THE GENERALIZED THEOREM OF STOKES*

BY

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The generalized theorem of Stokes is an identity between an integral over an orientable r-manifold, M_r, and an integral over the boundary, B_{r-1}, of M_r, where M_r is on an n-space, R_n. Proofs† heretofore given have taken for R_n the space of a single coordinate system and have either assumed M_r to be analytic or imposed conditions not known to be fulfilled save by analytic manifolds. The present paper contains a proof for the case where R_n is an n-manifold of class one,‡ and (M_r, B_{r-1}) are made up, in a manner specified below, of continuously differentiable manifolds on R_n.

1. **Statement of the theorem.** An n-manifold, R_n, of class one is defined‡ by means of overlapping coordinate systems, called *allowable* systems, with regular transformations§ of class one between them. An r-cell on R_n will be called *regular*, if its closure can be parametrically defined by equations of the form

$$y_i = f_i(u_1, \cdots, u_r) \qquad (i = 1, \cdots, n), \tag{1.1}$$

where $(y) \equiv (y_1, \cdots, y_n)$ is an allowable system, the f's have continuous first partial derivatives, and the matrix $(\partial f_i/\partial u_j)$ is of rank r.

An *r-manifold*, M_r, on R_n will mean a point set whose closure, $\overline{M}_r = M_r + B_{r-1}$, is compact and connected and has the following properties: (1) Every point of M_r has an r-cell for one of its neighborhoods on $\overline{M}_r$; (2) Any point, P, on B_{r-1} has for one of its neighborhoods on $\overline{M}_r$ the closure of an r-cell with P on its boundary. We will call B_{r-1} the *boundary* of M_r. If M_r has no boundary, we refer to it as *closed*. Any point of $\overline{M}_r$ will be called *regular* if it has a regular cell (open or closed) for one of its neighborhoods on $\overline{M}_r$. The manifold M_r will be called *regular* if (1) every point of $\overline{M}_r$ is regular and

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† For examples, see Poincaré, *Sur les résidus des intégrales doubles*, Acta Mathematica, vol. 9 (1887), pp. 321–380; Goursat, *Sur les invariants intégraux*, Journal de Mathématiques, (6), vol. 4 (1908), pp. 331–367; R. Weitzenböck, *Invariantentheorie*, 1923, Chapter XIV, §§11, 12, 14; F. D. Murnaghan, *Vector analysis and the theory of relativity*, 1922, pp. 29–33.

‡ Veblen and Whitehead, *A set of axioms for differential geometry*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 551–561; also *The Foundations of Differential Geometry*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 29, 1932, Chapter 6.

§ Such transformations are characterized by the existence of continuous first partial derivatives and of a non-vanishing jacobian.

(2) B_{r-1} is a set of distinct closed $(r-1)$-manifolds each made up of regular i-manifolds $(i=0, \cdots, r-1)$ with the same sort of incidence relations as the i-cells of an $(r-1)$-dimensional complex.* Since any O-manifold is a point, we have here a recurrent definition of regular i-manifolds $(i=0, \cdots, n)$.

The formulation and proof of Stokes' theorem will be given for a regular manifold, M_r, on R_n. The method depends on the existence of a triangulation (σ) of $\overline{M}_r$ into regular cells. This aspect of the work is treated in two papers† by the writer. The first paper constructs the triangulation in the n-space of a single coordinate system. An extension of the construction to make it applicable on an n-manifold of class one is given in the second paper, which deals explicitly only with the case where M_r is closed.

Let $Y_{i_1 \cdots i_{r-1}}$ be an alternating tensor such that the partial derivatives $\partial(Y_{i_1 \cdots i_{r-1}})/\partial y_i$ are defined and continuous in a neighborhood of $\overline{M}_r$. Then‡

$$D_{i_1 \cdots i_r} = \left(\frac{1}{r!}\right) \delta_{i_1 \cdots i_r}^{\alpha_1 \cdots \alpha_r} \frac{\partial Y_{\alpha_2 \cdots \alpha_r}}{\partial y_{\alpha_1}} \tag{1.2}$$

is called the *Stokes tensor* of $Y_{i_1 \cdots i_{r-1}}$. Here, and throughout the paper, we apply the summation convention of tensor analysis only to Greek indices.

Taking (1.1) as the definition of a typical r-cell of (σ) and

$$y_i = g_i(v_1, \cdots, v_{r-1}) \qquad (i = 1, \cdots, n) \tag{1.3}$$

as the definition of a typical $(r-1)$-cell of (σ) on B_{r-1}, we can formulate Stokes' theorem as the identity

$$\begin{aligned} &\int_{M_r} \epsilon D_{\alpha_1 \cdots \alpha_r} \frac{\partial(y_{\alpha_1} \cdots y_{\alpha_r})}{\partial(u_1 \cdots u_r)} du_1 \cdots du_r \\ &\qquad = \pm r \int_{B_{r-1}} \epsilon' Y_{\alpha_1 \cdots \alpha_{r-1}} \frac{\partial(y_{\alpha_1} \cdots y_{\alpha_{r-1}})}{\partial(v_1 \cdots v_{r-1})} dv_1 \cdots dv_{r-1}, \end{aligned} \tag{1.4}$$

where (1) the integrals are to be evaluated over the separate cells of the triangulation and the results summed, and (2) the ϵ's are, on each cell, $+1$ or -1 according as the orientation of the cell by the parameters (u), or (v), agrees or disagrees with arbitrarily preassigned orientations, of M_r and B_{r-1}.

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* O. Veblen, *Analysis Situs*, American Mathematical Society Colloquium Publications, vol. 5, 1931, pp. 76, 77.

† *On the triangulation of regular loci*, Annals of Mathematics, vol. 35 (1934), pp. 579–587; *Triangulation of the manifold of class one*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 549–553.

‡ For the generalized Kronecker deltas, see O. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 24. For the tensor character of $D_{i_1 \cdots i_r}$, see the same reference or R. Weitzenböck, *Invariantentheorie*, Chapter XIV, §12.

The $\pm$ in identity (1.4) depends on the relative orientations of M_r and B_{r-1}. It is necessary to note, in addition, that the integrals in the above identity are absolute integral invariants* under transformations of parameters and under transformations between allowable coordinate systems. The factor r in identity (1.4) drops out if the summations are made, on both sides, over all combinations of the α's instead of being made as the α's run independently from one to n.

Extensions of Stokes' theorem to various loci made up of regular manifolds immediately suggest themselves.

2. **Reduction to a special case.** We note first that it is sufficient to establish identity (1.4) for a typical r-cell, σ_r, of (σ), for if the identity in this special case be applied to the sum of the r-cells of (σ), the contributions from any $(r-1)$-cell common to the boundaries of a pair of r-cells add up to zero. Since each coordinate system (y) can be interpreted as a homeomorphism between its domain and a region of euclidean n-space, we lose no generality in assuming that σ_r is a regular r-cell in the euclidean n-space of a rectangular cartesian coordinate system (y). We can, furthermore, select (y), for any σ_r, arbitrarily from all the allowable systems whose domains contain σ_r. This last possibility, together with the restrictions involved in constructing the triangulation (σ), permits us, in the course of our proof, to impose further conditions on σ_r [cf. §3(A) and §4(A) below].

3. **The generalized divergence theorem.** In the case $r=n$, the manifold M_r becomes a region of the space in which it is imbedded. Stokes' theorem, for this case, is equivalent [see §4(B) below] to the generalized divergence theorem, in which a vector field $[Y_1(y), \cdots, Y_r(y)]$ plays the role of the tensor $Y_{i_1 \cdots i_{r-1}}$ and the divergence

$$\text{div } Y = \frac{\partial Y_\alpha}{\partial y_\alpha} \tag{3.1}$$

replaces the Stokes tensor. If we regard the y's as rectangular cartesian coordinates in a euclidean space (see §2 above), we can express the divergence theorem for the region σ_r in the form

$$\int_{\sigma_r} (\text{div } Y) dV = \int_{\beta_{r-1}} Y_\alpha \gamma^\alpha d\beta, \tag{3.2}$$

where β_{r-1} is the boundary of σ_r and the γ's are the direction cosines of the outer normal to β_{r-1} at any point.

The identity (3.2) will first be established for the special case of the field

* For a proof, see R. Weitzenböck, loc. cit., §11.

$(0, \cdots, 0, Y_r)$ and a region ρ_r of the following sort. Let ρ_{r-1} be the projection of ρ_r onto $(y_1, \cdots, y_{r-1})$-space. Then the boundary, b_{r-1}, of ρ_r is made up of three parts (b^1, b^2, b^3) as follows: b^1 and b^2 are $(r-1)$-dimensional surfaces definable by equations

$$(3.3) \qquad b^j: \quad y_r = f^j(y_1, \cdots, y_{r-1}) \qquad [j = 1, 2; (y_1, \cdots, y_{r-1}, 0) \text{ on* } \bar{\rho}_{r-1}],$$

where $f^2 > f^1$ on ρ_{r-1} and both the f's have continuous first partial derivatives on $\bar{\rho}_{r-1}$; b^3 is made up of the closed line-segments parallel to the y_r-axis which join the boundaries of b^1 and b^2. Some or all of these segments may be of length zero.

In the case of ρ_r, the divergence theorem for the field $(0, \cdots, 0, Y_r)$ reduces to

$$(3.4) \qquad \int_{\rho_r} \frac{\partial Y_r}{\partial y_r} dV = \int_{b_{r-1}} Y_r \gamma^r d\beta$$

an identity which follows, as in the proof† for three dimensions, from the equivalence of multiple and iterated integrals.

(A) We now require (see §2 above) that, for each value $j = 1, \cdots, r$, σ_r can be regarded as the sum of a finite number of distinct parts, each satisfying the above description of the region ρ_r, read with y_j in place of y_r.

This condition holds, for example, if (1) σ_r is a sufficiently close approximation to the r-simplex determined by its vertices and (2) the (y)-axes are suitably oriented. [Compare the "normal regions" of Kellogg's treatment.]†

In equation (3.4), we can now replace r by the general subscript j and (ρ_r, b_{r-1}) by (σ_r, β_{r-1}). Summing the resulting identities, we obtain the identity (3.2).

4. **The generalized theorem of Stokes.** Let $\gamma^{i_1 \cdots i_r}$ denote the direction cosines‡ of the tangent r-plane to σ_r (§2) at any point.

(A) We impose on σ_r the conditions (see §2) (1) that, for some orientation of the (y)-axes and for every set $(i_1, \cdots, i_r)$,

$$(4.1) \qquad \gamma^{i_1 \cdots i_r} \neq 0 \quad \text{on} \quad \bar{\sigma}_r$$

and (2) that the projection of σ_r on the r-space of $(y_{i_1}, \cdots, y_{i_r})$ shall satisfy

* The symbol for a point set, modified by a bar, denotes the closure of the set.

† O. D. Kellogg, *Foundations of Potential Theory*, 1929, Chapter 4. The writer's methods are similar to those used by Kellogg in 3-space.

‡ See the writer's paper, *The direction cosines of a p-space in euclidean n-space*, American Mathematical Monthly, vol. 39 (1932), pp. 518–523. We extend the definitions there given by the convention that the direction cosines be alternating in their indices. This paper is referred to hereafter as Dir. Cos.

the restrictions [§3(A)] imposed on σ_r in the proof of the generalized divergence theorem.

We will obtain Stokes' theorem by applying the divergence theorem to each projection and summing the resulting identities.

Let $\beta^{i_1\cdots i_{r-1}}$ be direction cosines of the tangent $(r-1)$-plane to the oriented boundary, β_{r-1}, of σ_r at any point. Then, for the euclidean case mentioned in §2, Stokes' theorem is equivalent (see Dir. Cos.) to the identity

$$(4.2)\qquad \int_{\sigma_r} D_{\alpha_1\cdots\alpha_r}\gamma^{\alpha_1\cdots\alpha_r}d\sigma = \pm\, r\int_{\beta_{r-1}} Y_{\alpha_1\cdots\alpha_{r-1}}\beta^{\alpha_1\cdots\alpha_{r-1}}d\beta.$$

(B) *The identity* (4.2), *read for $r=n$, suggests the following form for the divergence theorem, where we are using the vector field of* §3, *and where the β's are direction cosines of the tangent $(r-1)$-plane to β_{r-1}:*

$$(4.3)\qquad \int_{\sigma_r}\left(\sum_{i=1}^{r}(-1)^i\frac{\partial Y_i}{\partial y_i}\right)dV = \pm\int_{\beta_{r-1}}\left(\sum_{i=1}^{r}Y_i\beta^{1\cdots i-1,i+1\cdots r}\right)d\beta.$$

To make the work of §3 apply to identity (4.3), we need only show that

$$(4.4)\qquad \gamma^i = \pm\,(-1)^i\beta^{1\cdots i-1,i+1\cdots r}\qquad (i=1,\cdots,r),$$

where the $\pm$ depends on the orientation of β_{r-1}. Since the numerical equality of γ^i and $\beta^{1\cdots i-1,i+1\cdots r}$ follows easily from geometric considerations (see Dir. Cos.), we have only to show that the signs in (4.4) are correct. Using any point P on β_{r-1} as origin, let $(u_1,\cdots,u_r)$ be a coordinate system where (1) the u_1-axis is the outer normal to β_{r-1} and (2) the $(u_2,\cdots,u_r)$-axes are on the tangent $(r-1)$-plane, L_{r-1}, to β_{r-1} and orient L_{r-1} positively. Then the agreement or disagreement in orientation between the (u)-system and the (y)-system depends on the orientation of β_{r-1}. Let

$$(4.5)\qquad y_i = a_{i\alpha}u_\alpha\qquad (\alpha=1,\cdots,r)$$

be the transformation between the y's and the u's. If A_{i1} denote the minor of a_{i1} in the determinant $|a_{ij}|$, then, since the u_1-axis is perpendicular to all the other u-axes, a value k exists such that

$$(4.6)\qquad a_{i1} = (-1)^i k A_{i1},$$

where the sign of k depends on the orientation of β_{r-1}. Since the direction cosines $(\gamma^i, \beta^{1\cdots i-1,i+1\cdots r})$ have the signs of (a_{i1}, A_{i1}) respectively (see Dir. Cos.), the signs in equations (4.4) are correct and our demonstration is complete.

Now let $(j_1,\cdots,j_r)$ be a fixed set of r distinct numbers from the set $(1,\cdots,n)$, and let $(m_1,\cdots,m_{n-r})$ be the complement of $(j_1,\cdots,j_r)$ with respect to $(1, \cdots, n)$, where the m's are arranged in order of increasing magnitude. Let

$$y_{m_p} = f_{m_p}(y_{j_1}, \cdots, y_{j_r}) \qquad (p = 1, \cdots, n-r) \tag{4.7}$$

be defining equations of σ_r, where $(y_{j_1}, \cdots, y_{j_r})$ is on the projection, σ_r', of σ_r on the $\gamma^{j_1\cdots j_r}$-plane. Applying identity (4.3) in $(y_{j_1}, \cdots, y_{j_r})$-space to the vector field*

$$\begin{aligned} Z_{j_k}(y_{j_1}, \cdots, y_{j_r}) \equiv Y_{j_1\cdots j_{k-1}j_{k+1}\cdots j_r}[y_{j_1}, \cdots, y_{j_r}, f_{m_1}(y_{j_1}, \cdots, y_{j_r}),\\ \cdots, f_{m_{n-r}}(y_{j_1}, \cdots, y_{j_r})] \end{aligned} \tag{4.8}$$

we find

$$\int_{\sigma_r'} \sum_{k=1}^{r} (-1)^k \frac{\partial Z_{j_k}}{\partial y_{j_k}} d\sigma' = \pm \int_{\beta_{r-1}'} \sum_{k=1}^{r} Z_{j_k}\bar{\beta}^{j_1\cdots j_{k-1}j_{k+1}\cdots j_r} d\beta', \tag{4.9}$$

where β_{r-1}' is the boundary of σ_r' and hence the projection of β_{r-1}, and where the $\bar{\beta}$'s are direction cosines of the tangent $(r-1)$-plane to β_{r-1}', the orientation being determined by that of β_{r-1}. This identity is now to be interpreted in terms of the Y's, γ's, β's and integrals over σ_r and β_{r-1}.

By equation (4.8)

$$\frac{\partial Z_{j_k}}{\partial y_{j_k}} = \frac{\partial Y_{j_1\cdots j_{k-1}j_{k+1}\cdots j_r}}{\partial y_{j_k}} + \sum_{p=1}^{n-r}\left(\frac{\partial Y_{j_1\cdots j_{k-1}j_{k+1}\cdots j_r}}{\partial y_{m_p}}\right)\left(\frac{\partial f_{m_p}}{\partial y_{j_k}}\right) \qquad (k = 1, \cdots, r). \tag{4.10}$$

Hence

$$\sum_{k=1}^{r} (-1)^k \frac{\partial Z_{j_k}}{\partial y_{j_k}} = D_{j_1\cdots j_r} + \sum_{k=1}^{r}\sum_{p=1}^{n-r} (-1)^k \left(\frac{\partial Y_{j_1\cdots j_{k-1}j_{k+1}\cdots j_r}}{\partial y_{m_p}}\right)\left(\frac{\partial f_{m_p}}{\partial y_{j_k}}\right). \tag{4.11}$$

From equations (4.7) we obtain the following matrix, with rows permuted, for the positively oriented tangent r-plane to σ_r

$$(\operatorname{sgn} \gamma^{j_1\cdots j_r}) \left|\begin{matrix} 1 & 0 \cdots & 0 \\ 0 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & 1 \\ \dfrac{\partial f_{m_1}}{\partial y_{j_1}} & \cdots & \dfrac{\partial f_{m_1}}{\partial y_{j_r}} \\ \vdots & & \vdots \\ \dfrac{\partial f_{m_{n-r}}}{\partial y_{j_1}} & \cdots & \dfrac{\partial f_{m_{n-r}}}{\partial y_{j_r}} \end{matrix}\right| \begin{matrix} \\ \\ \text{rows } (j_1, \cdots, j_r) \\ \\ \\ \text{rows } (m_1, \cdots, m_{n-r}) \\ \\ \end{matrix}$$

* The arguments are permuted, for convenience, on the right side of equation (4.8).

Hence (see Dir. Cos.), if $d_{i_1\cdots i_r}$ denote the determinant of rows $(i_1, \cdots, i_r)$ in this matrix and $\Delta = (\sum_{i_1 < \cdots < i_r} d_{i_1\cdots i_r}^2)^{1/2}$, then

$$(4.12)\qquad \begin{cases} \gamma^{j_1\cdots j_r} = (\operatorname{sgn} \gamma^{j_1\cdots j_r}) \dfrac{1}{\Delta} \\ \gamma^{j_1\cdots j_{k-1} m_p j_{k+1}\cdots j_r} = (\operatorname{sgn} \gamma^{j_1\cdots j_r})\left(\dfrac{\partial f_{m_p}}{\partial y_{j_k}}\right)\left(\dfrac{1}{\Delta}\right) = \left(\dfrac{\partial f_{m_p}}{\partial y_{j_k}}\right)\gamma^{j_1\cdots j_r}. \end{cases}$$

Therefore, substituting in equations (4.11),

$$(4.13)\qquad \begin{aligned} \sum_{k=1}^{r} (-1)^k \frac{\partial Z_{j_k}}{\partial y_{j_k}} &= D_{j_1\cdots j_r} \\ &+ \sum_{k=1}^{r}\sum_{p=1}^{n-r} (-1)^k \left(\frac{\partial Y_{j_1\cdots j_{k-1} j_{k+1}\cdots j_r}}{\partial y_{m_p}}\right)\left(\frac{\gamma^{j_1\cdots j_{k-1} m_p j_{k+1}\cdots j_r}}{\gamma^{j_1\cdots j_r}}\right). \end{aligned}$$

By the definitions of direction cosines,

$$(4.14)\qquad \begin{cases} d\sigma' = \pm\, \gamma^{j_1\cdots j_r} d\sigma, \\ \bar\beta^{j_1\cdots j_{k-1} j_{k+1}\cdots j_r} d\beta' = \beta^{j_1\cdots j_{k-1} j_{k+1}\cdots j_r} d\beta. \end{cases}$$

Substituting from equations (4.13) and (4.14) into (4.9), we obtain

$$(4.15)\qquad \begin{aligned} &\int_{\sigma_r} D_{j_1\cdots j_r}\gamma^{j_1\cdots j_r} d\sigma \\ &\quad + \int_{\sigma_r} \sum_{k=1}^{r}\sum_{p=1}^{n-r} (-1)^k \left(\frac{\partial Y_{j_1\cdots j_{k-1} j_{k+1}\cdots j_r}}{\partial y_{m_p}}\right)\gamma^{j_1\cdots j_{k-1} m_p j_{k+1}\cdots j_r} d\sigma \\ &= \pm \int_{\beta_{r-1}} \sum_{k=1}^{r} Y_{j_1\cdots j_{k-1} j_{k+1}\cdots j_r}\beta^{j_1\cdots j_{k-1} j_{k+1}\cdots j_r} d\beta, \end{aligned}$$

where the $\pm$ is determined by the relative orientations of σ_r and β_{r-1} and is therefore the same for all sets $(j_1, \cdots, j_r)$. When we sum all the ${}_nP_r$ such identities as (4.15), the integrand of the second integral on the left becomes

$$(4.16)\qquad \sum_{j_1\cdots j_r}\sum_{k=1}^{r}\sum_{p=1}^{n-r} (-1)^k \left(\frac{\partial Y_{j_1\cdots j_{k-1} j_{k+1}\cdots j_r}}{\partial y_{m_p}}\right)\gamma^{j_1\cdots j_{k-1} m_p j_{k+1}\cdots j_r}.$$

Now let $(s_1, \cdots, s_r)$ be a fixed subset of $(1, \cdots, n)$. In the summation (4.16), the superscripts of γ become $(s_1, \cdots, s_r)$ whenever

$$(4.17)\qquad \begin{cases} \text{(a)} & j_i = s_i \qquad (i \neq k), \\ \text{(b)} & j_k \notin (s_1, \cdots, s_r), \\ \text{(c)} & m_p = s_k. \end{cases}$$

For each value of k, there are $(n-r)$ sets $(j_1, \cdots, j_r)$ satisfying (4.17a) and (4.17b). Corresponding to each such set of j's and value of k, there is just one value of p satisfying (4.17c). Hence the triple summation (4.16) reduces to

$$(4.18)\quad (n-r)\sum_{s_1\cdots s_r}\left[\sum_{k=1}^{r}(-1)^k\frac{\partial Y_{s_1\cdots s_{k-1}s_{k+1}\cdots s_r}}{\partial y_{s_k}}\right]\gamma^{s_1\cdots s_r} = (n-r)D_{\alpha_1\cdots\alpha_r}\gamma^{\alpha_1\cdots\alpha_r},$$

and the left side of identity (4.15) becomes

$$(4.19)\quad (n-r+1)\int_{\sigma_r} D_{\alpha_1\cdots\alpha_r}\gamma^{\alpha_1\cdots\alpha_r}d\sigma.$$

The term $Y_{t_1\cdots t_{r-1}}\beta^{t_1\cdots t_{r-1}}$, where the t's are a fixed subset of $(1, \cdots, n)$, appears in the identity (4.15) if and only if $(t_1, \cdots, t_{r-1})$ can be obtained from $(j_1, \cdots, j_r)$ by deleting one of the j's. There are r possible positions in the set $(j_1, \cdots, j_r)$, for the j which is to be deleted, and $(n-r+1)$ possible values. Hence $Y_{t_1\cdots t_{r-1}}\beta^{t_1\cdots t_{r-1}}$ appears in $(n-r+1)r$ identities such as (4.15). When we sum all the identities such as (4.15), we therefore find

$$(4.20)\quad (n-r+1)\int_{\sigma_r} D_{\alpha_1\cdots\alpha_r}\gamma^{\alpha_1\cdots\alpha_r}d\sigma = \pm(n-r+1)r\int_{\beta_{r-1}} Y_{\alpha_1\cdots\alpha_{r-1}}\beta^{\alpha_1\cdots\alpha_{r-1}}d\beta$$

which is equivalent to identity (4.2). As remarked in §2, the establishment of this identity completes our proof of the generalized theorem of Stokes.

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