists such that*

$$| g(u) | < Au^{-1} + A \qquad (0 < u < \infty). \tag{6.2}$$

To prove this result set $t=e^x$ and $u=e^y$ in the integral (6.1). It thus becomes

$$g(e^y) = \int_{-\infty}^\infty | \phi(e^{x+y}) - \phi(e^x) | \, e^x dx.$$

The transformation shows that the function $\psi(x)=\phi(e^x)e^x$ is integrable in $(-\infty, \infty)$. Furthermore,

$$| g(e^y) | \leqq \int_{-\infty}^\infty \{ | \psi(x+y)e^{-y} - \psi(x)e^{-y} | + | \psi(x)e^{-y} - \psi(x) | \} dx$$

$$= e^{-y} \int_{-\infty}^\infty | \psi(x+y) - \psi(x) | \, dx + | e^{-y} - 1 | \int_{-\infty}^\infty | \psi(x) | \, dx.$$

By appealing to a known result* we see that the right-hand side of this inequality approaches zero as y approaches zero. That is,

$$\lim_{u\to 1} g(u) = 0.$$

If we set

$$A = \int_0^\infty | \phi(u) | \, du,$$

the inequality (6.2) follows at once since

$$| g(u) | \leqq \int_0^\infty | \phi(ut) | \, dt + \int_0^\infty | \phi(t) \, dt | = A(1 + u^{-1}).$$

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* See, for example, N. Wiener, *The Fourier Integral and Certain of its Applications*, p. 14.

We are now able to prove

THEOREM 6.1. *Condition* D *is necessary and sufficient that a function* $f(x)$ *can be represented in the form*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt,$$

where $\phi(t)$ *is of class* L *in the interval* $(0, \infty)$.

Suppose that $f(x)$ has the above representation. Then simple computation shows that

$$\begin{aligned} | L_{k,t}[f(x)] - \phi(t) | &\leqq \left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-ku/t} \frac{u^k}{k!} | \phi(u) - \phi(t) | du \\ &= \frac{k^{k+1}}{k!} \int_0^\infty e^{-ku} u^k | \phi(ut) - \phi(t) | du. \end{aligned}$$

Hence

$$\int_0^\infty | L_{k,t}[f(x)] - \phi(t) | dt \leqq \int_0^\infty dt \frac{k^{k+1}}{k!} \int_0^\infty e^{-ku} u^k | \phi(tu) - \phi(t) | du$$

if the iterated integral converges. We see that it does by an interchange in the order of integration. If we define $g(u)$ as in the Lemma it may be seen by use of (6.2) that the resulting iterated integral converges for $k>0$. It follows that

$$\int_0^\infty | L_{k,t}[f(x)] - \phi(t) | dt \leqq \frac{k^{k+1}}{k!} \int_0^\infty e^{-ku} u^k g(u) du.$$

Corresponding to an arbitrary positive constant ϵ we now determine a positive number δ less than unity such that

$$| g(u) | < \epsilon/3 \qquad (| u - 1 | < \delta).$$

We then have

$$\frac{k^{k+1}}{k!} \int_{1-\delta}^{1+\delta} e^{-ku} u^k g(u) du < \frac{\epsilon}{3} \frac{k^{k+1}}{k!} \int_0^\infty e^{-ku} u^k du = \frac{\epsilon}{3}.$$

Furthermore, since $e^{-u}u$ is an increasing function of u in the interval (0, 1),

$$\frac{k^{k+1}}{k!} \int_0^{1-\delta} e^{-ku} u^k g(u) du < \frac{k^{k+1}}{k!} e^{-(k-1)(1-\delta)}(1 - \delta)^{k-1} \int_0^{1-\delta} e^{-u} A(u + 1) du < \frac{\epsilon}{3}$$
$$(k > k_0).$$

The last inequality holds for k_0 sufficiently large since

$$\lim_{k\to\infty} \frac{k^{k+1}}{k!} e^{-k(1-\delta)}(1 - \delta)^k = 0 \qquad (0 < \delta < 1).$$

In a similar way we have

$$\frac{k^{k+1}}{k!}\int_{1+\delta}^{\infty} e^{-ku}u^k g(u)du < \frac{k^{k+1}}{k!} e^{-(k-1)(1+\delta)}(1+\delta)^{k-1}\int_{1+\delta}^{\infty} e^{-u}A(u+1)du < \frac{\epsilon}{3}$$
$$(k > k_1)$$

for k_1 sufficiently large. Combining these results it follows that

$$\int_0^{\infty} | L_{k,t}[f(x)] - \phi(t) | dt < \epsilon \qquad (k > k_0,\ k > k_1).$$

Hence the necessity of Condition D is established.

To prove the sufficiency we note first that the convergence in the mean of $L_{k,t}[f(x)]$ as k becomes infinite implies the existence of a function $\psi(t)$ of class L in $(0, \infty)$ such that

$$\lim_{k\to\infty}\int_0^{\infty} | L_{k,t}[f(x)] - \psi(t) | dt = 0 \tag{6.3}$$

and

$$\lim_{k\to\infty}\int_0^{\infty} | L_{k,t}[f(x)] | dt = \int_0^{\infty} | \psi(t) | dt.$$

Hence

$$\int_0^{\infty} | L_{k,t}[f(x)] | dt \leqq \int_0^{\infty} | \psi(t) | dt + 1 = M \qquad (k \geqq k_0)$$

for some integer k_0 sufficiently large. Hence Condition A is satisfied by $f(x)$, and by Theorem 2.1 we see that

$$f(x) = \int_0^{\infty} e^{-xt}d\alpha(t),$$

where $\alpha(t)$ is a normalized function of bounded variation in the interval $(0, \infty)$. It remains to show that $\alpha(t)$ is an integral. Set

$$\alpha_k(t) = \int_0^{t} L_{k,u}[f(x)]du.$$

Then by Theorem 1.1

$$\lim_{k\to\infty}\alpha_k(t) = \alpha(t) \qquad (0 \leqq t < \infty).$$

But it is known that if the sequence $L_{k,u}[f(x)]$ converges in mean to $\psi(u)$ then

$$\lim_{k\to\infty}\int_0^t L_{k,u}[f(x)]du = \int_0^t \psi(u)du.$$

Hence

$$\alpha(t) = \int_0^t \psi(u)du,$$

and

$$f(x) = \int_0^\infty e^{-xt}\psi(t)dt,$$

with $\psi(t)$ of class L in $(0, \infty)$.

COROLLARY 1. *If*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt$$

where $\phi(t)$ *belongs to the class* L *in* $(0, \infty)$, *then*

$$\lim_{k\to\infty}\int_0^\infty |L_{k,t}[f(x)]|\,dt = \int_0^\infty |\phi(t)|\,dt.$$

This is a known result of the mean convergence of the sequence $L_{k,t}[f(x)]$ to $\phi(t)$.

COROLLARY 2. *If*

$$\text{l.i.m.}\ L_{k,t}[f(x)] = \phi(t),$$

then

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \phi(t)$$

almost everywhere.

This follows at once from an earlier result* of the author.

We can also prove

COROLLARY 3. *A necessary and sufficient condition that*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt, \tag{6.4}$$

where $\phi(t)$ *is of class* L *in the interval* $(0, R)$ *for every* $R>0$, *the integral converging absolutely for* $x>0$, *is that* $f(x+\epsilon)$ *should satisfy Condition* D *for every positive* ϵ.

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* III, p. 122.

For, suppose that $f(x)$ has the representation (6.4). Then the integral

$$\int_0^\infty e^{-xt} | \phi(t) | \, dt$$

converges and $e^{-\epsilon t}\phi(t)$ is of class L in $(0, \infty)$. Hence $f(x+\epsilon)$ has the representation of Theorem 6.1 with $\phi(t)$ of that theorem replaced by the function $e^{-\epsilon t}\phi(t)$. It follows that $f(x+\epsilon)$ satisfies Condition D.

Conversely, if $f(x+\epsilon)$ satisfies Condition D we have

$$f(x + \epsilon) = \int_0^\infty e^{-xt}\phi_\epsilon(t)dt, \tag{6.5}$$

where $\phi_\epsilon(t)$ belongs to the class L in $(0, \infty)$ for each $\epsilon > 0$. Then

$$f(x) = \int_0^\infty e^{-\epsilon t}\phi(t)dt, \tag{6.6}$$

$$\phi(t) = e^{\epsilon t}\phi_\epsilon(t).$$

The uniqueness theorem shows that $\phi(t)$ is independent of ϵ. It is clear from the definition of $\phi(t)$ that it belongs to the class L in $(0, R)$ for each $R>0$. By Theorem 6.1, (6.5) converges absolutely for $x \geqq 0$. Hence (6.6) converges absolutely for $x \geqq \epsilon$ for each $\epsilon > 0$, that is, for $x > 0$.

This corollary enables us to treat such familiar integrals as

$$\frac{\Gamma(a + 1)}{x^{a+1}} = \int_0^\infty e^{-xt}t^a dt \qquad (a > -1),$$

$$\frac{x}{x^2 + 1} = \int_0^\infty e^{-xt} \cos t \, dt,$$

$$\frac{1}{x^2 + 1} = \int_0^\infty e^{-xt} \sin t \, dt.$$

COROLLARY 4. *A necessary and sufficient condition that*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt,$$

where $\phi(t)$ is absolutely continuous and of bounded variation in $(0, \infty)$ *and vanishes at $t = 0$, is that the function $xf(x)$ should satisfy Condition* D.

The proof may easily be supplied.

7. **The general Laplace-Stieltjes integral.** In this section we shall obtain necessary and sufficient conditions for the representation of a function $f(x)$ in the form

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is of bounded variation in $0 \leqq t \leqq R$ for every $R > 0$, the integral converging for x sufficiently large. It will be sufficient to restrict attention to convergence for $x > 0$, since a suitable change of variable reduces the general problem to this case. We first introduce

CONDITION E. *A function $f(x)$ satisfies Condition* E *if and only if*

(a) *it is of class C^∞ in $0 < x < \infty$,*

(b) *a constant M exists such that*

$$\left| \int_0^R L_{k,t}[f(x)] dt \right| \leqq M \qquad (R > 0,\ k = 1, 2, \cdots),$$

(c) *a positive function $N(t)$, defined for $t > 0$, exists such that*

$$\int_0^R | L_{k,t}[f(x)] | dt \leqq N(R) \qquad (R > 0;\ k = 1, 2, \cdots).$$

We can now prove

THEOREM 7.1. *Condition* E *is necessary and sufficient that $f(x)$ can be represented in the form*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a normalized function of bounded variation in $0 \leqq t \leqq R$ for every positive R and is bounded in $0 \leqq t < \infty$.

In the definition of Condition E it is to be understood that for each positive value of t, $N(t)$ is defined as a finite number. To prove the necessity of the condition we note first that if $f(x)$ has the representation described in the theorem, we may also write

$$f(x) = x \int_0^\infty e^{-xt} \alpha(t) dt \qquad (x > 0).$$

Moreover, since $\alpha(t)$ is of bounded variation in the neighborhood of $t = 0$, we have

$$\lim_{t \to 0+} \frac{1}{t} \int_0^t \alpha(u) du = \alpha(0+).$$

Hence we are in a position to apply a known result* to be assured of the existence of a constant M such that

* III, Theorem 21, p. 152.

(7.1) $$\left|\int_x^\infty \frac{u^{k-1}}{(k-1)!} f^{(k)}(u)du\right| \leqq M \qquad (x > 0;\ k = 1, 2, \cdots).$$

By the change of variable $u = k/t$ this becomes

$$\left|\frac{1}{k!}\int_0^{k/x} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1} dt\right| = \left|\int_0^{k/x} L_{k,t}[f(x)]dt\right| \leqq M.$$

Since x was an arbitrary positive number in (7.1) we have

$$\left|\int_0^R L_{k,t}[f(x)]dt\right| \leqq M \qquad (R > 0,\ k = 1, 2, \cdots)$$

for an arbitrary R. Hence $f(x)$ satisfies Condition E (b). That it satisfies Condition E (a) is obvious. To show that it also satisfies Condition E (c) we make use of Theorem 1.2. Assuming $\alpha(t)$ normalized we have

$$\lim_{k\to\infty} \int_0^R |L_{k,t}[f(x)]|\, dt = V(R) - |f(\infty)|.$$

It follows that for each positive R we can determine a number $N(R)$ such that

$$\int_0^R |L_{k,t}[f(x)]|\, dt \leqq N(R) \qquad (k = 1, 2, \cdots),$$

so that the necessity of Condition E is completely established.

To prove the sufficiency we note first that Condition E (b) implies* the existence of a function $\alpha(t)$ bounded in $(0, \infty)$ and such that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \alpha(u)du$$

exists, and that

$$f(x) = x\int_0^\infty e^{-xt}\alpha(t)dt \qquad (x > 0).$$

It remains to show that $\alpha(t)$ is of bounded variation in $(0, R)$ for every positive R to be assured that

(7.2) $$f(x) = \int_0^\infty e^{-xt}d\alpha(t) \qquad (x > 0).$$

To show this we recall† first that

* III, Theorem 21, p. 152.

† III, Theorem 4, p. 122.

$$\lim_{k\to\infty} L_{k,t}\left[\frac{f(x)}{x}\right] = \alpha(t)$$

for almost all values of t in $(0, \infty)$. Set

$$\alpha_k(t) = L_{k,t}[f(x)/x] \qquad (t > 0),$$
$$\alpha_k(0) = 0.$$

Then

$$\lim_{k\to\infty} \alpha_k(t) = \alpha(t)$$

when t lies in a set of points E of the interval $(0, \infty)$ whose complement with respect to that interval is of measure zero. But we are at once able to conclude* that

$$L_{k,t}[f(x)/x] = f(\infty) + \frac{(-1)^{k+1}}{k!}\int_{k/t}^{\infty} u^k f^{(k+1)}(u)du.$$

By the change of variable $u=(k+1)/v$ this equation becomes

$$L_{k,t}[f(x)/x] = f(\infty) + \int_0^{(k+1)t/k} L_{k+1,v}[f(x)]dv,$$

whence

$$\int_0^R |d\alpha_k(t)| \leqq |f(\infty)| + \int_0^{2R} |L_{k+1,v}[f(x)]|\, dv \leqq |f(\infty)| + N(2R).$$

That is, the set of functions $\alpha_k(t)$ is of uniformly bounded variation in the interval $(0, R)$. As k becomes infinite $\alpha_k(t)$ approaches $\alpha(t)$ on the set E, so that $\alpha(t)$ is of bounded variation on that part of E which lies in the interval $(0, R)$. By a result† from the theory of functions of a real variable we conclude that $\alpha(t)$ coincides on E with a function $\bar{\alpha}(t)$ which is of bounded variation on $(0, R)$ for every positive R. Hence, if we redefine $\alpha(t)$ on the set complementary to E as $\bar{\alpha}(t)$, a process which may be carried out without changing the value of $f(x)$, it becomes a function of bounded variation on $(0, R)$ for every positive R. In particular we may define $\alpha(t)$ so as to be normalized. Then

$$x\int_0^R e^{-xt}\alpha(t)dt = e^{-xR}\alpha(R) + \int_0^R e^{-xt}d\alpha(t).$$

* III, Theorem 14, Corollary 1, p. 140.

† See, for example, S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 149, Theorem 1.

Now let R become infinite. Since $\alpha(t)$ is bounded in $(0, \infty)$

$$\lim_{R\to\infty} e^{-xR}\alpha(R) = 0 \qquad (x > 0).$$

Hence the integral (7.2) converges for $x>0$ and our theorem is proved.

This theorem fails of complete generality in that the function $\alpha(t)$ is assumed uniformly bounded in $(0, \infty)$. Complete generality is attained in

THEOREM 7.2. *A necessary and sufficient condition that $f(x)$ should have the representation*

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t), \tag{7.3}$$

where $\alpha(t)$ is of bounded variation in $(0, R)$ for every positive R, the integral converging for $x>0$, is that for each positive ϵ the function $f(x+\epsilon)$ should satisfy Condition E.

For, if $f(x)$ has the representation (7.3) then

$$f(x+\epsilon) = \int_0^\infty e^{-xt}d\beta_\epsilon(t), \qquad \beta_\epsilon(t) = \int_0^t e^{-\epsilon u}d\alpha(u). \tag{7.4}$$

But, since (7.3) converges for $x>0$, to each positive ϵ corresponds* a constant K_ϵ such that

$$|\alpha(t)| < K_\epsilon e^{\epsilon t} \qquad (0 \leqq t < \infty).$$

By means of this inequality it is easily shown that $\beta_\epsilon(t)$ is uniformly bounded in $(0, \infty)$. Hence Theorem 7.1 is applicable and $f(x+\epsilon)$ satisfies Condition E. Conversely, if $f(x+\epsilon)$ satisfies E, (7.4) holds with $\beta_\epsilon(t)$ bounded. That is,

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t) \qquad (x > \epsilon), \tag{7.3}$$

$$\alpha(t) = \int_0^t e^{\epsilon u}d\beta_\epsilon(u).$$

By the uniqueness theorem $\alpha(t)$ is independent of ϵ, and, since ϵ is arbitrary, the integral (7.3) converges for $x>0$.

We can give Theorem 7.2 a different form which may be more useful in the application of the condition. We state the result in the

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* I, Lemma 2 of §3, p. 703.

COROLLARY. *A necessary and sufficient condition that $f(x)$ should have the representation*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

the integral converging for $x>0$, is that for each positive ϵ there should exist a constant M_ϵ and a function $N_\epsilon(t)$, defined for $0 \leqq t < \infty$, such that

$$\left| \int_0^R L_{k,t}[f(x)] \left(1 - \frac{t\epsilon}{k}\right)^{k+1} dt \right| \leqq M_\epsilon \qquad (k \geqq 1, 0 < R < k/\epsilon),$$

$$\int_0^R \left| L_{k,t}[f(x)] \left(1 - \frac{t\epsilon}{k}\right)^{k+1} \right| dt \leqq N_\epsilon(R) \qquad (k \geqq 1, 0 < R < k/\epsilon).$$

The proof of this follows from Theorem 7.2 by a change of variable. If

$$\frac{k}{t} + \epsilon = \frac{k}{u},$$

we have

$$\int_0^R L_{k,t}[f(x+\epsilon)] dt = \int_0^{kR/(k+\epsilon R)} L_{k,u}[f(x)] \left(1 - \frac{u\epsilon}{k}\right)^{k+1} du,$$

and, since R is an arbitrary positive constant,

$$0 < \frac{kR}{k+\epsilon R} < \frac{k}{\epsilon}.$$

We are now able to improve on the generality of Corollary 3 of Theorem 6.1. It failed of complete generality in that it had to do with absolutely convergent integrals. The general case is treated in

THEOREM 7.3. *A necessary and sufficient condition that $f(x)$ should have the representation*

$$f(x) = \int_0^\infty e^{-xt} \phi(t) dt,$$

where $\phi(t)$ is a function of class L in $(0, R)$ for every positive R, is that for every positive ϵ the function $f(x+\epsilon)$ should satisfy Condition E *and that the limit*

$$\lim_{k\to\infty} \int_0^t L_{k,u}[f(x)] du,$$

which will then exist for every positive t, should be absolutely continuous.

The result follows at once from Theorem 7.2, and the proof is omitted.

8. **Determining function possessing derivatives.** In this section we are able to discuss the case in which the determining function $\alpha(t)$ has a derivative of order n which is itself of bounded variation in the interval $(0, \infty)$. A necessary and sufficient condition on the generating function will be obtained. As a preliminary to the main result we prove the

LEMMA. *If $f(x)$ is of class C^{n+1} in the interval $(0, \infty)$, and if*

$$\int_0^\infty \frac{u^n}{n!} \left| f^{(n+1)}(u) \right| du$$

converges, then the constants $A_0, A_1, \cdots, A_n$ in the function

$$F(x) = f(x) - A_0 - A_1 x - A_2 \frac{x^2}{2!} - \cdots - A_n \frac{x^n}{n!}$$

can be so determined that the integrals

$$\int_0^\infty \frac{u^k}{k!} \left| F^{(k+1)}(u) \right| du \qquad (k = 0, 1, \cdots, n)$$

converge.

If the constants A_i are arbitrary,

$$F^{(n+1)}(x) = f^{(n+1)}(x),$$

so that the integral

$$\int_0^\infty \frac{u^n}{n!} \left| F^{(n+1)}(u) \right| du$$

converges by hypothesis. Also we have

$$F^{(n)}(x) = f^{(n)}(x) - A_n.$$

But

$$\int_1^\infty \left| F^{(n+1)}(u) \right| du \leqq \int_1^\infty u^n \left| F^{(n+1)}(u) \right| du,$$

so that the integral

$$\int_1^\infty F^{(n+1)}(u) du$$

converges absolutely, and

$$\lim_{x \to \infty} F^{(n)}(x) = \lim_{x \to \infty} f^{(n)}(x) - A_n$$

exists regardless of the manner in which A_n is determined. We now determine it so that

$$A_n = \lim_{x\to\infty} f^{(n)}(x).$$

Then

$$\lim_{x\to\infty} F^{(n)}(x) = 0.$$

Under these circumstances we have

$$F^{(n)}(x) = -\int_x^\infty F^{(n+1)}(u)du,$$

$$\int_0^\infty \frac{x^n}{(n-1)!}\left| F^{(n)}(x)\right| dx \leqq \int_0^\infty \frac{x^{n-1}}{(n-1)!}\, dx \int_x^\infty \left| F^{(n+1)}(u)\right| du$$
$$= \int_0^\infty \left| F^{(n+1)}(u)\right| \frac{u^n}{n!}\, du.$$

We will now show that the constants A_i can be determined by the recursion formula

$$(8.1) \qquad A_k = \lim_{x\to\infty}\left[f^{(k)}(x) - \frac{A_n x^{n-k}}{(n-k)!} - \frac{A_{n-1}x^{n-k-1}}{(n-k-1)!} - \cdots - A_{k+1}x\right]$$
$$(k = 0, 1, \cdots, n-1).$$

Assume that we have proved the existence of the above limits for $k=n$, $n-1, \cdots, p+1$, and that we have shown the convergence of the integrals

$$\int_0^\infty \frac{u^k}{k!}\left| F^{(k+1)}(u)\right| du \qquad (k = n, n-1, \cdots, p).$$

Then the previous argument shows that

$$\lim_{x\to\infty} F^{(p)}(x) = \lim_{x\to\infty}\left[f^{(p)}(x) - A_p - A_{p+1}x - \cdots - \frac{A_n x^{n-p}}{(n-p)!}\right]$$

exists regardless of the manner in which $A_0, A_1, \cdots, A_p$ are determined. Here the limit (8.1) exists for $k=p$, and if we define A_p by (8.1) with $k=p$ we have

$$\lim_{x\to\infty} F^{(p)}(x) = 0.$$

Then applying the earlier argument we see that

$$\int_0^\infty \frac{u^{p-1}}{(p-1)!}\left| F^{(p)}(u)\right| du$$

also converges. We are now in a position to complete the proof of the lemma by induction.

As an immediate consequence we see that the limits

$$\lim_{x\to\infty} f^{(k)}(x)/x^{n-k} \qquad (k = 0, 1, 2, \cdots, n)$$

exist under the hypothesis of the lemma.

We make use of this Lemma to prove

THEOREM 8.1. *A necessary and sufficient condition that $f(x)$ can be represented in the form*

$$f(x) = \sum_{k=0}^{n} A_k \frac{x^k}{k!} + \int_0^\infty e^{-xt} d\alpha(t),$$

where $A_0, A_1, \cdots, A_n$ are constants and $\alpha(t)$ is of bounded variation in $(0, \infty)$, *is that*

$$\int_0^\infty \frac{u^k}{k!} \left| f^{(k+1)}(u) \right| du \leqq M \qquad (k = n, n+1, \cdots). \tag{8.2}$$

To prove the necessity of the condition apply Theorem 2.1 to the function

$$F(x) = f(x) - \sum_{k=0}^{n} A_k \frac{x^k}{k!}. \tag{8.3}$$

Since

$$F^{(k+1)}(x) = f^{(k+1)}(x) \qquad (k = n, n+1, \cdots),$$

the necessity of (8.2) is established.

For the sufficiency we first determine a function $F(x)$ in the form (8.3) determining the constants A_i as in the lemma in such a way that the integrals

$$\int_0^\infty \frac{u^k}{k!} \left| F^{(k+1)}(u) \right| du \qquad (k = 0, 1, \cdots, n) \tag{8.4}$$

all converge. If we choose M' greater than M and greater than each of the integrals (8.4) we have

$$\int_0^\infty \frac{u^k}{k!} \left| F^{(k+1)}(u) \right| du < M' \qquad (k = 0, 1, \cdots).$$

Hence, by another application of Theorem 2.1, we have

$$F(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is of bounded variation in the interval $(0, \infty)$. The result leads at once to the

COROLLARY. *A necessary and sufficient condition that $f(x)$ can be represented in the form*

$$f(x) = \sum_{k=0}^{n} A_k \frac{x^k}{k!} + \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is of bounded variation in the interval $(0, \infty)$, is that

$$\int_0^\infty | L_{k,t}[f(x)] | dt \leqq M \qquad (k = n+1, n+2, \cdots).$$

We come now to the principal result of the present section. For convenience in stating the result let us extend the definition of $L_{k,t}[f(x)]$ to the case $k=0$ in such a way that

$$\int_0^\infty \left| \frac{d^{n+1}}{dt^{n+1}} L_{0,t}[f(x)] \right| dt = \int_0^\infty \left| \frac{d^{n+1}}{dt^{n+1}} [t^{n+1} f(t)] \right| \frac{t^n}{n!} dt.$$

Then the result may be stated as follows:

THEOREM 8.2. *A necessary and sufficient condition that $f(x)$ can be represented in the form*

$$f(x) = \int_0^\infty e^{-xt} \phi(t) dt, \tag{8.5}$$

where $\phi(t)$ has a derivative of order n which is of bounded variation in the interval $(0, \infty)$, is that there should exist a constant M such that

$$\int_0^\infty \left| \frac{d^{n+1}}{dt^{n+1}} L_{k,t}[f(x)] \right| dt \leqq M \qquad (k = 0, 1, 2, \cdots). \tag{8.6}$$

To prove the necessity let $f(x)$ have the representation (8.5). Then integration by parts gives

$$xf(x) = \phi(0) + \int_0^\infty e^{-xt} \phi'(t) dt,$$

provided

$$\lim_{t \to \infty} \phi'(t) e^{-xt} = 0.$$

But since $\phi^{(n)}(t)$ is of bounded variation in $(0, \infty)$ the integral

$$H(x) = \int_0^\infty e^{-xt} d\phi^{(n)}(t)$$

converges for $x \geqq 0$ and has the value

$$H(x) = -\phi^{(n)}(0) + x\int_0^\infty e^{-xt}\phi^{(n)}(t)dt \qquad (x > 0).$$

Another integration by parts gives

$$H(x) = -\phi^{(n)}(0) - x\phi^{(n-1)}(0) + x^2\int_0^\infty e^{-xt}\phi^{(n-1)}(t)dt \qquad (x > 0).$$

That

$$\lim_{t\to\infty} e^{-xt}\phi^{(n-1)}(t) = 0 \qquad (x > 0) \tag{8.7}$$

follows from the fact that $\phi^{(n)}(t)$ is bounded. For, since

$$\phi^{(n)}(t) = O(1) \qquad (t \to \infty),$$

then

$$\phi^{(n-1)}(t) = \int_0^t \phi^{(n)}(u)du + \phi^{(n-1)}(0) = O(t) \qquad (t \to \infty),$$

and this result is sufficient to insure (8.7). In a similar way we see that

$$\phi^{(k)}(t) = O(t^{n-k}) \qquad (t \to \infty,\ k = 0, 1, \cdots, n),$$

so that

$$\lim_{t\to\infty} e^{-xt}\phi^{(k)}(t) = 0 \qquad (k = 0, 1, \cdots, n;\ x > 0).$$

Hence

$$H(x) = \sum_{i=0}^n \phi^{(i)}(0)x^{n-i} + x^{n+1}\int_0^\infty e^{-xt}\phi(t)dt,$$

or

$$x^{n+1}f(x) = \sum_{i=0}^n \phi^{(i)}(0)x^{n-i} + \int_0^\infty e^{-xt}d\phi^{(n)}(t).$$

Hence if

$$F(x) = x^{n+1}f(x) - \sum_{i=0}^n \phi^{(i)}(0)x^{n-i},$$

we see that $F(x)$ must satisfy Condition A. Consequently a constant M' exists such that

$$\int_0^\infty \left| F^{(k+1)}(u) \right| \frac{u^k}{k!} du \leqq M' \qquad (k = 0, 1, 2, \cdots).$$

For $k \geqq n$ this becomes

$$(8.8)\qquad \int_0^\infty \left| \left[x^{n+1}f(x)\right]^{(k+1)} \right| \frac{x^k}{k!}\,dx \leqq M' \qquad (k = n, n+1, \cdots).$$

Now consider the integral

$$\int_0^\infty \frac{1}{t^{n+1}} \left| L_{k,t}\left[(x^{n+1}f(x))^{(n+1)}\right] \right| dt \qquad (k = 1, 2, \cdots).$$

By definition of the operator $L_{k,t}[f(x)]$ it has the value

$$\int_0^\infty \frac{u^{n+k}}{k^n k!} \left| \left[u^{n+1}f(u)\right]^{(n+k+1)} \right| du.$$

Since

$$\lim_{k\to\infty} \frac{(n+k)!}{k!k^n} = 1,$$

we see by (8.8) for $k = n+1, n+2, \cdots$ that there exists a constant M'' such that

$$\int_0^\infty \frac{1}{t^{n+1}} \left| L_{k,t}\left[x^{n+1}f(x)\right] \right| dt \leqq M'' \qquad (k = 1, 2, \cdots).$$

By a previous result* this becomes

$$\int_0^\infty \left| \frac{d^{n+1}}{dt^{n+1}} L_{k,t}[f(x)] \right| dt \leqq M'' \qquad (k = 1, 2, \cdots).$$

Setting $k = n$ in (8.8) and making use of the convention established concerning $L_{0,t}[f(x)]$ we see that for a suitable constant M

$$\int_0^\infty \left| \frac{d^{n+1}}{dt^{n+1}} L_{k,t}[f(x)] \right| dt \leqq M \qquad (k = 0, 1, 2, \cdots).$$

The necessity of the condition is thus established.

Conversely, if inequalities (8.6) hold, then inequalities (8.8) are also satisfied. By Theorem 8.1 constants $A_0, A_1, \cdots, A_n$ and a function $\alpha(t)$ of bounded variation in the interval $(0, \infty)$ exist such that

$$x^{n+1}f(x) = \sum_{i=0}^{n} A_i \frac{x^i}{i!} + \int_0^\infty e^{-xt}d\alpha(t).$$

No loss of generality results from assuming $\alpha(0) = A_0$.

Integration by parts gives

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* III, p. 135, Lemma.

$$x^{n+1}f(x) = \sum_{i=1}^{n} A_i \frac{x^i}{i!} + x \int_0^\infty e^{-xt}\alpha(t)dt.$$

Set

$$\phi_1(t) = A_1 + \int_0^t \alpha(u)du,$$

$$\phi_2(t) = \frac{A_2}{2!} + \int_0^t \phi_1(u)du,$$

. . . .

$$\phi(t) = \phi_n(t) = \frac{A_n}{n!} + \int_0^t \phi_{n-1}(u)du.$$

Then successive integration by parts gives

$$x^{n+1}f(x) = \frac{A_k x^k}{k!} + \cdots + \frac{A_n x^n}{n!} + x^k \int_0^\infty e^{-xt}\phi_{k-1}(t)dt \qquad (k = 2, 3, \cdots, n),$$

$$x^{n+1}f(x) = x^{n+1} \int_0^\infty e^{-xt}\phi(t)dt.$$

But $\phi(t)$ is of class C^{n-1} in $(0, \infty)$. In fact it has an nth derivative

$$\phi^{(n)}(t) = \alpha(t)$$

which is of bounded variation in $(0, \infty)$, so that the sufficiency of our condition is also established.

9. **Continuous determining function.** By use of the operator $l_{k,t}[f(x)]$ defined in §1 we now deduce

THEOREM 9.1. *If $f(x)$ has the representation*

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t),$$

where $\alpha(t)$ is a normalized function of bounded variation in $(0, R)$ for every positive R, the integral converging for some x, then a necessary and sufficient condition that $\alpha(t)$ should be continuous for $t>0$ is that

$$\lim_{k\to\infty} l_{k,t}[f(x)] = 0 \qquad (t > 0). \tag{9.1}$$

The proof follows at once from Theorem 1.3. We also point out that (9.1) may be replaced by

$$\lim_{k\to\infty} (et)^k f^{(k)}(kt) = 0 \qquad (t > 0).$$

COROLLARY. *Under the conditions of the theorem $\alpha(t)$ is continuous for $0 \leqq t < \infty$ if and only if*

$$\lim_{k\to\infty} (et)^k f^{(k)}(kt) = 0 \qquad (t > 0),$$

$$\lim_{k\to\infty} f(x) = 0.$$

This follows since

$$\alpha(0+) = f(\infty).$$

As an example of the type of result that can now be proved we state

THEOREM 9.2. *A necessary and sufficient condition that $f(x)$ can be represented in the form*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \qquad (\alpha(0) = 0),$$

where $\alpha(t)$ is of bounded variation in $(0, \infty)$ and is continuous in $0 \leqq t < \infty$, is that Condition A *should be satisfied and that*

$$\lim_{k\to\infty} l_{k,t}[f(x)] = 0 \qquad (t > 0),$$

$$\lim_{x\to\infty} f(x) = 0.$$

10. **The Stieltjes integral equation.** In this section we shall discuss the class of generating functions whose determining functions have first derivatives which are completely monotonic. Since a completely monotonic function is a Laplace integral we are treating generating functions whose determining functions are themselves generating functions. We have shown that such functions $f(x)$ can be represented in the form

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

and such an equation is known as an integral equation of Stieltjes. We shall obtain a necessary and sufficient condition that it shall have a non-decreasing solution $\alpha(t)$. The result is contained in

THEOREM 10.1. *A necessary and sufficient condition that $f(x)$ can be expressed in the form*

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t}, \tag{10.1}$$

where $\alpha(t)$ *is uniformly bounded and non-decreasing in* $(0, \infty)$, *is that*

$$(-1)^k[x^nf(x)]^{(n+k)} \geqq 0 \qquad (x > 0;\ n,\ k = 0, 1, 2, \cdots), \tag{10.2}$$

$$\lim_{x\to\infty} xf(x) \textit{ exists.} \tag{10.3}$$

To prove the necessity of the condition we note first that if $\alpha(t)$ is uniformly bounded and non-decreasing, then the integral (10.1) converges for $x>0$. For such x we have

$$\begin{aligned}[x^nf(x)]^{(n)} &= \int_0^\infty \frac{\partial^n}{\partial x^n}\frac{x^n}{x+t}\,d\alpha(t)\\ &= n!\int_0^\infty \frac{t^n d\alpha(t)}{(x+t)^{n+1}} \qquad (n = 0, 1, 2, \cdots),\end{aligned}$$

and

$$(-1)^k[x^nf(x)]^{(n+k)} = \int_0^\infty \frac{t^n d\alpha(t)}{(x+t)^{n+k+1}} \qquad (n,\ k = 0, 1, 2, \cdots).$$

Differentiation under the integral sign is easily justified for $x>0$. One sees by inspection of this integral that it is non-negative for positive x so that (10.2) is established. To establish (10.3) we have

$$\left| xf(x) - \int_0^\infty d\alpha(t) \right| \leqq \frac{R}{x+R}\alpha(R) + [\alpha(\infty) - \alpha(R)]$$

for every positive R. From this it is clear that

$$\lim_{x\to\infty} xf(x) = \int_0^\infty d\alpha(t) = \alpha(\infty).$$

Conversely, we see that (10.2) implies that for each fixed non-negative integer n the function $[x^nf(x)]^{(n)}$ is completely monotonic for $x>0$. Hence by Theorem 3.1

$$[x^nf(x)]^{(n)} = \int_0^\infty e^{-xt}d\beta_n(t) \qquad (x > 0;\ n = 0, 1, 2, \cdots), \tag{10.4}$$

where each $\beta_n(t)$ is a normalized non-decreasing function and the integral converges for $x>0$. Since

$$[x^{n+1}f(x)]^{(n+1)} = x[x^nf(x)]^{(n+1)} + (n+1)[x^nf(x)]^{(n)},$$

we have by differentiating (10.4)

$$\int_0^\infty e^{-xt}d\beta_{n+1}(t) = -x\int_0^\infty e^{-xt}td\beta_n(t) + (n+1)\int_0^\infty e^{-xt}d\beta_n(t).$$

This becomes, after integration by parts,

$$\int_0^\infty e^{-xt}\beta_{n+1}(t)dt = -\int_0^\infty e^{-xt}td\beta_n(t) + (n+1)\int_0^\infty e^{-xt}\beta_n(t)dt.$$

The uniqueness theorem for Laplace* integrals now shows that

$$\int_0^t \beta_{n+1}(u)du = -\int_0^t ud\beta_n(u) + (n+1)\int_0^t \beta_n(u)du,$$

or, again integrating by parts, that

$$t\beta_n(t) = (n+2)\int_0^t \beta_n(u)du - \int_0^t \beta_{n+1}(u)du \qquad (n = 0, 1, 2, \cdots).$$

This equation shows at once that all of the $\beta_n(t)$ are continuous for $t>0$. Hence the equation may be differentiated, showing that all $\beta_n'(t)$ exist and are continuous for $t>0$. Thus,

$$t\beta_n'(t) = (n+1)\beta_n(t) - \beta_{n+1}(t).$$

Since all $\beta_n(t)$ are now known to be of class C', we see that the $\beta_n''(t)$ exist and are continuous for $t>0$ and that

$$t\beta_n''(t) = n\beta_n'(t) - \beta_{n+1}'(t). \qquad (10.5)$$

Thus, we can show by induction that the $\beta_n(t)$ are of class C^∞ for $t>0$.

We now rewrite equation (10.5) as follows:

$$\beta_{n+1}'(t)/t^{n+1} = -\frac{d}{dt}[\beta_n'(t)/t^n] \qquad (t > 0).$$

Repeated application of this formula to itself yields

$$\frac{\beta_{n+1}'(t)}{t^{n+1}} = (-1)^{n+1}\frac{d^{n+1}}{dt^{n+1}}[\beta_0'(t)] \qquad (n = 0, 1, 2, \cdots).$$

Since all $\beta_n(t)$ are known to be non-decreasing, the left-hand side is non-negative for positive t. Hence $\beta_0'(t)$ is completely monotonic for $t>0$. A further application of Theorem 3.1 shows that

$$\beta_0'(t) = \int_0^\infty e^{-tu}d\alpha(u) \qquad (t > 0),$$

* I, p. 705.

where $\alpha(u)$ is non-decreasing and the integral converges for $t>0$.

We show next that $\alpha(u)$ is bounded in $(0, \infty)$. By (10.3) we see that

$$\beta_0(0+) = \lim_{x\to\infty} f(x) = 0,$$

so that $\beta_0(t)$ is continuous in the interval $0 \leqq t < \infty$. Moreover, since $\beta_0(t)$ is non-decreasing, a familiar Tauberian theorem* is applicable, and we have

$$\lim_{t\to 0+} \frac{\beta_0(t)}{t} = A = \lim_{x\to\infty} xf(x). \tag{10.6}$$

Now $\beta_0'(t)$, being completely monotonic, is a decreasing function of t in $(0, \infty)$. Then (10.6) shows that

$$\alpha(\infty) = \beta'(0+) = \lim_{t\to 0+} \frac{\beta_0(t)}{t}\cdot$$

Hence $\alpha(u)$ is bounded in $(0, \infty)$. That is,

$$\begin{aligned} f(x) &= \int_0^\infty e^{-xt}dt \int_0^\infty e^{-tu}d\alpha(u) \\ &= \int_0^\infty d\alpha(u) \int_0^\infty e^{-(x+u)t}dt \\ &= \int_0^\infty \frac{d\alpha(u)}{x+u}\cdot \end{aligned}$$

This completes the proof of the theorem.

Corollary. *Condition* (10.2) *may be replaced by*

$$\begin{aligned} &(-1)^n L_{k,t}^{(n)}[f(x)] \geqq 0 \qquad (t > 0;\ k = 1, 2, \cdots;\ n = 0, 1, 2, \cdots), \\ &[t^n f(t)]^{(n)} \geqq 0 \qquad (t > 0;\ n = 0, 1, 2, \cdots). \end{aligned} \tag{10.7}$$

The upper index (n) in (10.7) indicates the nth derivative of $L_{k,t}[f(x)]$ with respect to t. The equivalence of this condition with (10.2) is established by use of the identity†

$$L_{k,t}[\{x^n f(x)\}^{(n)}] = (-1)^n t^n L_{k,t}^{(n)}[f(x)].$$

This form of the theorem suggests the following result:

---

* See, for example, J. Karamata, *Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen*, Journal für die Reine und Angewandte Mathematik, vol. 164 (1931), p. 29.

† III, p. 135.

THEOREM 10.2. *If $f(x)$ has the representation*

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

where $\alpha(t)$ is bounded and non-decreasing in the interval $(0, \infty)$, *then there exist non-decreasing functions $\gamma_1(y), \gamma_2(y), \cdots$ such that*

$$L_{k,t}[f(x)] = \int_0^\infty e^{-yt} d\gamma_k(y) \qquad (k = 1, 2, \cdots),$$

the integrals converging for $x>0$. In fact

$$\gamma_k'(y) = \int_0^\infty L_{k,u}[e^{-xy}] d\alpha(u). \tag{10.8}$$

In the operation $L_{k,u}[e^{-xy}]$ the function e^{-xy} is regarded as a function of x, y being a parameter. The existence of the functions $\gamma_k(y)$ follows at once from Theorem 3.1 since (10.7) implies that $L_{k,t}[f(x)]$ is a completely monotonic function of t. To determine the $\gamma_k(y)$ explicitly we show first that

$$\int_0^\infty e^{-uy} L_{k,t}[e^{-xy}] dy = \int_0^\infty e^{-ty} L_{k,u}[e^{-xy}] dy.$$

The first of these integrals is equal to

$$\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-uy} e^{-ky/t} y^k dy.$$

Make the change of variable $uy = tz$, and it becomes

$$\frac{1}{k!}\left(\frac{k}{u}\right)^{k+1} \int_0^\infty e^{-tz} e^{-kz/u} z^k dz = \int_0^\infty e^{-ty} L_{k,u}[e^{-xy}] dy.$$

Now returning to the computation of $\gamma_k(y)$ we have

$$L_{k,t}[f(x)] = \int_0^\infty L_{k,t}\left[\frac{1}{x+u}\right] d\alpha(u).$$

In view of

$$\frac{1}{x+u} = \int_0^\infty e^{-xy} e^{-uy} dy,$$

this becomes

$$L_{k,t}[f(x)] = \int_0^\infty d\alpha(u) \int_0^\infty e^{-ty} L_{k,u}[e^{-xy}] dy.$$

Interchanging the order of integration, we have

$$L_{k,t}[f(x)] = \int_0^\infty e^{-ty}dy \int_0^\infty L_{k,u}[e^{-xy}]d\alpha(u),$$

so that the uniqueness theorem for Laplace integrals yields (10.8).

The integral (10.1) is of use in the theory of continued fractions. It is of particular interest there when the function $\alpha(t)$ is non-decreasing and of such a nature that the integrals

$$\int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, \cdots)$$

all converge. In this connection we establish

THEOREM 10.3. *A necessary and sufficient condition that $f(x)$ should have the representation*

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

where $\alpha(t)$ is non-decreasing and of such a nature that the integrals

$$\int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, \cdots) \tag{10.9}$$

all converge, is that

$$(-1)^k[x^n f(x)]^{(n+k)} \geqq 0 \qquad (x > 0;\ n, k = 0, 1, 2, \cdots)$$

and that $f(x)$ should have an asymptotic development

$$f(x) \sim \frac{A_1}{x} - \frac{A_2}{x^2} + \frac{A_3}{x^3} - \cdots \qquad (x \to \infty). \tag{10.10}$$

To prove the necessity of the condition it remains only to show the necessity of (10.10). To prove this it is sufficient to show that

$$\lim_{x\to\infty} \int_0^\infty \frac{t^n d\alpha(t)}{x+t} = 0 \qquad (n = 1, 2, \cdots),$$

a result which is established under the hypothesis (10.9) in much the same way as was done for the case $n=1$ in the proof of Theorem 10.1.

For the sufficiency of the condition we have as in the proof of Theorem 10.1

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} = \int_0^\infty e^{-xt}\beta_0'(t)dt. \tag{10.11}$$

Moreover, $\beta_0'(0+) = A_1$. Hence integration by parts gives

$$f(x) = \frac{A_1}{x} + \int_0^\infty e^{-xt}\beta_0''(t)dt. \tag{10.12}$$

By (10.10)

$$\lim_{x\to\infty} x^2\left[f(x) - \frac{A_1}{x}\right] = A_2.$$

Applying the same arguments to the integral (10.12) as were applied to the integral (10.11) we find that

$$\beta_0''(0+) = A_2.$$

It is now clear how we can prove by induction that

$$\beta_0^{(n)}(0+) = A_n \qquad (n = 1, 2, \cdots).$$

But

$$\beta_0^{(n)}(t) = \int_0^\infty e^{-ut}d\gamma(u),$$

$$\gamma(t) = \int_0^t (-u)^n d\alpha(u),$$

and

$$\beta_0^{(n)}(0+) = \gamma(\infty) = (-1)^n \int_0^\infty t^n d\alpha(t),$$

so that the sufficiency of the condition is established.

11. **Dirichlet series.** The methods of the present paper are well adapted to the discussion of what functions can be represented in Dirichlet series. We prove

THEOREM 11.1. *A necessary and sufficient condition that $f(x)$ can be represented in a Dirichlet series*

$$f(x) = \sum_{n=0}^\infty a_n e^{-\lambda_n x} \tag{11.1}$$

convergent for $x>0$ is that for each positive number ϵ the function $f(x+\epsilon)$ should satisfy Condition E *and that*

$$\lim_{k\to\infty} L_{k,t}[f(x)] = 0$$

in a set of intervals

$$
(11.2) \qquad \begin{aligned} 0 = \lambda_0 < t < \lambda_1, \\ \lambda_1 < t < \lambda_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \lim_{n\to\infty} \lambda_n = \infty, \end{aligned}
$$

the approach being uniform in any closed sub-interval.

For, if $f(x)$ has the representation (11.1) it also has the form

$$
(11.3) \qquad f(x) = \int_0^\infty e^{-xt} d\alpha(t),
$$

where $\alpha(t)$ is constant in each of the intervals (11.2). Hence $f(x+\epsilon)$ satisfies Condition E for each positive ϵ by Theorem 7.2. By use of an earlier result*

$$
\lim_{k\to\infty} L_{k,t}[f(x)] = 0
$$

in each of the intervals (11.2).

To prove the approach uniform in the interval $a \leqq t \leqq b$, where

$$
\lambda_n < a < b < \lambda_{n+1},
$$

we have by simple computations

$$
\begin{aligned} L_{k,t}[f(x)] &= \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-ku/t} u^k d\alpha(u) \qquad \left(\frac{k}{t} > 0\right) \\ &= \frac{1}{k!}\left(\frac{k}{t}\right)^{k+2} \int_0^\infty e^{-ku/t} u^{k-1}(u - t)[\alpha(u) - \alpha(t)]du. \end{aligned}
$$

Since $\alpha(u)$ is constant in the interval $(\lambda_n, \lambda_{n+1})$, we have for t in (a, b)

$$
L_{k,t}[f(x)] = I_1 + I_2,
$$

$$
I_1 = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+2} \int_0^{\lambda_n} e^{-ku/t} u^{k-1}(u - t)[\alpha(u) - \alpha(t)]du,
$$

$$
I_2 = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+2} \int_{\lambda_{n+1}}^\infty e^{-ku/t} u^{k-1}(u - t)[\alpha(u) - \alpha(t)]du.
$$

But

$$
|I_1| \leqq \frac{1}{k!}\left(\frac{k}{t}\right)^{k+2} [e^{-\lambda_n/t}\lambda_n]^{k-1} \int_0^{\lambda_n} (b - u)(|\alpha(u)| + M)du,
$$

where M is an upper bound of $|\alpha(t)|$ in (a, b). Since $e^{-\lambda_n/t}(\lambda_n/t)$ is a decreasing function of t in (a, b), we have

* III, p. 137, Theorem 13.

$$|I_1| \leqq \frac{1}{k!}\left(\frac{k}{a}\right)^3 [ke^{-\lambda_n/a}(\lambda_n/a)]^{k-1} \int_0^{\lambda_n} (b-u)(|\alpha(u)|+M)du.$$

It is easily seen that the right-hand member of this inequality tends to zero with $1/k$. Since it is independent of t, I_1 approaches zero uniformly with $1/k$. By a similar argument, I_2 does likewise. Hence the necessity of the condition is established.

Conversely, if the conditions of the theorem are satisfied, $f(x)$ has the form (11.3) by Theorem 7.2. It remains to show that $\alpha(t)$ is a step-function. By the lemma of §5 we have for any two positive numbers t_1 and t_2

$$\alpha(t_2) - \alpha(t_1) = \lim_{k\to\infty} \int_{t_1}^{t_2} L_{k,u}[f(x)]du. \tag{11.4}$$

Since $L_{k,t}[f(x)]$ approaches zero uniformly in any closed sub-interval of the interval $\lambda_n < t < \lambda_{n+1}$, it follows that for any fixed points t_1 and t_2 of that interval we may take the limit under the sign of integration in (11.4), so that

$$\alpha(t_1) = \alpha(t_2).$$

This proves that $\alpha(t)$ is a step-function of the type required to make (11.3) a Dirichlet series, so that the theorem is proved.

In a similar way we could prove the result stated in

THEOREM 11.2. *A necessary and sufficient condition that $f(x)$ can be expressed as a Dirichlet series*

$$f(x) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n x}$$

absolutely convergent for $x \geqq 0$ is that $f(x)$ should satisfy Condition A *and that*

$$\lim_{k\to\infty} L_{k,t}[f(x)] = 0$$

in a set of intervals

$$0 = \lambda_0 < t < \lambda_1,$$
$$\lambda_1 < t < \lambda_2,$$
$$\cdots\cdots$$
$$\lim_{n\to\infty} \lambda_n = \infty,$$

the approach being uniform in any closed sub-interval.

12. **Determining function of bounded variation.** We give here the proof of Theorem 2.1 omitted in §2. A preliminary result we state as

LEMMA 1. *As x approaches zero the function*

$$H(x, y) = e^{-xy} - (1 - x)^y$$

approaches zero uniformly in the interval $0 \leqq y < \infty$.

The function $H(x, y)$ can clearly be expressed as an integral as follows:

$$H(x, y) = \int_x^{-\log(1-x)} e^{-ty} y\, dt.$$

Since

$$0 \leqq e^{-u} u \leqq e^{-1} \qquad (0 \leqq u < \infty),$$

we have

$$0 \leqq H(x, y) \leqq e^{-1} \int_x^{-\log(1-x)} \frac{dt}{t} = e^{-1} \log\left(\frac{-\log(1-x)}{x}\right).$$

The right-hand side of this inequality is independent of y and approaches zero with x, so that the lemma is proved.

An immediate consequence is

LEMMA 2. *For each non-negative value of x*

$$\lim_{a\to\infty} \left[\left(1 - \frac{x}{a}\right)^n - e^{-nx/a}\right] = 0$$

uniformly for all non-negative integers n.

We next introduce

CONDITION A′. *A function $f(x)$ satisfies Condition* A′ *in the interval* $(0, \infty)$ *if*

(a) *$f(x)$ is of class C^∞ in the interval* $0 < x < \infty$,

(b) *a constant M exists such that*

$$\sum_{k=0}^{\infty} \left| f^{(k)}(x) \right| x^k/k! \leqq M \qquad (0 < x < \infty).$$

We now prove

THEOREM 12.1. *A function $f(x)$ which satisfies Condition* A′ *in* $(0, \infty)$ *is analytic for* $0 < x < \infty$, *and*

$$\lim_{x\to 0+} f(x) = \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(-a)^k}{k!}$$

for every positive number a.

For the Taylor expansion of $f(x)$ about the point $x = a > 0$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k,$$

clearly converges absolutely as a result of Condition A′ in the interval $0<x<2a$. That it converges to $f(x)$ one sees by an investigation of the remainder, noting that Condition A′ implies that

$$| f^{(k)}(x) | \leqq \frac{Mk!}{x^k} \qquad (0 < x < \infty).$$

The above series is also seen to converge absolutely for $x=0$. Hence by Abel's theorem $f(x)$ is continuous on the right at $x=0$ if we define $f(0)$ as

$$f(0) = \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(-a)^k}{k!}.$$

In the remainder of this section we shall take this as the definition of $f(0)$.

We turn next to the proof of

THEOREM 12.2. *If $f(x)$ satisfies Condition* A′ *in* $(0, \infty)$, *then for each non-negative x the series*

$$(12.1) \qquad \phi_a(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (-a)^k e^{-xk/a} \qquad (a > 0)$$

converges and

$$(12.2) \qquad \lim_{a\to\infty} \phi_a(x) = f(x).$$

The series (12.1) is clearly dominated by the convergent series

$$\sum_{k=0}^{\infty} | f^{(k)}(a) | \frac{a^k}{k!}$$

so that its convergence is assured. By Theorem 12.1

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(x-a)^k}{k!} \qquad (| x - a | \leqq a).$$

Hence

$$f(x) - \phi_a(x) = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} f^{(k)}(a) \left[\left(1 - \frac{x}{a}\right)^k - e^{-xk/a}\right]$$

for $|x-a| \leqq a$, $a>0$. Then by Lemma 2 of the present section we have

$$\lim_{a\to\infty} [f(x) - \phi_a(x)] = 0$$

for each fixed value of x greater than zero. This completes the proof of the theorem.

By use of this result we are able to prove

THEOREM 12.3. *Condition* A′ *is necessary and sufficient that* $f(x)$ *can be expressed in the form*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \qquad (x > 0), \tag{12.3}$$

where $\alpha(t)$ *is of bounded variation in the interval* $(0, \infty)$.

We first prove the necessity of the condition. Let $f(x)$ have the form (12.3). Then the integral converges for $x \geqq 0$ and $f(x)$ is analytic* in $0 < x < \infty$, so that A′ (a) is satisfied.

It remains to prove A′ (b). We have at once

$$\sum_{k=0}^\infty |f^{(k)}(x)| \frac{x^k}{k!} \leqq \sum_{k=0}^\infty \int_0^\infty e^{-xt} \frac{(tx)^k}{k!} dV(t)$$

where $V(x)$ is the total variation of $\alpha(t)$ in the interval $0 \leqq t \leqq x$. Applying any one of several familiar tests, we see that it is permissible to integrate the series

$$\sum_{k=0}^\infty e^{-xt} \frac{(tx)^k}{k!}$$

term by term with respect to $V(t)$ and obtain

$$\sum_{k=0}^\infty |f^{(k)}(x)| \frac{x^k}{k!} \leqq \int_0^\infty dV(t) = V(\infty) = M,$$

so that Condition A′ (c) is satisfied.

To prove the sufficiency of the condition we note first that the function $\phi_a(x)$ of Theorem 12.2 can be expressed as a Laplace-Stieltjes integral

$$\phi_a(x) = \int_0^\infty e^{-xt} d\alpha_a(t),$$

where

$$\alpha_a(0) = 0,$$

$$\alpha_a(t) = \sum_{k < at} f^{(k)}(a) \frac{(-a)^k}{k!} \qquad (t > 0). \tag{12.4}$$

* I, p. 702.

The total variation of this function in $(0, \infty)$ is clearly equal to

$$\sum_{k=0}^{\infty} |f^{(k)}(a)| \frac{a^k}{k!},$$

and this by hypothesis is not greater than M. By Theorem 12.2

$$f(x) = \lim_{a\to\infty} \int_0^{\infty} e^{-xt} d\alpha_a(t).$$

By a theorem of E. Helly* we can pick from the set of functions $\alpha_a(t)$ $(a=0, 1, 2, \cdots)$, all of variation not greater than M, a sub-set $\alpha_{a_i}(t)$ $(i=0, 1, 2, \cdots)$, which approaches a limit $\alpha(t)$, also of variation not greater than M in $(0, \infty)$. Then

$$f(x) = \lim_{i\to\infty} \int_0^{\infty} e^{-xt} d\alpha_{a_i}(t).$$

By the Helly-Bray Theorem† we may take the limit under the sign of integration and obtain

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t)$$

as stated in the theorem. The proof of the theorem shows that the smallest possible value of M in Condition A′ is the total variation of $\alpha(t)$ in $(0, \infty)$.

In addition to giving a new proof of Theorem 2.1 the present methods give a new proof of Theorem 3.1. For, if $f(x)$ is completely monotonic in $0 \leq x < \infty$ we see at once by Taylor's series that

$$\sum_{k=0}^{\infty} |f^{(k)}(a)| \frac{a^k}{k!} = f(0+) \qquad (0 < a < \infty),$$

so that Condition A′ is satisfied. Hence $f(x)$ has the form (12.3). By (12.4) $\alpha_a(t)$ is non-decreasing and

$$\alpha_a(\infty) = f(0).$$

The same must therefore be true of the limit function $\alpha(t)$ so that the proof of Theorem 3.1 is complete.

Finally, we prove

THEOREM 12.4. *Condition* A *and* A′ *are equivalent.*

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* E. Helly, *Über lineare Funktionaloperationen*, Wiener Sitzungsberichte, vol. 121 (1921), p. 265.

† See, for example, G. C. Evans, *The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems*, Colloquium Publications of the American Mathematical Society, vol. 6, 1927, p. 15.

Let us first show that A′ implies A. Assuming that A′ is satisfied we have from Theorem 12.1

$$f^{(k+1)}(u) = \sum_{n=0}^{\infty} f^{(k+n+1)}(a)\frac{(u-a)^n}{n!} \qquad (0 < u \leqq a).$$

Hence

$$| f^{(k+1)}(u) | \leqq \sum_{n=0}^{\infty} | f^{(k+n+1)}(a) | \frac{(a-u)^n}{n!} \qquad (0 < u \leqq a),$$

and

$$\int_{\epsilon}^{a} \frac{u^k}{k!} | f^{(k+1)}(u) | du \leqq \sum_{n=0}^{\infty} \frac{| f^{(k+n+1)}(a) |}{n!k!} \int_{\epsilon}^{a} (a-u)^n u^k du$$

for any positive ϵ less than a. The inequality is strengthened if ϵ is replaced by 0 on the right-hand side, and the series continues to converge, for its value is

$$\sum_{n=k+1}^{\infty} | f^{(k)}(a) | \frac{a^k}{k!}.$$

This series converges by virtue of A′. Hence we have

$$\int_{0}^{a} \frac{u^k}{k!} | f^{(k+1)}(u) | du \leqq \sum_{k=0}^{\infty} | f^{(k)}(a) | \frac{a^k}{k!} \leqq M,$$

so that

$$\int_{0}^{\infty} \frac{u^k}{k!} | f^{(k+1)}(u) | du \leqq M,$$

and A is satisfied.

We show conversely that A implies A′. Supposing that A is satisfied, we see* that

$$\lim_{x\to\infty} x^k f^{(k)}(x) = 0 \qquad (k = 0, 1, 2, \cdots),$$

and hence that

$$f(a) - f(\infty) = (-1)^{k+1} \int_{a}^{\infty} \frac{(t-a)^k}{k!} f^{(k+1)}(t) dt \qquad (k = 0, 1, 2, \cdots).$$

Further we see by differentiation p times, or by applying the above formula to $f^{(p)}(a)$, that

* III, p. 139.

$$f^{(p)}(a) = (-1)^{k+p+1}\int_a^\infty \frac{(t-a)^{k-p}}{(k-p)!} f^{(k+1)}(t)dt \qquad (p \leqq k).$$

Consequently

$$\frac{a^p}{p!}\left| f^{(p)}(a) \right| \leqq \int_a^\infty \frac{(t-a)^{k-p}a^p}{(k-p)!p!}\left| f^{(k+1)}(t) \right| dt \qquad (p = 1, 2, \cdots, k),$$

and

$$\sum_{p=0}^{k} \frac{a^p}{p!}\left| f^{(p)}(a) \right| \leqq \int_a^\infty \frac{1}{k!}\sum_{p=0}^{k}\binom{k}{p}(t-a)^{k-p}a^p \left| f^{(k+1)}(t) \right| dt + \left| f(\infty) \right|$$

$$= \int_a^\infty \frac{t^k}{k!}\left| f^{(k+1)}(t) \right| dt + \left| f(\infty) \right| \leqq M + \left| f(\infty) \right|.$$

Hence

$$\sum_{p=0}^{\infty} \frac{a^p}{p!}\left| f^{(p)}(a) \right| \leqq M + \left| f(\infty) \right|,$$

so that the theorem is proved.

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