ON PHRAGMÉN-LINDELÖF'S PRINCIPLE*

BY

LARS V. AHLFORS

In the first part of this paper we give a proof of Phragmén-Lindelöf's now classical principle,† which is simpler and yields more detailed information than any of the proofs hitherto known. Our procedure consists in proving a theorem in finite terms, similar to Hadamard's three circles theorem, from which the asymptotic statement is shown to follow by a very simple and transparent reasoning. A certain symmetry in the result is obtained by allowing the functions considered to have two possible singularities, one at 0 and one at ∞. The ultimate theorem is the sharpest possible and contains all previous results, including those of the brothers Nevanlinna.‡

In Part II we generalize Phragmén-Lindelöf's principle to harmonic functions of n variables. The methods of Part I are seen to carry over without any difficulties. The result is particularly interesting in so far as the symmetry of the two-dimensional case is not maintained, the extremal functions corresponding to the two singularities being now essentially different.

PART I

1. Let $f(s)=f(x+iy)$ be analytic in the strip T: $-\pi/2 \leqq y \leqq \pi/2$, and suppose further that $|f(x\pm i\pi/2| \leqq 1$ for every x. Under these fundamental assumptions two alternatives are possible:

(a) The inequality $|f(s)| \leqq 1$ holds for all s in T. (The situation is then governed by theorems such as the lemmas of Schwarz and Julia, etc., and shall not concern us further.)

(b) The set of points s in T with $|f(s)| > 1$ is not void.

In the sequel we shall always suppose that the second alternative takes place.

Some of our results are stated for a finite interval (x_1, x_2). In that case it will be noted that the arguments and results are not based on any assump-

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† E. Phragmén and E. Lindelöf, *Sur une extension d'un principe classique de l'analyse et sur quelque propriétés des fonctions monogènes dans le voisinage d'un point singulier*, Acta Mathematica, vol. 31 (1908), pp. 381–406.

‡ F. and R. Nevanlinna, *Über die Eigenschaften analytischer Funktionen in der Umgebung einer singularen Stelle oder Linie*, Acta Societatis Scientiarum Fennicae, vol. 50 (1922), No. 5.

R. Nevanlinna, *Über die Eigenschaften meromorpher Funktionen in einem Winkelraum*, Acta Societatis Scientiarum Fennicae, vol. 50 (1922), No. 12.

tions regarding the function outside of the interval with which we are concerned. For the sake of convenience we shall not further repeat this remark.

Consider now the set of points for which $|f(s)|>1$. Obviously this set is composed by certain open regions Ω, none of which is finite. We choose arbitrarily a region Ω and propose to study the behavior of $f(s)$ within this region. As a preliminary result we notice that $|f(s)|=1$ on the boundary of Ω, and that this boundary is made up of analytic arcs, some of which may be segments of the lines $y=\pm\pi/2$. We also remark that the boundary may form an angle, namely, at points where $f'(s)=0$.

Denote by Δ_t the set in which the line $x=t$ intersects the region Ω. We introduce the notation

$$m(x) = \int_{\Delta_x} \log|f(x+iy)| \cos y\, dy \tag{1}$$

and complete the definition by setting $m(x)=0$ if Δ_x is void. The function $m(x)$ is evidently non-negative and continuous.*

Differentiating (1) at a point x, where the boundary of Ω has no vertical tangent and forms no angle, we first obtain

$$m'(x) = \int_{\Delta_x} \frac{\partial}{\partial x} \log|f(x+iy)| \cos y\, dy,$$

since $\log|f|=0$ at the endpoints of the segments forming Δ_x. One more differentiation gives

$$m''(x) = \int_{\Delta_x} \frac{\partial^2}{\partial x^2} \log|f| \cos y\, dy + \frac{\partial}{\partial x}\log|f| \cos y \frac{dy}{dx}\Big|_1^2, \tag{2}$$

where the sign $|_1^2$ means that we have to form the sum of the values of the expression under the sign for all upper endpoints of the intervals Δ_x and subtract the corresponding sum for the lower endpoints.

Observing that $\Delta \log|f|=0$ the integral may be transformed and integrated by parts as follows:

$$\begin{aligned}\int_{\Delta_x} \frac{\partial^2}{\partial x^2}\log|f|\cos y\, dy &= -\int_{\Delta_x}\frac{\partial^2}{\partial y^2}\log|f|\cos y\, dy \\ &= -\int_{\Delta_x}\frac{\partial}{\partial y}\log|f|\sin y\, dy - \frac{\partial}{\partial y}\log|f|\cos y\Big|_1^2 \\ &= \int_{\Delta_x}\log|f|\cos y\, dy - \frac{\partial}{\partial y}\log|f|\cos y\Big|_1^2.\end{aligned}$$

* The idea of introducing a quantity like $m(x)$ is due to the brothers Nevanlinna (loc. cit.) who consider $\int_{-\pi/2}^{\pi/2}\overset{+}{\log}|f|\cos y\, dy$. This integral evidently represents the sum of the $m(x)$ for all regions Ω and is consequently not less than every single $m(x)$.

Substituting in (2) we finally obtain the formula

$$m''(x) = m(x) + \left(\frac{\partial}{\partial x}\log|f|\frac{dy}{dx} - \frac{\partial}{\partial y}\log|f|\right)\cos y\,\Big|_1^2.$$

Here the terms of the finite sum are all seen to be non-negative, and we obtain

$$m''(x) \geqq m(x). \tag{3}$$

We still have to consider the obviously isolated irregular values where one of the exceptional circumstances occurs. First of all it may happen that a line $x=t$ contains a whole segment lying on the boundary of Ω. It is then easy to prove that the one-sided derivatives $m'(t-0)$ and $m'(t+0)$ exist, and that $m'(t-0)<m'(t+0)$. At a point with vertical tangent and at a corner $m'(x)$ is seen to be continuous, while $m''(x)$ may fail to exist.

These remarks complete the proof of

THEOREM 1. *The function $m(x)$ satisfies the differential inequality*

$$m''(x) \geqq m(x) \tag{3}$$

except at certain isolated points. At these exceptional points $m'(x)$ is continuous or presents a positive jump.

The sign of equality holds if and only if Δ_x coincides with the segment $(-\pi/2, \pi/2)$ or is vacuous.

2. The solutions of the differential equation $\phi''(x)=\phi(x)$ are $\phi(x)=\alpha e^x+\beta e^{-x}$. We are going to show that the curve C: $Y=m(x)$ is convex with respect to this family of functions.

Suppose that the curves $Y=m(x)$ and $Y=\phi(x)$ intersect in two points with the coordinates x_1 and $x_2(>x_1)$, and let these be the nearest points of intersection so that there are no common points between x_1 and x_2. The difference $\omega(x)=m(x)-\phi(x)$ has a constant sign in (x_1, x_2) and satisfies the differential inequality $\omega''(x)\geqq\omega(x)$. This leads to a contradiction if $\omega(x)$ is constantly positive. In fact, if $t_1<t_2<\cdots<t_k$ are the irregular points between x_1 and x_2, we have

$$\omega'(x_1+0) + [\omega'(t_1+0) - \omega'(t_1-0)] + \cdots \\ + [\omega'(t_k+0) - \omega'(t_k-0)] - \omega'(x_2-0) \geqq 0,$$

since all the terms are non-negative. Using the mean-value theorem we can rewrite the expression in the form

$$[\omega'(x_1+0) - \omega'(t_1-0)] + \cdots + [\omega'(t_k+0) - \omega'(x_2-0)] \\ = (x_1 - t_1)\omega''(\xi_1) + \cdots + (t_k - x_2)\omega''(\xi_k), \\ (x_1 < \xi_1 < t_1, \cdots, t_k < \xi_k < x_2)$$

and now conclude that it is negative. It follows that C lies under the curve $Y=\phi(x)$.

More precisely, if x_1 and x_2 are any two roots of the equation $m(x)=\phi(x)$ we may conclude that either the strict inequality $m(x)<\phi(x)$ holds for all $x_1<x<x_2$ or $m(x)\equiv\phi(x)$ in the whole interval. For if x' is a common point between x_1 and x_2, the development of $\omega(x)$ at x' must be of the form $\omega(x)=-c(x-x')^{2k}+\cdots$, $c>0$, $k\geqq 1$, whence $\omega''(x)=-2k(2k-1)c(x-x')^{2k-2}+\cdots$, and this is clearly incompatible with the inequality $\omega''(x)\geqq\omega(x)$.

The particular solution $\phi(x)$ which intersects $Y=m(x)$ at x_1 and x_2 is calculated from the equation

$$\begin{vmatrix} \phi(x) & e^{x} & e^{-x} \\ m(x_1) & e^{x_1} & e^{-x_1} \\ m(x_2) & e^{x_2} & e^{-x_2} \end{vmatrix} = 0.$$

Expressing the inequality $m(x)\leqq\phi(x)$ we get

THEOREM 2. *For* $x_1<x<x_2$

$$\begin{vmatrix} m(x) & e^{x} & e^{-x} \\ m(x_1) & e^{x_1} & e^{-x_1} \\ m(x_2) & e^{x_2} & e^{-x_2} \end{vmatrix} \geqq 0 \tag{4}$$

or

$$m(x) \leqq \frac{m(x_1)\sinh(x_2 - x) + m(x_2)\sinh(x - x_1)}{\sinh(x_2 - x_1)}.$$

If the sign of equality holds for one x *between* x_1 *and* x_2, *then it holds for all. Supposing* $m(x_1)$ *and* $m(x_2)>0$ *the equality holds if and only if* $|f(s)|>1$ *at all points between* x_1 *and* x_2.

The last assertion is an immediate consequence of the remark concerning the sign of equality in the condition (3).

The simplest functions for which the limits are actually attained are those of the form

$$f(s) = \exp(\alpha e^{s} + \beta e^{-s}),$$

where α and β are constants. If, for example, $\alpha>0$ and $\beta<0$ it should be noticed that $m(x)\equiv 0$ for $x<\frac{1}{2}\log(-\beta/\alpha)$, and that $m'(x)$ has a jump at the endpoint of this interval.

3. In order to study the possible shapes of the curve C: $Y=m(x)$ we fix an arbitrary point x_0 with $m(x_0)>0$. For convenience we suppose that $x_0=0$

and write $m_0=m(0)$, $m_0'=m'(0)$. The curves $Y=\phi(x)$ passing through the point $(0, m_0)$ are given by $\phi(x)=\alpha e^x+(m_0-\alpha)e^{-x}$, the slope at $x=0$ being $2\alpha-m_0$. Through every point with $x\neq 0$ passes one and only one curve of the field with $\alpha=\alpha(x)$. For $0<\alpha<m_0$ the curves tend towards infinity in both directions. The two curves $\alpha=0$ and $\alpha=m_0$ are exponential curves, and the curves $\alpha<0$ or $>m_0$ intersect the x-axis at $x=\frac{1}{2}\log[(\alpha-m_0)/\alpha]$.

The slope m_0' at $x=0$ corresponds to $\alpha=\frac{1}{2}(m_0+m_0')$. From Theorem 2 we infer that C does not cut any of the curves $\alpha>\frac{1}{2}(m_0+m_0')$ in the half-plane $x>0$, nor any of the curves $\alpha<\frac{1}{2}(m_0+m_0')$ in the half-plane $x<0$. Moreover, the values of $\alpha(x)$ are steadily non-decreasing as x increases. Hence $\alpha(x)$ must tend to definite limits as x tends to plus or minus infinity.

From these remarks we can easily deduce all the statements in

THEOREM 3. *For every C one of the following mutually exclusive statements is true:*

(a) $\eta_1=\lim_{x\to\infty}m(x)e^{-x}$ *and* $\eta_2=\lim_{x\to-\infty}m(x)e^{x}$ *exist and are positive*;

(b) $\lim_{x\to-\infty}m(x)=0$ *and* $\eta_1=\lim_{x\to\infty}m(x)e^{-x}>0$;

(c) $\lim_{x\to\infty}m(x)=0$ *and* $\eta_2=\lim_{x\to-\infty}m(x)e^{x}>0$.

If $|m_0'|<m_0$, *C is of type* (a); *if* $m_0'\geqq m_0$, *C is of type* (a) *or* (b); *and if* $m_0'\leqq -m_0$, *C is of type* (a) *or* (c).

The proof follows immediately from the fact that α is increasing. The normal situation in the cases (b) and (c) is of course that $m(x)$ vanishes identically from a certain point on.

If C is known to be of the type (b), then by Theorem 3 the inequality $m'(x)\geqq m(x)$ must hold for all x, for $x=0$ was only a representative for an arbitrary point. This differential inequality is equivalent with the fact that $m(x)e^{-x}$ is a non-decreasing function. Hence we obtain the more precise

THEOREM 4. *If the curve C is of type* (b) *the function $m(x)e^{-x}$ is non-decreasing, and for a curve of type* (c) *$m(x)e^{x}$ is non-increasing.*

All these results refer to the $m(x)$ of a single region Ω. The same proof can, however, be given also if $m(x)$ is formed with respect to any number of regions Ω, including the case considered by the brothers Nevanlinna, where $m(x)$ refers to all regions Ω.

4. We shall finally interpret our main result for the more familiar case of a function analytic in a half-plane.

THEOREM 5. *Let the function $f(z)$ be analytic in the closed right half-plane and suppose that $|f(z)|\leqq 1$ on the imaginary axis. If we write*

$$m(r) = \int_{-\pi/2}^{\pi/2} \overset{+}{\log}\,|f(re^{i\phi})|\cos\phi\, d\phi.$$

the function $m(r)/r$ is non-decreasing and consequently tends to a positive limit as $r\rightarrow\infty$, unless $m(r)\equiv 0$.

The proof is obtained by taking $\log z$ as independent variable. There is only one alternative, since the curve C must obviously be of the type (b).

Remark. It is interesting to show that $m(r)\equiv\eta r$ implies $f(z)\equiv Ae^{2\eta z}$, where A is a constant of absolute value 1. In §4 we already proved that $Y=m(x)$ coincides with a curve $Y=\phi(x)$ if and only if $|f|>1$ for all interior points. Hence we must have $|f(x)|>1$ in the whole half-plane and the function can be continued to the left half-plane with $|f(z)|<1$. The entire function $f(z)$ has no zeros and is easily seen to be of the first order at most. Consequently it is of the form Ae^{cz}, where $|A|=1$ and c must be real and positive. The computation of $m(r)$ yields $c=2\eta$.

In the classical principle of Phragmén-Lindelöf the object of consideration is the maximum modulus $M(r)=\max_{|z|=r}|f(z)|$. Comparing with $m(r)$ we have $M(r)\geqq\frac{1}{2}m(r)$, whence we obtain

$$\liminf_{r\rightarrow\infty}\frac{\log M(r)}{r}>0.$$

Whether the limit of $(\log M(r))/r$ always exists we have not been able to decide.

The conditions of Phragmén-Lindelöf's principle are usually weakened by supposing that $f(z)$ is analytic only in the open half-plane and that $\limsup|f(z)|\leqq 1$ as z approaches a point on the imaginary axis. The most convenient way of extending Theorem 1, and hence all subsequent results, to this case would be first to consider a slightly smaller rectangle $x_1<x<x_2$, $|y|<\pi/2-\epsilon$ and to apply the theorem to the function $f(s)/\mu$, where μ is the upper bound of $|f|$ on the horizontal sides of the rectangle. A simple passage to the limit would then yield the desired result.

Part II

5. Phragmén-Lindelöf's principle is essentially a theorem on harmonic functions of two variables. It is natural to ask whether a similar theorem holds for harmonic functions of n variables. We shall show that our method can easily be generalized to this case.

Let the function $u(x_1,\cdots,x_n)$ be harmonic in some part of the n-dimensional space and consider one of the regions Ω in which u is positive. We shall suppose that Ω lies entirely in the half-space $x_1\geqq 0$. In addition we require that u shall be regular, and consequently $=0$, at all boundary points of Ω, except possibly at the origin and at infinity. The problem consists in studying the behavior of u in the region Ω.

Consider the intersection Δ_r or Ω with the sphere $x_1^2+x_2^2+\cdots+x_n^2=r^2$. In close analogy with the two-dimensional case we define

$$m(r) = \int_{\Delta_r} u \cos\phi\, d\omega,$$

where $\sin\phi = x_1/r$ and $d\omega$ is the central projection of the surface element on the sphere of radius 1.

A first differentiation is easy to carry out and yields

$$m'(r) = \int_{\Delta_r} \frac{\partial u}{\partial r} \cos\phi\, d\omega$$

since u vanishes on the boundary of Ω. The direct computation of the second derivative is rather intricate, so we prefer to make use of Green's formula which is well known to hold in n dimensions.

According to Green's formula we have

$$0 = \int \left(u\,\frac{\partial x_1}{\partial n} - x_1 \frac{\partial u}{\partial n}\right) d\sigma,$$

when the integral is extended over a closed surface. Take this surface to be the boundary of the part of Ω lying in the region $r_1 < r < r_2$ and let the normals be outer normals. Then Δ_{r_1} and Δ_{r_2} contribute together

$$\Big|_{r_1}^{r_2} r^{n-1}m(r) - r^n m'(r).$$

On the boundary of Ω we have $u=0, x \geqq 0$, and $\partial u/\partial n \leqq 0$. The corresponding part of the integral is thus seen to be non-positive, and we conclude that

$$r^{n-1}(m(r) - rm'(r))$$

is a non-increasing function of r.

In order to get a simple differential inequality we introduce $x = \log r$ as independent variable. We then get

$$\frac{d}{dx}\left[e^{(n-1)x}\left(m - \frac{dm}{dx}\right)\right] = e^{(n-1)x}\left((n-1)m - (n-2)\frac{dm}{dx} - \frac{d^2m}{dx^2}\right) \leqq 0$$

and finally

$$\frac{d^2m}{dx^2} + (n-2)\frac{dm}{dx} - (n-1)m \geqq 0.$$

This inequality holds whenever m'' exists. At irregular points one proves readily that m' is continuous or has a positive jump.

The easiest way of handling the differential inequality is to make the substitution $\mu = me^{(n/2-1)x}$. The inequality goes over into

$$\frac{d^2\mu}{dx^2} \geqq \left(\frac{n}{2}\right)^2 \mu,$$

and we are now in a position to apply the results already proved. In the first place we find that the curve C: $Y = \mu(x)$ is convex with respect to the family of curves $Y = \alpha e^{nx/2} + \beta e^{-nx/2}$. Expressing this result in terms of r we get

THEOREM 7. *In n dimensions the relation* (4) *is replaced by*

$$\begin{vmatrix} m(r)r^{n/2-1} & r^{n/2} & r^{-n/2} \\ m(r_1)r_1^{n/2-1} & r_1^{n/2} & r_1^{-n/2} \\ m(r_2)r_2^{n/2-1} & r_2^{n/2} & r_2^{-n/2} \end{vmatrix} \geqq 0.$$

We note that the sign of equality holds for harmonic functions of the form $\alpha x_1 + \beta x_1/r^n$.

Theorem 3 is immediately carried over if we replace $m(x)$ by $\mu(x)$ and the exponentials $e^{\pm x}$ by $e^{\pm nx/2}$. The numbers η_1 and η_2 of the theorem are thus defined by $\eta_1 = \lim_{r\to\infty} m(r)/r$ and $\eta_2 = \lim_{r\to 0} m(r)r^{n-1}$.

We conclude the paper by a complete statement of the assertions corresponding to Theorems 4–5 of the first section.

THEOREM 8. *Let $u(x_1, \cdots, x_n)$ be harmonic and positive in a region Ω contained in the half-space $x_1 \geqq 0$. Suppose further that u is regular and $=0$ at all boundary points of Ω, except possibly at the origin and at infinity. The expression*

$$m(r) = \int_{\Delta_r} u \cos\phi \, d\omega$$

has the following properties:

The limits

$$\eta_1 = \lim_{r\to\infty} \frac{m(r)}{r} \quad \textit{and} \quad \eta_2 = \lim_{r\to 0} m(r)r^{n-1}$$

exist and one at least is positive.

If $\eta_1 = 0$ the function $m(r)/r$ is non-decreasing, and $\eta_2 = 0$ implies that $m(r)r^{n-1}$ is non-increasing.

We remark that the maximum modulus $M(r)$ is greater than $m(r)$ multiplied by a certain constant. Hence we conclude that one at least of the relations $\lim_{r\to\infty} M(r)/r > 0$ and $\lim_{r\to 0} M(r)r^{n-1} > 0$ is true. The extremal functions are x_1 and x_1/r^n. In two dimensions one of these is obtained from the other by an inversion, but in higher dimensions they are essentially different.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.