REMARKS ON THE PRECEDING PAPER OF
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1. In the preceding paper Clarkson has introduced the interesting concept of a Banach space $X$ with a uniformly convex norm and has shown that for such spaces the following theorem holds.

**Theorem.** If the additive function $F(R)$ which is defined for elementary figures $R$ contained in a fixed figure $R_0$ in Euclidean space of $n$ dimensions and has its values in the Banach space $X$, is of bounded variation,† then it has a derivative $F'(P)$ for almost all points $P$ in $R_0$. $F'(P)$ is summable‡ on $R_0$ and if $F(R)$ is absolutely continuous§ then for every elementary figure $R$ in $R_0$ we have

$$F(R) = \int_R F'(P) dP.$$ 

In this paper it is shown that the theorem holds for all Banach spaces $X$ with a base¶ $\{\phi_i\}$ which satisfies the following postulate:

(A) If $a_1, a_2, \cdots$ is any sequence of real numbers such that $\sup_n ||\sum_{i=1}^n a_i \phi_i|| < \infty$, then the series $\sum_{i=1}^\infty a_i \phi_i$ converges.

It is obvious that $l_p$ ($p \geq 1$) or any Hilbert space satisfies (A).

In §3 it is shown that $L_p$ ($p > 1$) does likewise. The method of proof is entirely different from that of Clarkson. In §4 it is shown that if $X$ is any Banach space having the property that every function on a linear interval to $X$, which satisfies a Lipschitz condition, is differentiable almost everywhere, then also every function of bounded variation from the linear interval to $X$ is differentiable almost everywhere and its derivative is summable.

2. **Proof of the theorem.** It should first be noted that it is no loss of generality to assume besides (A) the property

$$\left( B \right) \left\| \sum_{i=1}^n a_i \phi_i \right\| \leq \left\| \sum_{i=1}^{n+1} a_i \phi_i \right\| \text{ for any constants } a_i, i = 1, 2, \cdots .$$

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† I.e., $\sum_{i=1}^r ||F(R_i)||$ is bounded for all finite sets of non-overlapping elementary figures $R_i, \cdots, R_k$ contained in $R_0$.
‡ Here, as well as elsewhere in this note, the concept of summability is that of Bochner or of Dunford. The two notions are equivalent.
§ I.e., $\lim_{|R| \rightarrow 0} F(R) = 0$.
¶ See Banach, Théorie des Opérations Linéaires, Warsaw, 1932, p. 110.
This is made evident by the following two considerations. If a Banach space $Y$ is isomorphic* to $X$ then the theorem holds for $X$ if and only if it holds for $Y$. Now associated with $X$ is a space $Y$ composed of all sequences $y = \{y_i\}$ such that $\sum_{i=1}^{\infty} y_i \phi_i$ converges. If the metric in $Y$ is defined by $\|y\| = \sup_n \|\sum_{i=1}^{n} y_i \phi_i\|$ then $Y$ is a Banach space and isomorphic to $X$. By taking $y_k = \{\eta^k_i\}$ where $\eta^k_i = 0$ if $k \neq i$, $\eta^i_i = 1$, it is easily seen that $y_i$ forms a base for $Y$ which has the properties (A) and (B). Thus it will be assumed in what follows that $X$ has the properties (A) and (B). Now

$$F(R) = \sum_{i=1}^{\infty} a_i(R)\phi_i,$$

where the coefficients $a_i(R)$ are given by means of the limited linear functionals† $T_i$ defined on $X$ according to the equation $a_i(R) = T_i F(R)$. Thus it is immediate that the functions $a_i(R)$ are additive, real functions of bounded variation with summable derivatives $a_i'(P)$. The functions $F_n(R) = \sum_{i=1}^{n} a_i(R)\phi_i$ thus have derivatives $F_n'(P) = \sum_{i=1}^{n} a_i'(P)\phi_i$ summable on $R_0$. Let $V(F_n, R)$ be the total variation of $F_n$ on $R$, then the positive real function $S_n(R) = V(F_n, R)$ is additive and of bounded variation on $R_0$ with $S_n(R) \geq \|F_n(R)\|$. Hence $S_n'(P) \leq \|F_n'(P)\|$ and thus

$$V(F_n, R) = S_n(R) \geq \int_{R} S_n'(P) dP \geq \int_{R} \|F_n'(P)\| dP. \tag{1}$$

Now from postulate (B) and (1)

$$\|F_n'(P)\| \leq \|F_{n+1}'(P)\|, \quad \int_{R} \|F_n'(P)\| dP \leq V(F_n, R_0) \leq V(F, R_0). \tag{2}$$

If we let $b(P) = \lim_n \|F_n'(P)\|$, the inequality (2) shows that $b(P)$ is summable and hence finite almost everywhere. Postulate (A) then insures the convergence of the series

$$G(P) = \sum_{n=1}^{\infty} a_n'(P)\phi_n$$

for almost all $P$ in $R_0$. Since $\|G(P)\| = b(P)$ and $G(P)$ is measurable it is summable on $R_0$. It will now be shown that $G(P)$ is the derivative of $F(R)$. To do this we need the following lemmas which, in the case of real-valued functions, are well known.

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† Banach, loc. cit., p. 111.
‡ Banach, loc. cit., p. 111.
LEMMA 1. Let $G(R)$ be an additive function of bounded variation defined for elementary figures $R$ contained in a fixed figure $R_0$ and with values in an arbitrary Banach space. Then $G(R)$ has a derivative equal to zero at almost all points of $R_0$ if and only if for every $\epsilon > 0$ there is an open set $E$ in $R_0$ with measure less than $\epsilon$ such that $V(G, E) = V(G, R_0)$. 

By the variation $V(G, D)$ of $G$ on an open set $D$ in $R_0$ is meant the upper bound of all finite sums $\sum_{i=1}^{k} ||G(R_i)||$ where $R_1, R_2, \ldots, R_k$ are non-overlapping elementary figures in $D$. Now suppose $|E| < \epsilon$ and $V(G, E) = V(G, R_0)$. Define $S(C) = V(G, C)$ where $C$ is either an open set or an elementary figure and let $R_n$ be a sequence of elementary figures such that 

$$R_n \subset R_{n+1} \subset E, \quad S(R_n) \to S(E).$$

Let $R'_n = R_0 - R_n$, $E' = R_0 - E$, so that 

$$S(R_0) \geq S(R_n) + S(R'_n) \geq S(R_n) + \int_{R_n} S'(P)dP.$$ 

Thus since $R'_n \supset E'$ we have 

$$S(R_0) \geq S(R_n) + \int_{E'} S'(P)dP$$

or 

$$\int_{E'} S'(P)dP \leq S(R_0) - S(R_n) \to 0.$$ 

Whence it follows that $S'(P) = 0$ almost everywhere on $E'$, and since $|R_0 - E'| < \epsilon$ we conclude that $S'(P) = 0$ almost everywhere on $R_0$ which implies $G'(P) = 0$ almost everywhere on $R_0$.

To prove the converse let $\eta > 0$ and let $R_1, R_2, \ldots, R_k$ be non-overlapping elementary figures contained in $R_0$ with 

$$\left| \sum_{i=1}^{k} R_i \right| > |R_0| - \frac{\epsilon}{2}, \quad \sum_{i=1}^{k} ||G(R_i)|| \geq V(G, R_0) - \eta.$$ 

Now define the set $E_n$ as follows: a point $P$ is in $E_n$ if for every cube $I$ containing $P$ with $|I| \leq 1/n$ it follows that $||G(I)||/|I| \leq \eta$. Thus the set $\lim E_n$ contains all points at which $G'(P) = 0$, i.e., almost all points in $R_0$. Hence $\lim E_n \sum_{i=1}^{k} R_i$ contains almost all points of $\sum_{i=1}^{k} R_i$. Consequently there is an $n_0$ and a closed set $C$ contained in $E_{n_0} \sum_{i=1}^{k} R_i$ for which $|C| > |R_0| - \epsilon$. Thus $C \subset \sum_{i=1}^{k} R_i$ and $||G(I)||/|I| \leq \eta$ for any cube $I$ containing a point of $C$ and having $|I| \leq 1/n_0$. Let $d$ be the distance from $C$ to the boundary of $\sum_{i=1}^{k} R_i$ and $I_1, I_2, \ldots, I_t$ be non-overlapping cubes satisfying the conditions
\[
\sum_{i=1}^{l} I_i \supset R_0; \quad |I_i| \leq 1/n_0, \quad \text{diameter } I_i < d, \quad (i = 1, 2, \ldots, l).
\]

Define \( \delta_i (j = 1, 2, \ldots, l) \) to be 0 or 1 according as \( I_i C \) is null or non-null and let \( \delta'_i = 1 - \delta_i \). Now \( D = D(\epsilon, \eta) \), the complement of \( C \) with respect to \( R_0^\delta \), is open and since \( R_i I_j \) is null or a cube if \( \delta_i = 1 \), it follows that

\[
V(G, R_0^\delta) - \eta \leq \sum_{i=1}^{k} \|G(R_i)\| \leq \sum_{i=1}^{k} \sum_{j=1}^{l} \|G(R_i I_j)\| \leq \sum_{i=1}^{k} \sum_{j=1}^{l} \delta_i \|G(R_i I_j)\| + \sum_{i=1}^{k} \sum_{j=1}^{l} \delta'_i \|G(R_i I_j)\| \leq \sum_{i=1}^{k} \sum_{j=1}^{l} \eta \left| R_i I_j \right| + V(G, E) \leq \eta \left| R_0 \right| + V(G, D).
\]

By defining \( E = \sum_{n=1}^{\infty} D(\epsilon/2^n, 1/n) \) the conclusion is immediate.

**Lemma 2.** If \( F(R) \) is an additive function of bounded variation defined for elementary figures \( R \) contained in a fixed figure \( R_0 \) with values in a Banach space satisfying postulates (A) and (B) then there exist additive functions \( \alpha(R), \beta(R) \) such that

\[
F(R) = \alpha(R) + \beta(R),
\]

\( \alpha(R) \) is an indefinite integral and \( \beta(R) \) has a derivative equal to zero almost everywhere on \( R_0 \).

Let \( F(R) = \sum_{i=1}^{\infty} a_i(R) \phi_i \) and define

\[
\alpha(R) = \int_R \left( \sum_{i=1}^{\infty} a_i(P) \phi_i \right) dP, \quad \beta(R) = F(R) - \alpha(R).
\]

If we write \( \beta(R) = \sum_{i=1}^{\infty} b_i(R) \phi_i \) and denote \( \sum_{i=1}^{n} b_i(R) \phi_i \) by \( \beta_n(R) \) then \( \beta'_n(P) = 0 \) almost everywhere on \( R_0 \). Take \( \epsilon > 0 \) and let (Lemma 1) \( E_n \) be an open set with \( |E_n| < \epsilon/2^n \) such that \( V(\beta_n, E_n) = V(\beta_n, R_0^\delta) \) \((n = 1, 2, \ldots)\). Let \( E = E_1 + E_2 + E_3 + \cdots \). Now the \( |E| < \epsilon \) and \( V(\beta_n, E) = V(\beta_n, R_0^\delta) \). Since \( V(\beta_n, R_0^\delta) \rightarrow V(\beta, R_0^\delta) \) by semi-continuity,* we have

\[
V(\beta, E) \geq V(\beta_n, E) \rightarrow V(\beta, R_0^\delta).
\]

Hence \( V(\beta, E) = V(\beta, R_0^\delta) \) so that by Lemma 1, \( \beta'(P) = 0 \) almost everywhere on \( R_0 \). This completes the proof of Lemma 2.

Returning to the argument of the theorem itself we see immediately that

* It is well known that the relation \( \beta_n(R) \rightarrow \beta(R) \) for \( R \subset R_0 \) implies \( \lim \inf_n V(\beta_n, R_0) \geq V(\beta, R_0) \). It is likewise readily seen that \( \lim \inf_n V(\beta_n, R_0^\delta) \geq V(\beta, R_0^\delta) \).
$F(R)$ has a derivative $F'(P) = G(P)$ almost everywhere on $R_0$. This follows from the fact that $\alpha(R)$, being an indefinite integral, is differentiable with $\alpha'(P) = G(P)$ almost everywhere on $R_0$. Now if $F(R)$ is absolutely continuous so are the functions $a_i(R)$ and hence

$$\int_R F'(P) dP = \sum_{i=1}^{\infty} \phi_i T_i \int_R F'(P) dP = \sum_{i=1}^{\infty} \phi_i \int_R T_i F'(P) dP$$

$$= \sum_{i=1}^{\infty} \phi_i \int_R a_i'(P) dP = \sum_{i=1}^{\infty} a_i(R) \cdot \phi_i = F(R)$$

for every elementary figure $R$ in $R_0$.

3. $L_p(p > 1)$ has the property (A). Let $\{\phi_i\}$ be the orthonormal sequence of Haar. Schauder† has shown that $\{\phi_i\}$ is a base for $L_p (p \geq 1)$. The sequence $\{\phi_i\}$ also determines the sequence $\{T_i\}$ of linear functionals on $L_p$ by the formula

$$T_i \psi = \int_0^1 \phi_i(t) \psi(t) dt.$$ 

If $p > 1$ this sequence forms a fundamental set in $\overline{L_p}$ (the space conjugate to $L_p$) in the sense that every point in $\overline{L_p}$ can be approached by finite linear combinations of the elements of the sequence $\{T_i\}$. Now suppose $a_1, a_2, \ldots$ is an arbitrary sequence of real numbers such that $\|a_n\|$ is bounded, where $x_n = \sum_{i=1}^{n} a_i \phi_i$. We have

$$T_i x_n = a_i,$$ (i ≤ n),

and so $x_n$ is a weakly convergent sequence in $L_p$. Since $L_p$ is weakly complete there is a point $x = \sum_{i=1}^{\infty} T_i x \phi_i$ in $L_p$ such that $T x_n \rightarrow T x$ for every $T$ in $\overline{L_p}$. Now from (5) $a_i = \lim_n T_i x_n = T_i x$ and so $x = \sum_{i=1}^{\infty} a_i \phi_i$, which was to be proved.

4. Differentiability of functions of bounded variation. It is the purpose of this paragraph to prove the final assertion in the introduction. Let $f(t)$ be of bounded variation on $(0, 1)$ to $X$, and let $E$ be the set of functional values of the strictly monotone real function

$$\sigma(t) = t + V(f; 0, t),$$ (0 ≤ t ≤ 1).

The symbol $V(f; a, b)$ stands for the total variation of $f$ on $a \leq t \leq b$. Let $\tau(s)$ be the interchange of $T_i$ and $\int_R$ see Garrett Birkhoff, these Transactions, vol. 38 (1935), p. 371.


§ Banach, loc. cit., p. 133, Theorem 1. This theorem needs to be modified so as to apply to weakly convergent sequences rather than sequences weakly convergent to a point.
on $E$ to $(0, 1)$ be the inverse of $\sigma(t)$ and let $g(s)$ be defined on $E$ by the equation $g(s) = f(\tau(s))$. Now for any two points $s < s'$ in $E$,

$$\|g(s') - g(s)\| \leq V(f; \tau(s), \tau(s'))$$

(6)

$$\leq \tau(s') - \tau(s) + V(f; \tau(s), \tau(s'))$$

By first extending the domain of definition of $g(s)$ to $\bar{E}$ (the closure of $E$) in the natural way and then in a linear fashion on each of the intervals which make up the complement of $E$ with respect to the interval $0 \leq s \leq 1 + V(f; 0, 1)$ it is seen that the extended function satisfies the same Lipschitz condition (6) on the whole of $0 \leq s \leq 1 + V(f; 0, 1)$. Thus $g(s)$ has a derivative almost everywhere on $(0, 1 + V(f; 0, 1))$ and hence almost everywhere on $E$ with respect to $E$. Now $\tau(s)$ satisfies the Lipschitz condition $|\tau(s') - \tau(s)| \leq |s' - s|$, and hence if we let $E^*$ be those points of $E$ at which $g$ has a derivative with respect to $E$ we have $m[\tau(E - E^*)] = 0$. That is, for almost all $t$ in $(0, 1)$ the point $\sigma(t)$ is in $E^*$. Thus for almost all $t$ in $(0, 1)$ we have

$$\lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = \lim_{h \to 0} \frac{g(\sigma(t + h)) - g(\sigma(t))}{\sigma(t + h) - \sigma(t)} \lim_{h \to 0} \frac{\sigma(t + h) - \sigma(t)}{h}$$

$$= g'(\sigma(t)) \cdot \sigma'(t),$$

so that $f(t)$ has a derivative at almost all points of $(0, 1)$, and since this derivative is the product of a bounded measurable function and a real summable function it follows that $f'(t)$ is summable.

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