ON COMPLEMENTARY MANIFOLDS AND PROJECTIONS IN SPACES $L_p$ AND $l_p$\dagger

BY

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Introduction. If $\Lambda$ is a Banach space, $\mathcal{M}$ a closed linear subset of $\Lambda$, then a closed linear subset $\mathcal{N}$ such that every $f \in \Lambda$ is uniquely expressible as $g + h$, $g \in \mathcal{M}$, $h \in \mathcal{N}$, is called a complementary manifold to $\mathcal{M}$.

In his treatise on linear operations, Banach\ddagger presents the following two problems ((B), pp. 244-245).

(a) To every closed linear subset $\mathcal{M}$ in $L_p$, $1 < p \neq 2$, does there exist a complementary manifold?

(b) To every closed linear subset $\mathcal{M}$ in $l_p$, $1 < p \neq 2$, does there exist a complementary manifold?

We show in this paper that the answer to both questions is "no."

In Chapter 1 of this paper we show that if a certain limit has the value $\infty$, then the answer is negative.§ In Chapters 2 and 3, it is proved that this is indeed the case. In the concluding section we discuss the relation of various other problems to (a) and (b).

Chapter 1. $C(\mathcal{M})$ and $\overline{C}(\Lambda)$

1.1. Let $\Lambda$ denote a separable space with a $p$-norm, i.e., $\Lambda$ is either $L_p$ or $l_p$ or the set of ordered $n$-tuples of real numbers $\{(a_1, \ldots, a_n)\}$, $l_p$, with the norm $\| (a_1, \ldots, a_n) \| = (\sum_{i=1}^{n} |a_i|^p)^{1/p}$. We also let $l_p, \infty = l_p$. The notation $p' = 1/(p-1), 1/p + 1/p' = 1$ will be used throughout.

Let $\mathcal{M}$ be a closed linear manifold in $\Lambda$. Let $R$ denote the set of real numbers $0 \leq a \leq \infty$, and let $r(a, b) = a/(1 + a) - b/(1 + b), (\infty/(1 + \infty) \equiv 1)$. It is easy to see that $R$ with the metric $|r(a, b)|$ is a complete metric space and is homeomorphic to the closed interval $(0, 1)$.

If $\mathcal{M}$ is a closed linear manifold in $\Lambda$, a limited transformation $E$ such that $E\Lambda = \mathcal{M}$, $E^2 = E$, is said to project $\Lambda$ on $\mathcal{M}$.

If $E$ is a limited transformation, we denote by $|E|$ the bound of $E$.

**Lemma 1.1.1.** Let $\mathcal{M}$ be a closed linear manifold in $\Lambda$. The existence of a
complementary manifold \( \mathfrak{N} \) to \( \mathfrak{M} \) is equivalent to the existence of a projection \( E \) of \( \Lambda \) on \( \mathfrak{M} \).

Suppose \( \mathfrak{N} \) exists. Let \( E \) be the transformation which is such that \( f = g + h, \ g \in \mathfrak{M}, \ h \in \mathfrak{N} \), then \( Ef = g \). Owing to the properties of \( \mathfrak{N} \), \( E \) is single-valued, additive, homogeneous and defined everywhere. Now let \( \{ f_i \} \) be a sequence, which approaches \( f \) and such that if \( f_i = g_i + h_i, \ g_i \in \mathfrak{M}, \ h_i \in \mathfrak{N} \) the \( g_i \) form a convergent sequence with limit \( g' \). Then \( g' \in \mathfrak{M} \) and the sequence \( h_i = f_i - g_i \) also converges to a \( h' \in \mathfrak{N} \). By continuity we have \( f = g' + h' \). The uniqueness of the resolution of \( f \) now implies that \( Ef = g' \) or that \( E \) is closed. Theorem 7 of (B), Chapter III, p. 41, now implies that \( E \) is bounded. Since the range of \( E \) is included in \( \mathfrak{M} \) and for every \( f \in \mathfrak{M}, \ Ef = f \), we see that the range of \( E \) is \( \mathfrak{M} \) and \( E^2 = E \) or \( E \) is a projection of \( \Lambda \) on \( \mathfrak{M} \).

Now suppose \( E \) exists. Let \( \mathfrak{N} \) be the set of \( g \)'s in \( \Lambda \) for which \( Eg = 0 \). Since \( E \) is limited and linear, \( \mathfrak{N} \) is a closed linear manifold. If \( f \in \Lambda, \ f = Ef + (1 - E)f \) where \( Ef \in \mathfrak{M} \) and \( (1 - E)f \in \mathfrak{N} \), since \( E(1 - E)f = (E - E^2)f = 0 \). On the other hand if \( h \in \mathfrak{M}, \ h = Ef \) for some \( f \in \Lambda \), and hence \( Eh = E^2f = Ef = h \). Thus if \( h \in \mathfrak{N} \cdot \mathfrak{M}, \ 0 = Eh = h \), or \( \mathfrak{M} \cdot \mathfrak{N} = \{ 0 \} \). Now let \( f \) again be \( e \Lambda, \ f = g + h = g' + h', \ g, g' \in \mathfrak{M}, \ h, h' \in \mathfrak{N} \). Then \( g - g' = h' - h \), and since \( h - h' \in \mathfrak{N}, \ g' - g \in \mathfrak{M}, \) and \( \mathfrak{M} \cdot \mathfrak{N} = \{ 0 \} \), this implies \( g - g' = h' - h = 0 \). This shows that \( f \in \Lambda \) can only be expressed in one way as \( h + g, \ h \in \mathfrak{N}, \ g \in \mathfrak{M} \).

We may therefore consider problems (a) and (b) in the following equivalent form.

(A) To every closed linear manifold \( \mathfrak{M} \) of \( L_p, \ 1 < p \neq 2 \), is there a projection of \( L_p \) on \( \mathfrak{M} \)?

(B) To every closed linear manifold \( \mathfrak{M} \) of \( l_p, \ 1 < p \neq 2 \), is there a projection of \( l_p \) on \( \mathfrak{M} \)?

Let \( \Lambda_1, \ldots, \Lambda_n, \ n = 1, 2, \ldots, \infty, \Lambda_\infty = \Lambda \) be a set of spaces. Let \( \sum_{a=1}^n \oplus \Lambda_a = \Lambda_1 \oplus \cdots \oplus \Lambda_n \) denote the space of ordered sets of elements \( \{ f_1, f_2, \ldots, f_n \} \) \( (f_\infty = f) f_\alpha \in \Lambda_\alpha \), such that \( \sum_{a=1}^n \| f_\alpha \|_p < \infty \), with a norm defined by the equation

\[
\| \{ f_1, f_2, \ldots, f_n \} \| = \left( \sum_{a=1}^n \| f_\alpha \|_p \right)^{1/p}.
\]

\( \Lambda_\alpha \cong \Lambda_\beta \) is to mean that there exists a one-to-one isometric mapping of \( \Lambda_\alpha \) on \( \Lambda_\beta \).

**Lemma 1.1.2.** (a) \( \sum_{a=1}^n \oplus l_p, n_a = 1, 2, \ldots, \infty, \) if \( \sum_{a=1}^n n_a = m \).

(b) \( \sum_{a=1}^n \oplus \Lambda_\alpha \cong L_p \) if \( \Lambda_\alpha = L_p \), for each \( \alpha \).

The proof of this lemma may be left to the reader.

1.2. Let \( \mathfrak{M} \) be a closed linear manifold in \( \Lambda \). We define a function \( C(\mathfrak{M}) \),
which takes on values in $R$ as follows. If there exists no projection of $\Lambda$ on $M$, then $C(M) = \infty$. Otherwise $C(M) = \text{g.l.b.} (|E|; EA = M, E^2 = E)$. Similarly we define the function $C(\Lambda)$ as $\text{l.u.b.} (C(M), M \subseteq \Lambda)$.

**Lemma 1.2.1.** Let $\Lambda_1$ and $\Lambda_2$ be such that $\Lambda_1$ is equivalent [[(B), p. 180]] to a closed linear manifold $M$ of $\Lambda_2$. Let $M$ be such that there exists a projection $E$ of $\Lambda_2$ on $M$, with $|E| = 1$, $R$ the set of $S$'s $\Delta$, for which $Ef = 0$. Let $\Psi$ be any closed linear manifold of $\Lambda_2$, such that if $f \in \Psi$, then $f = g + h$, $g \in \Psi \cdot M$, $h \in \Psi \cdot R$. Let $\Psi_1$ in $\Lambda_1$ be the manifold which corresponds to $\Psi \cdot M$. Then $C(\Psi_1) \leq C(\Psi)$.

If $C(\Psi) = \infty$, our statement is true. Suppose $C(\Psi)$ is $< \infty$. Let $F$ be any projection of $\Lambda_2$ on $\Psi$. Then $EF$ is a projection on $\Psi \cdot M$. For if $f \in \Lambda_2$, $f = Ff = g + h$, $g \in \Psi \cdot M$, $h \in \Psi \cdot R$, and $EFf = g$ or the range of $EF$ is included in $\Psi \cdot M$. Also for every $h \in \Psi \cdot R$, we have $EFh = Eh = h$. This with our previous statement shows that $(EF)^2 = EF$ and that the range of $EF$ is exactly $\Psi \cdot M$.

Let $(EF)'$ be $EF$ considered only on $M$. Obviously $(EF)'$ projects $M$ on $\Psi \cdot M$. Let $G$ be the corresponding transformation on $\Lambda_1$. Then $C(\Psi_1) \leq |G| = |(EF)'| \leq |EF| \leq |E| \cdot |F| = |F|$ or $C(\Psi_1) \leq |F|$. Since $F$ was any projection on $\Psi$, $C(\Psi_1) \leq C(\Psi)$.

**Lemma 1.2.2.** If $\Lambda_1$ and $\Lambda_2$ are as in Lemma 1.2.1, $C(\Lambda_1) \leq C(\Lambda_2)$. In particular if $\Lambda_2 = \Lambda_0 \oplus \Lambda_1$, $C(\Lambda_1) \leq C(\Lambda_2)$.

Let $\Psi_1$ be any closed linear manifold of $\Lambda_1$, $\Psi$ the corresponding set of elements in $M$. $\Psi$ is a closed linear manifold satisfying the conditions given in Lemma 1.2.1, since $\Psi \cdot M = \Psi$, $\Psi \cdot R = \{0\}$. Lemma 1.2.1 now implies that $C(\Psi_1) \leq C(\Psi) \leq C(\Lambda_2)$. But $\Psi_1$ was any closed linear manifold in $\Lambda_1$, hence $C(\Lambda_1) \leq C(\Lambda_2)$.

To show the second statement, we take $M \subseteq \Lambda_0 \oplus \Lambda_1$ as the set of elements $\{0, f\}$ of $\Lambda_0 \oplus \Lambda_1$, $E$ as the transformation of $\Lambda_0 \oplus \Lambda_1$, such that $E\{f, g\} = \{0, g\}$. One readily sees that $M$ is equivalent to $\Lambda_1$ and that $E$ projects $\Lambda_0 \oplus \Lambda_1$ on $M$ and $|E| = 1$. We may now apply the first part of this lemma to obtain the desired result.

**Lemma 1.2.3.** If $\Lambda \equiv \bigoplus_{a=1}^{\infty} \Lambda_a$ and $k$ is $\text{lim sup}_{a \to \infty} C(\Lambda_a)$, then there exists a manifold $\Psi \subseteq \Lambda$, such that $C(\Psi) \geq k$.

It follows from the definition of $k$, that if $\varepsilon$ is $>0$, then there exists an infinite number of the $\alpha$’s for which $r(C(\Lambda_\alpha), k) \geq -\varepsilon$. Thus we can find a sequence of integers $\{\alpha_i\}$ such that $\alpha_i < \alpha_{i+1}$, for which $r(C(\Lambda_{\alpha_i}), k) \geq -2^{-i-1}$.

Now since $r(C(\Lambda_{\alpha_i}), k) \geq -2^{-i-1}$, we can find a $\Psi_{\alpha_i}$ in $\Lambda_{\alpha_i}$ such that $r(C(\Psi_{\alpha_i}), k) > -2^{-i}$. Let $\Psi$ be the closed linear manifold consisting of those elements $\{f_1, f_2, f_3, \ldots\} \in \Lambda$, such that $f_\beta = 0$ if $\beta$ is not $\varepsilon\{\alpha_i\}$ and $f_\alpha \in \Psi_{\alpha_i}$. As we saw in the proof of Lemma 1.2.2, $\Lambda_{\alpha_i}$ and $\Lambda$ are as $\Lambda_1$ and $\Lambda_2$ in Lemma 1.2.1.
and it is easily seen that $\mathfrak{B}$ satisfies the conditions given in Lemma 1.2.1 also. Thus Lemma 1.2.1 now implies that $C(\mathfrak{B}) \cong C(\mathfrak{B}_\infty)$. Hence $r(C(\mathfrak{B}), k) \cong -2^{-i}$ for every $i$. This implies that $r(C(\mathfrak{B}), k) \cong 0$, $C(\mathfrak{B}) \cong k$.

1.3. We now prove the following lemma.

**Lemma 1.3.1.** $\overline{C}(L_p) \cong \overline{C}(l_{p,\infty})$.

In (B), Theorem 9, Chapter XII, p. 206, it is shown that the manifold $\mathfrak{M} \subseteq L_p$, determined by the functions $y_i$, is equivalent to $l_p$ when

$$y_i(t) = 2^{i/p} \text{ for } 1/2^i \leq t \leq 1/2^{i-1}, \quad y_i(t) = 0, \quad \text{otherwise}.$$  

Now for any $z(t) \in L_p$, let

$$E(z(t)) = \sum_{i=1}^{\infty} \int_0^1 z(s) y_i^{p-1}(s) ds \cdot y_i(t).$$

Then by a direct calculation one can verify that $|E| = 1$ and that if $z \in \mathfrak{M}$ (i.e., if $z = \sum \alpha_i y_i$, $\sum |\alpha_i|^p < \infty$), then $Ez = z$. Hence $E$ projects $L_p$ on $\mathfrak{M}$ and we may apply Lemma 1.2.2 so that it yields $\overline{C}(L_p) \cong \overline{C}(l_{p,\infty})$.

**Lemma 1.3.2.** There exists a linear manifold $\mathfrak{M} \subseteq L_p$, such that $C(\mathfrak{M}) = \overline{C}(L_p)$.

This follows from Lemma 1.1.2, (b) (with $n = \infty$) and Lemma 1.2.3 for $k$ is in this case $\overline{C}(L_p)$.

**Lemma 1.3.3.** There exists a linear manifold $\mathfrak{M} \subseteq l_{p,\infty}$, such that $C(\mathfrak{M}) = \overline{C}(l_{p,\infty})$.

In Lemma 1.1.2, (a), let $n_\alpha = \infty$ for every $\alpha$. Then apply Lemma 1.2.3.

**Lemma 1.3.4.** $\overline{C}(l_{p,n}) \cong \overline{C}(l_{p,m})$ if $n \geq m$.

This follows from Lemma 1.1.2, (a) and Lemma 1.2.2.

**Theorem I.** $C(\mathfrak{M})$ and $\overline{C}(\Lambda)$ are to be as in §1.2. There exists an $\mathfrak{M}$ in $L_p$, and an $\Lambda$ in $l_p$, such that $C(\mathfrak{M}) = \overline{C}(L_p)$ and $C(\mathfrak{M}) = \overline{C}(l_p)$. Furthermore

$$1 = \overline{C}(l_{p,1}) \leq \overline{C}(l_{p,2}) \leq \cdots \leq \overline{C}(l_p) \leq \overline{C}(L_p).$$

The lemmas of this section imply this theorem.

Now if we are able to show that $\lim_{n \to \infty} \overline{C}(l_{p,n}) = \infty$, it follows from this theorem that $\overline{C}(L_p) = \overline{C}(l_p) = \infty$ and then in each of them we have a manifold $\mathfrak{M}$ for which $C(\mathfrak{M}) = \infty$. Hence from the definition of $C(\mathfrak{M})$, we can answer problems (a) and (b) negatively. The next two chapters of this paper contain the proof of the fact that $\lim_{n \to \infty} \overline{C}(l_{p,n}) = \infty$.

**Chapter 2.** $\mathfrak{M}$ in Situation A

2.1. Let $f = \{a_1, \ldots, a_n\}$ be an $n$-dimensional vector, which may be regarded as $\in l_{p,n}$. We define for $k > 0$
\{f\}^k = \{ |a_1|^k \text{ sign } a_1, \ldots, |a_n|^k \text{ sign } a_n\}

\begin{align*}
\{f\}^* &= \{ |a_1|^k, \ldots, |a_n|^k\},
\end{align*}

which may be regarded as elements of \(l_{p/k,n}\). If \(g = \{b_1, \ldots, b_n\}\), we define \((f, g)\) = \(\sum_{i=1}^n a_i b_i\). The linearity and homogeneity of this expression will be used without comment.

The following two lemmas can be easily shown.

**Lemma 2.1.1.** If \(p > 1\),
\[
\frac{d}{dt} \|f + tg\|^p \bigg|_{t=0} = p(\{f\}^{p-1}, g).
\]

**Lemma 2.1.2.** If \(p > 2\),
\[
\frac{d^2}{dt^2} \|f + tg\|^p = p(p - 1)(\{f + tg\}^{p-2}, [g]^2).
\]

We now prove

**Lemma 2.1.3.** If \(p > 2\), and \(\{f\}^{p-1}, g\) \(\geq 0\), then \(\|f + g\|^p \geq \|f\|^p\).

By Lemma 2.1.2, \(H(t) = \|f + tg\|^p\) is convex in \(t\) and hence increasing for \(t \geq 0\) since \(dH/dt\bigg|_{t=0} = 0\) by Lemma 2.1.1.

2.2. If \(\mathcal{M}\) is a linear manifold in \(l_{p,n}\), let \(\mathcal{M}^k\) consist of those elements \(g \in l_{p',n}, 1/p + 1/p' = 1\), for which \((f, g) = 0\) for all \(f \in \mathcal{M}\). If \(\mathcal{M}\) is \(k\)-dimensional, it is well known that \(\mathcal{M}^k\) is \((n - k)\)-dimensional and also that \((\mathcal{M}^k)^\perp = \mathcal{M}\). The following lemma is of a standard type in the theory of linear manifolds of a finite number of dimensions and the proof of it may be omitted.

**Lemma 2.2.1.** Let \(\mathcal{M}\) be a \(k\)-dimensional linear manifold in \(l_{p,n}\). Let \(\phi_1, \ldots, \phi_k\) be \(k\) linearly independent elements of \(\mathcal{M}\). If \(E\) is a projection of \(l_{p,n}\) on \(\mathcal{M}\), there exist \(k\) elements \(\psi_1, \ldots, \psi_k\), of \(l_{p',n}\) such that for every \(f \in l_{p,n}\),
\[
E f = \sum_{i=1}^k (\psi_i, f) \phi_i
\]

and \((\psi_i, \phi_i) = \delta_{i,j}\). If \(\psi_i\)'s with this last property are given, the \(E\) defined by (\(\alpha\)) is a projection. If \(E'\) is any other projection of \(l_{p,n}\) on \(\mathcal{M}\), then \(\psi_i = \psi_i + g_i\), \(i=1, \ldots, k\), where \(g_i \in \mathcal{M}^k\).

2.3. If \(E\) is a linear transformation in \(l_{p,n}\), we denote by \(E^*\) (the adjoint of \(E\)) the transformation in \(l_{p',n}\), such that if \(g\) and \(g^* \in l_{p',n}\), are related so that for every \(f \in l_{p,n}\), \((Ef, g) = (f, g^*)\), then \(E^* g = g^*\). It is well known that \(E^*\) is linear, \(|E| = |E^*|\) and \((FE)^* = E^*F^*\).
Lemma 2.3.1. If \( M \) and \( E \) are as in Lemma 2.2.1, then

\[
E^*g = \sum_{i=1}^{k} (\phi_i, g)\psi_i
\]

for all \( g \in l_{p',n} \).

For every \( f \in l_{p,n} \), we have

\[
(Ef, g) = \left( \sum_{i=1}^{k} (\psi_i, f)\phi_i, g \right) = \sum_{i=1}^{k} (\psi_i, f)(\phi_i, g) = \left( f, \sum_{i=1}^{k} (\phi_i, g)\psi_i \right).
\]

Lemma 2.3.2. If \( M \) and \( E \) are as in Lemma 2.2.1, then \( 1 - E^* \) is a projection on \( M^4 \).

The range of \( 1 - E^* \) is \( M^4 \). For if \( f = (1 - E^*)g \) and \( h \) is \( \epsilon M \), then \( Eh = h \) and

\[
(h, f) = (h, (1 - E^*)g) = (h, g) - (h, E^*g) = (Eh, g) - (h, E^*g) = 0.
\]

Hence \( f \) is \( \epsilon M^4 \) or the range of \( 1 - E^* \) is included in \( M^4 \). Furthermore by Lemma 2.3.1, if \( f \) is \( \epsilon M^4 \),

\[
E^*f = \sum_{i=1}^{k} (\phi_i, f)\psi_i = 0
\]

and \( (1 - E^*)f = f \). This with our previous result shows that the range \( 1 - E^* \) is exactly \( M^4 \) and \( (1 - E^*)^2 = 1 - E^* \).

Lemma 2.3.3. \( \overline{C}(l_{p',n}) \leq \overline{C}(l_{p,n}) + 1 \).

Let \( M' \) be any linear manifold of \( l_{p',n} \). Let \( M = M'^k \). Then if \( \epsilon > 0 \), there exists a projection \( E \) of \( l_{p,n} \) on \( M \) with \( |E| \leq C(M) + \epsilon \leq \overline{C}(l_{p,n}) + \epsilon \). By Lemma 2.3.2, \( 1 - E^* \) is a projection on \( M^4 = (M^k)^4 = M' \). Thus \( C(M') \leq |1 - E^*| \leq 1 + |E^*| = 1 + |E| \leq 1 + \overline{C}(l_{p,n}) + \epsilon \), which implies our lemma.

Of course \( p \) and \( p' \) are interchangeable and so we see that the answer to our question is the same for both \( p \) and \( p' \). Thus we may confine ourselves to the case \( p < 2 \). This is not an essential step in our proof but merely a convenient one. We suppose from now on that \( p \) is \( < 2 \).

2.4. We say that Situation A holds in a \( k \)-dimensional manifold \( M \) of \( l_{p,n} \), if

(a) we have \( k \) linearly independent elements, \( \phi_1, \ldots, \phi_k, \epsilon M \), and \( k \) elements of \( l_{p',n}, \psi_1, \ldots, \psi_k \) such that \( (\phi_i, \psi_j) = \delta_{i,j} \) (the transformation \( E \) given by the equation \( Ef = \sum_{i=1}^{k} (\psi_i, f)\phi_i \) is a projection of \( l_{p,n} \) on \( M \));

(b) we have \( r \) elements \( h_1, \ldots, h_r \) of \( M \), with \( ||h_i|| = 1 \);

(c) there exists a constant \( C > 1 \), such that \( ||E^*\{h_i\}||^p = C \) for every \( i \);

(d) there exist \( r \) constants \( c_1, \ldots, c_r \), \( c_i > 0 \), such that for every \( f \in M \) and \( g \in M^k \).
$$\sum_{i=1}^{r} c_i(h_i)^{p-1} f(\{E^*\{h_i\}^{p-1}\}) g = 0.$$ 

Lemma 2.4.1. If $M$ is in Situation A, then $C(M) \geq C$ (cf. (c) above).

Since $|E| = |E^*|$, we must show that for every projection $E'$ of $l_{p,n}$ on $M$, $|E'^*| \geq C$. Since $\|h_i\| = 1$ and hence $\|\{h_i\}^{p-1}\| = 1$, it will be sufficient to show that $\|E'^*\{h_i\}^{p-1}\| \geq C$ for at least one $i$.

Now

$$E'^*\{h_i\}^{p-1} = \sum_{j=1}^{k} (\{h_i\}^{p-1}, \phi_j) \psi_j^*,$$

where $\psi_j^* = \psi_j + g_j$, where $g_j$ is an element of $M^\perp$ (Lemmas 2.2.1 and 2.3.1). Let $E_i$ be the projection of $l_{p,n}$ on $M$ given by

$$E_i f = \sum_{i=1}^{k} (\phi_i, g_i) (\psi_i + t g_i).$$

By Lemma 2.3.1,

$$E_i^* g = \sum_{i=1}^{k} (\phi_i, g) (\psi_i + t g_i)$$

and

$$E_i^* \{h_i\}^{p-1} = \sum_{i=1}^{k} (\phi_i, \{h_i\}^{p-1}) (\psi_i + t g_i)$$

$$= \sum_{i=1}^{k} (\phi_i, \{h_i\}^{p-1}) \psi_i + t \sum_{i=1}^{k} (\phi_i, \{h_i\}^{p-1}) g_i$$

$$= E^* \{h_i\}^{p-1} + t \sum_{i=1}^{k} (\phi_i, \{h_i\}^{p-1}) g_i.$$ 

Now by Lemma 2.1.1

$$\frac{d}{dt} \|E_i^* \{h_i\}^{p-1}\|_{t=0}^p = \left(\{E^* \{h_i\}^{p-1}\}^{p-1}, \sum_{i=1}^{k} (\phi_i, \{h_i\}^{p-1}) g_i\right)$$

$$= \sum_{i=1}^{k} (\{E^* \{h_i\}^{p-1}\}^{p-1}, g_i) (\phi_i, \{h_i\}^{p-1}).$$

Since $g_i \in M^\perp$, $\phi_i \in M$, (d) implies that

$$\sum_{i=1}^{r} c_i \frac{d}{dt} \|E_i^* \{h_i\}^{p-1}\|_{t=0}^p = \sum_{i=1}^{r} c_i \sum_{i=1}^{k} (\{E^* \{h_i\}^{p-1}\}^{p-1}, g_i) (\phi_i, \{h_i\}^{p-1}) = 0.$$
Since $c_i > 0$, for $i = 1, \ldots, r$ this implies that there must be an $i'$ such that $d \|E^* \{h_{i'} \}^{p-1} \|v'/dt\|_{t=0}$ is $\geq 0$. Hence by the above $d \|E^* \{h_{i'} \}^{p-1} + tg_{i'} \|^{p'/dt\|^p} \geq 0$, when $g_{i'} = \sum_{i=1}^{r} (\phi_{i'} \{h_{i'} \}^{p-1}) g_{i}$. Lemma 2.1.3 now yields

$$\|E^* \{h_{i'} \}^{p-1} + g_{i'} \|^{p'} \geq \|E^* \{h_{i'} \}^{p-1} \|^p$$

since $p' > 2$. But

$$E^* \{h_{i'} \}^{p-1} = E^* \{h_{i'} \}^{p-1} = E^* \{h_{i'} \}^{p-1} + g_{i'}$$

and $\|E^* \{h_{i'} \}^{p-1} \|^p = C$. Substituting these values on both sides of our inequality, we get $\|E^* \{h_{i'} \}^{p-1} \|^p \geq C^p$ or $\|E^* \{h_{i'} \}^{p-1} \| \geq C$. As we remarked at the beginning of the proof this is sufficient.

Chapter 3. The Product of $l_{p,n}$ and $l_{p,m}$

3.1. We define $l_{p,n} \otimes l_{p,m}$ as $l_{p,nm}$. If $f = \{a_1, \ldots, a_n \} \in l_{p,n}$, and $g = \{b_1, \ldots, b_m \} \in l_{p,m}$, we define $f \otimes g$ as $\{a_1b_1, a_1b_2, \ldots, a_1b_m, a_2b_1, a_2b_2, \ldots, a_2b_m, \ldots, a_nb_1, a_nb_2, \ldots, a_nb_m \}$ or if $f \otimes g = \{c_1, \ldots, c_{nm} \}$, then $c_{(a-1)m+t} = a_t b_t$. The proofs of the following Lemmas 3.1.1-3.1.4 do not present any difficulty and may be left to the reader.

**Lemma 3.1.1.** (i) $\|f \otimes g\| = \|f\| \cdot \|g\|$, (ii) $(\phi_1 \otimes \phi_2, f \otimes g) = (\phi_1, f)(\phi_2, g)$, (iii) $\{f \otimes g\}^k = \{f\}^k \otimes \{g\}^k$, (iv) $\alpha(f^{(1)} \otimes g) + \beta(f^{(2)} \otimes g) = (\alpha f^{(1)} + \beta f^{(2)}) \otimes g$.

**Lemma 3.1.2.** Let $f_r = \{a_{1,r}, \ldots, a_{n,r} \}$, $r = 1, \ldots, k$, $k \leq n$, be $k$ linearly independent elements of $l_{p,n}$, and $g = \{b_{1,r}, \ldots, b_{m,r} \}$ be $k$ elements of $l_{p,m}$, such that $\sum_{r=1}^{k} f_r \otimes g_r = 0$. Then $g_r = 0$, $r = 1, \ldots, k$.

**Lemma 3.1.3.** Let $f_r, r = 1, \ldots, k$, $k \leq n$, be $k$ linearly independent elements of $l_{p,n}$ and for every $r = 1, \ldots, k$ let $g_{r,s}, s = 1, \ldots, k, k, k \leq m$, be $k$ linearly independent elements of $l_{p,m}$. Then the set of elements $f_r \otimes g_{r,s}, r = 1, \ldots, k, s = 1, \ldots, k$, are linearly independent.

**Lemma 3.1.4.** Let $f_1, \ldots, f_n$ be $n$ linearly independent elements of $l_{p,n}$, $g_1, \ldots, g_m$ linearly independent elements of $l_{p,m}$. Then the set of elements $f_i \otimes g_j, i = 1, \ldots, n, j = 1, \ldots, m$, determine $l_{p,mn}$.

3.2. Let $\mathbb{M}^{(1)} \subseteq l_{p,n}$ and $\mathbb{M}^{(2)} \subseteq l_{p,m}$ be linear manifolds. We define $\mathbb{M}^{(1)} \otimes \mathbb{M}^{(2)}$ as the linear manifold in $l_{p,mn}$ determined by the elements $f \otimes g, f \in \mathbb{M}^{(1)}$, $g \in \mathbb{M}^{(2)}$.

**Lemma 3.2.1.** If $\phi_1^{(1)}, \ldots, \phi_n^{(1)}$ is a set of linearly independent elements which determine $\mathbb{M}^{(1)}$ and $\phi_1^{(2)}, \ldots, \phi_m^{(2)}$ is a set of linearly independent elements which determine $\mathbb{M}^{(2)}$, then $\phi_i^{(1)} \otimes \phi_j^{(2)}, i = 1, \ldots, k^{(1)}, j = 1, \ldots, k^{(2)}$, determine the manifold $\mathbb{M}^{(1)} \otimes \mathbb{M}^{(2)}$ which is $k^{(1)}k^{(2)}$-dimensional.
The proof is easily derived from Lemma 3.1.3.

**Lemma 3.2.2.** Let $\mathcal{M}(1)$ and $\mathcal{M}(2)$ be as above. Then $(\mathcal{M}(1) \otimes \mathcal{M}(2))^1$ is determined by the elements of the form $f \otimes g$, where either $f \in \mathcal{M}(1)$ or $g \in \mathcal{M}(2)$.

We first show that if $f \otimes g$ is such that either $f \in \mathcal{M}(1)$ or $g \in \mathcal{M}(2)$, then $f \otimes g \in (\mathcal{M}(1) \otimes \mathcal{M}(2))^1$. Indeed by the definition of $\mathcal{M}(1) \otimes \mathcal{M}(2)$ and the linearity of the operation $(, )$, if $(\phi(1) \otimes \phi(2), f \otimes g) = 0$ for all $\phi(1) \in \mathcal{M}(2)$ and $\phi(2) \in \mathcal{M}(2)$, then $f \otimes g \in (\mathcal{M}(1) \otimes \mathcal{M}(2))^1$. But Lemma 3.1.1, (ii) implies that $(\phi(1) \otimes \phi(2), f \otimes g) = (\phi(1), f)(\phi(2), g)$. Since either $(\phi(1), f) = 0$ or $(\phi(2), g) = 0$, we obtain that $f \otimes g \in (\mathcal{M}(1) \otimes \mathcal{M}(2))^1$.

Now let $\tilde{\phi}_1^{(1)}, \cdots, \tilde{\phi}_{n-k}^{(1)}$ and $\tilde{\phi}_1^{(2)}, \cdots, \tilde{\phi}_{n-k}^{(2)}$ be sets of $n-k(1)$ and $m-k(2)$ linearly independent elements of $\mathcal{M}(1)^1$ and $\mathcal{M}(2)^1$ respectively. Let $\psi_1^{(1)}, \cdots, \psi_1^{(2)}$, $\psi_2^{(1)}, \cdots, \psi_2^{(2)}$ be elements of $l_p$, $\psi_3^{(1)}, \cdots, \psi_3^{(2)}$ be elements of $l_p$, such that $\tilde{\phi}_i^{(1)}$'s and $\psi_i^{(1)}$'s together determine $l_p$, $\tilde{\phi}_i^{(2)}$'s and $\psi_i^{(2)}$'s determine $l_p$. In Lemma 3.1.3, let $f_r = \psi_r^{(1)}$ for $r = 1, \cdots, k(1), f_r = \phi_r^{(1)}$ for $r = 1, \cdots, n-k(1)$; $k = n$. For $r = 1, \cdots, k(1)$, let $k_r = m-k(2)$, $g_{r,s} = \tilde{\phi}_s^{(2)}$, $s = 1, \cdots, m-k(2)$, and for $r = k+1, \cdots, n-k(2)$; $k = m$, $g_{r,s} = \psi_s^{(2)}$, $s = 1, \cdots, k(2)$, $g_{r,s} = \phi_s^{(2)}$, $s = 1, \cdots, m-k(2)$. Thus the $f_r \otimes g_{r,s}$ are linearly independent and such that either $f_r \in \mathcal{M}(1)^1$, or $g_{r,s} \in \mathcal{M}(2)^1$. Since there are $k(1)(m-k(2)) + (n-k(1))m = mn - k(1)k(2)$ of them and the dimensionality of $(\mathcal{M}(1) \otimes \mathcal{M}(2))^1$ is $k(1)k(2)$ by Lemma 3.2.1, the $f_r \otimes g_{r,s}$ determine $(\mathcal{M}(1) \otimes \mathcal{M}(2))^1$.

3.3. Let $T(1)$ be a linear transformation in $l_p,n$ and $T(2)$ a linear transformation in $l_p,m$. Let $\phi_1^{(1)}, \cdots, \phi_n^{(1)}$ be $n$ linearly independent elements of $l_p,n$ and $\phi_1^{(2)}, \cdots, \phi_m^{(2)}$, $m$ linearly independent elements of $l_p,m$. Then the elements $\phi_i^{(1)} \otimes \phi_j^{(2)}$, $i = 1, \cdots, n$, $j = 1, \cdots, m$ determine $l_{p,m}$ by Lemma 3.1.4 and are linearly independent. Hence given any set of $mn$ elements $f_i,j \in l_{p,m}$, there exists a unique linear transformation $T'$ such that $T'(\phi_i^{(1)} \otimes \phi_j^{(2)}) = f_i,j$ $(i = 1, \cdots, n, j = 1, \cdots, m)$.

Now let $f_i,j = T(1)\phi_i^{(1)} \otimes T(2)\phi_j^{(2)}$ and denote the corresponding $T'$ by $T(1) \otimes T(2)$. Apparently this definition of $T(1) \otimes T(2)$ depends on the choice of the $\phi_i^{(1)}$'s and $\phi_j^{(2)}$'s, but this is not the case as is shown by the following

**Lemma 3.3.1.** If $f \in l_p,n$, $g \in l_p,m$, then $T(1) \otimes T(2)f \otimes g = T(1)f \otimes T(2)g$. Thus $T(1) \otimes T(2)$ does not depend on the choice of the $\{\phi_i^{(1)}\}$ or the $\{\phi_j^{(2)}\}$.

The proof follows immediately from the definition of $T(1) \otimes T(2)$ and Lemma 3.1.1, (iv).

We also have

**Lemma 3.3.2.** A linear transformation $T'$ of $l_{p,m}$ equals $T(1) \otimes T(2)$ if and only if $T'f \otimes g = T(1)f \otimes T(2)g$ for every $f \in l_p,n$, and $g \in l_p,m$. 

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The sufficiency of the condition follows from the definition. Lemma 3.3.1 implies its necessity.

**Lemma 3.3.3.** \((T^{(1)} \otimes T^{(2)})^* = (T^{(1)})^* \otimes (T^{(2)})^*\)

By Lemma 3.3.2, it suffices to show that if \(f \in l_{p',n}, \ g \in l_{p',m}\), then \((T^{(1)} \otimes T^{(2)})^* f \otimes g = T^{(1)}f \otimes T^{(2)}g\). Now if \(h \in l_{p,mn}\), it follows from Lemma 3.1.4, that \(h = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} \phi_i^{(1)} \otimes \phi_j^{(2)}\) and by the definition of \(T^{(1)} \otimes T^{(2)}\),

\[
T^{(1)} \otimes T^{(2)} h = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} T^{(1)}(\phi_i^{(1)}) \otimes T^{(2)}(\phi_j^{(2)}).
\]

By the definition of \(T^*\) (cf. §2.3) and Lemma 3.1.1, (ii), we have

\[
(T^{(1)} \otimes T^{(2)} h, f \otimes g) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} (T^{(1)}(\phi_i^{(1)}) \otimes T^{(2)}(\phi_j^{(2)}), f \otimes g)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} (T^{(1)}(\phi_i^{(1)})^*, f)(T^{(2)}(\phi_j^{(2)})^*, g)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} (\phi_i^{(1)}(T^{(1)}f), T^{(2)}g)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} (\phi_j^{(2)}(T^{(1)}f), T^{(2)}g)
\]

\[
= (\sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} \phi_i^{(1)}(T^{(1)}f), T^{(2)}g)
\]

\[
= (h, T^{(1)}f \otimes T^{(2)}g).
\]

Or for every \(h\) of \(l_{p,mn}\),

\[
(T^{(1)} \otimes T^{(2)} h, f \otimes g) = (h, T^{(1)}f \otimes T^{(2)}g).
\]

The definition of \(T^*\), then implies

\[
(T^{(1)} \otimes T^{(2)})^* f \otimes g = T^{(1)}f \otimes T^{(2)}g
\]

which is the desired result.

3.4. We have the following lemma.

**Lemma 3.4.1.** (i) If \(E^{(1)}\) is a projection on \(M^{(1)} \subseteq l_{p,n}\), and \(E^{(2)}\) a projection on \(M^{(2)} \subseteq l_{p,m}\), then \(E^{(1)} \otimes E^{(2)}\) is a projection of \(l_{p,mn}\) on \(M^{(1)} \otimes M^{(2)}\).

(ii) Let \(\phi_i^{(1)}\) and \(\psi_i^{(1)}\) (\(\phi_i^{(2)}\) and \(\psi_i^{(2)}\)) be in the same relation to \(E^{(1)}\) (\(E^{(2)}\)) as \(\phi_i\) and \(\psi_i\) are to \(E\) in Lemma 2.2.1, i.e.,

\[
E^{(1)}f = \sum_{i=1}^{k^{(1)}} (\psi_i^{(1)}, f)\phi_i^{(1)}; \quad \quad E^{(2)}f = \sum_{j=1}^{k^{(2)}} (\psi_j^{(2)}, f)\phi_j^{(2)}.
\]

If \(E\) is the transformation on \(l_{p,mn}\) defined by the equation
\[ Eh = \sum_{i=1}^{k^{(1)}} \sum_{j=1}^{k^{(2)}} (\psi_i^{(1)} \otimes \psi_j^{(2)}, h)\phi_i^{(1)} \otimes \phi_j^{(2)}, \]

then \( E \) is a projection of \( l_{p,m} \) on \( \mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)} \) and \( E = E^{(1)} \otimes E^{(2)} \).

(iii) \( E^* = E^{(1)^*} \otimes E^{(2)^*} \).

Since (ii) implies (i) and (ii) and Lemma 3.3.3 implies (iii), we need only prove (ii).

We have \((\psi_i^{(1)} \otimes \psi_j^{(2)}, \phi_i^{(1)} \otimes \phi_k^{(2)}) = (\psi_i^{(1)}, \phi_i^{(1)})(\psi_j^{(2)}, \phi_k^{(2)}) = \delta_i \delta_j \delta_k, \) by Lemma 3.1.1 and Lemma 2.2.1. By Lemma 3.2.1 the \( \phi_i^{(1)} \otimes \phi_k^{(2)} \) determine \( \mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)} \). Lemma 2.2.1 now implies that \( E \) is a projection on \( \mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)} \).

It remains to show that \( E = E^{(1)} \otimes E^{(2)} \). If \( f \in \mathcal{K}_{p,m}, g \in \mathcal{K}_{p,m}, \) then by Lemma 3.1.1, (ii) and (iv), and Lemma 2.2.1,

\[ (Ef \otimes g) = \sum_{i=1}^{k^{(1)}} \sum_{j=1}^{k^{(2)}} (\psi_i^{(1)} \otimes \psi_j^{(2)}, f \otimes g)\phi_i^{(1)} \otimes \phi_j^{(2)} \]

\[ = \sum_{i=1}^{k^{(1)}} \sum_{j=1}^{k^{(2)}} (\psi_i^{(1)}, f)(\psi_j^{(2)}, g)\phi_i^{(1)} \otimes \phi_j^{(2)} \]

\[ = \left( \sum_{i=1}^{k^{(1)}} (\psi_i^{(1)}, f)\phi_i^{(1)} \right) \otimes \left( \sum_{j=1}^{k^{(2)}} (\psi_j^{(2)}, g)\phi_j^{(2)} \right) = E^{(1)}f \otimes E^{(2)}g. \]

Lemma 3.3.2 now implies that \( E = E^{(1)} \otimes E^{(2)} \).

3.5. Next we prove

**Lemma 3.5.1.** If \( \mathcal{M}^{(1)} \) in \( l_{p,m} \) and \( \mathcal{M}^{(2)} \) in \( l_{p,m} \) are in Situation A (cf. §2.4), then \( \mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)} \) is in Situation A with

(a) \( \phi_{(s-1)k^{(1)}+t} = \phi_s^{(1)} \otimes \phi_t^{(2)}, \) \( t = 1, \ldots, k^{(1)}, \) \( s = 1, \ldots, k^{(2)}; \)

(b) \( \psi_{(s-1)k^{(1)}+t} = \psi_s^{(1)} \otimes \psi_t^{(2)}, \) \( t = 1, \ldots, k^{(1)}, \) \( s = 1, \ldots, k^{(2)}; \)

(c) \( C = C^{(1)}C^{(2)}; \)

(d) \( c_{(s-1)k^{(1)}+t} = c_s^{(1)}c_t^{(2)}, \) \( t = 1, \ldots, r^{(1)}; \) \( s = 1, \ldots, r^{(2)}. \)

That the \( \phi_s^{(1)} \otimes \phi_t^{(2)}, \) \( t = 1, \ldots, k^{(1)}, \) \( s = 1, \ldots, k^{(2)} \) are linearly independent and determine \( \mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)} \) has been shown in Lemma 3.1.3 and Lemma 3.2.1. The remaining statements of (a) were shown in the proof of Lemma 3.4.1.

To prove (b) we have \( h_t^{(1)} \otimes h_t^{(2)} \in \mathcal{K}^{(1)} \otimes \mathcal{K}^{(2)} \) by definition (cf., §3.2).

Also by Lemma 3.1.1 \( \| h_t^{(1)} \otimes h_t^{(2)} \| = \| h_t^{(1)} \| \cdot \| h_t^{(2)} \| = 1. \)

Now consider (c). If \( i = (s-1)r^{(1)}+t, \) then by Lemma 3.1.1, (ii); Lemma 3.4.1, (iii); Lemma 3.3.2; Lemma 3.1.1, (i); and §2.4, (c),
\[ \| E^* \{ h_1 \} p^{-1} \| = \| E^* \{ h_{1(1)} \otimes h_{2(2)} \} p^{-1} \| = \| E^* \{ h_{1(1)} \} p^{-1} \otimes \{ h_{2(2)} \} p^{-1} \| 
\]
\[ = \| E(1)* \otimes E(2)* \{ h_{1(1)} \} p^{-1} \otimes \{ h_{2(2)} \} p^{-1} \| 
\]
\[ = \| E(1)* \{ h_{1(1)} \} p^{-1} \otimes E(2)* \{ h_{2(2)} \} p^{-1} \| 
\]
\[ = \| E(1)* \{ h_{1(1)} \} p^{-1} \| \cdot \| E(2)* \{ h_{2(2)} \} p^{-1} \| . \]

We now prove (d). If \( h \in (M(1) \otimes M(2))^\perp \), then by Lemma 3.2.2, \( h \) is a linear combination of elements in the form \( f(1) \otimes g(2) \), where either \( f(1) \) is in \( M(1)^\perp \) or \( g(2) \) is in \( M(2)^\perp \). Hence since

\[
\sum_{r=1}^{r(1)+r(2)} c_i(\{ h_i \} p^{-1}, f(h, \{ E^* \{ h_i \} p^{-1} \} p^{-1} ) ,
\]

\( f \in M(1) \otimes M(2) \), \( h \in (M(1) \otimes M(2))^\perp \), is linear in \( h \), it is enough to show (d) for \( h \) in the form \( f(1) \otimes g(2) \), where either \( f(1) \) is in \( M(1)^\perp \) or \( g(2) \) is in \( M(2)^\perp \). It is also linear in \( f \); hence by §3.2.2 it suffices to show (d) for \( f \) in the form \( \phi(1) \otimes \phi(2) \), \( \phi_1 \in M(1) \), \( \phi_2 \in M(2) \).

Now it was shown in the proof of (c) above that \( E^* \{ h_i \} p^{-1} = E(1)* \{ h_{1(1)} \} p^{-1} \otimes E(2)* \{ h_{2(2)} \} p^{-1} \). Then by Lemma 3.1.1, (iii),

\[
\{ E^* \{ h_i \} p^{-1} \} p' = \{ E(1)* \{ h_{1(1)} \} p^{-1} \otimes E(2)* \{ h_{2(2)} \} p^{-1} \} p'
\]

and \( \{ h_i \} p^{-1} = \{ h_{1(1)} \} p^{-1} \otimes \{ h_{2(2)} \} p^{-1} \). Hence by Lemma 3.1.1, (ii), we see that

\[
\sum_{r=1}^{r(1)+r(2)} c_i(\{ h_i \} p^{-1}, \phi(1) \otimes \phi(2) )(f(1) \otimes g(2), \{ E^* \{ h_{1(1)} \} p^{-1} \} p' )
\]

\[ = \left( \sum_{r=1}^{r(1)} c_{i(1)} (\{ h_{1(1)} \} p^{-1}, \phi(1)) (f(1), \{ E(1)* \{ h_{1(1)} \} p^{-1} \} p' ) \right) \]

\[ \times \left( \sum_{r=1}^{r(2)} c_{i(2)} (\{ h_{2(2)} \} p^{-1}, \phi(2)) (g(2), \{ E(2)* \{ h_{2(2)} \} p^{-1} \} p' ) \right) . \]

Since either \( f(1) \in M(1)^\perp \), or \( g(2) \in M(2)^\perp \), this is zero for \( M(1) \) and \( M(2) \) are in Situation A.

3.6. Now it follows from Lemma 2.4.1 and Lemma 3.5.1, that we can show that \( \lim_{n \to \infty} (C(l_p,n)) = \infty \) if we can find a manifold \( M \) in \( l_p \) in Situation A (cf. §2.4, (c)). For let \( N \) be any integer >0. Then using Lemma 3.5.1 we can find a manifold \( M_N \) in \( l_p^{*n} \) in Situation A with \( C_{M_N} = C_{M_N}^N \). Lemma 2.4.1 now implies that \( C(M_N) \geq C_{M_N}^N \) and since

\[
C(l_p,n) \geq C(M_N), \quad \lim_{n \to \infty} (C(l_p,n)) = \infty .
\]

As we remarked in §1.3, this implies that the answer to both (a) and (b) is negative.
Let \( \mathcal{M} \) be the manifold in \( l_{\rho,s} \), determined by the vectors \((1,1,0)\) and \((0,1,1)\). Let
\[
\phi_1 = (2^{-1/\rho}, 2^{-1/\rho}, 0), \quad \phi_2 = (0, 2^{-1/\rho}, 2^{-1/\rho}),
\]
\[
\psi_1 = (2^{1/\rho+1/3}, 2^{1/\rho}/3, -2^{1/\rho}/3), \quad \psi_2 = (-2^{1/\rho}/3, 2^{1/\rho}/3, 2^{1/\rho+1/3});
\]
if \( \alpha = 1/(2+2\rho)^{1/\rho} \), \( h_1 = (\alpha, -\alpha, -2\alpha) \), \( h_2 = (\alpha, 2\alpha, \alpha) \), \( h_3 = (2\alpha, \alpha, -\alpha) \). Also
\[
C = ((2^{p'-1} + 1)/3)^{1/p'}((2^{p'-1} + 1)/3)^{1/p},
\]
\[
c_1 = c_2 = c_3 = 1.
\]

We show that \( \mathcal{M} \) is in Situation A (§2.4) and thus complete the proof.

We have (a) \( (\phi_i, \psi_i) = \delta_{i,j} \).

(b) \( h_1 = \alpha 2^{1/p}(\phi_1 - 2\phi_2) \), \( h_2 = \alpha 2^{1/p}(\phi_1 + \phi_2) \), \( h_3 = \alpha 2^{1/p}(2\phi_1 - \phi_2) \), and thus
\( h_i \in \mathcal{M}, i = 1, 2, 3 \). We also have \( ||h_i|| = 1, i = 1, 2, 3 \).

Before showing (c) and (d) we make certain calculations. From the definitions in §2.1, we get
\[
\{h_1\}^{-1} = \alpha^{-1}(1, -1, -2\rho^{-1}), \quad \{h_2\}^{-1} = \alpha^{-1}(1, 2\rho^{-1}, 1), \quad \{h_3\}^{-1} = (2^{p'-1}, 1, 1),
\]
\[
\{h_1\}^{-1} = (\{h_2\}^{-1}, \{h_3\}^{-1}) = (\{h_3\}^{-1}, \{h_1\}^{-1}, \{h_2\}^{-1}) = 2^{-1}(1 + 2^{p'-1})^{1/p}.
\]

By Lemma 2.3.1
\[
E^*\{h_1\}^{-1} = (\{h_1\}^{-1}, \phi_1)\psi_1 + (\{h_1\}^{-1}, \phi_2)\psi_2 = (\{h_1\}^{-1}, \phi_2)\psi_2
\]
\[
= -3^{-1/2^{1/p}(2^{p-1} + 1)^{1/p}}(1, 1, 1, 2),
\]
\[
E^*\{h_2\}^{-1} = (\{h_2\}^{-1}, \phi_1)\psi_1 + (\{h_2\}^{-1}, \phi_2)\psi_2 = 2^{-1}(2^{p-1} + 1)^{1/p}(\psi_1 + \psi_2)
\]
\[
= 3^{-1/2^{1/p}(2^{p-1} + 1)^{1/p}(1, 1, 1)}.
\]

Similarly
\[
E^*\{h_3\}^{-1} = 3^{-1/2^{1/p}(2^{p-1} + 1)^{1/p}(2, 1, -1)}.
\]

Finally (cf. §2.1)
\[
\{E^*\{h_1\}^{-1}\}^{-1} = -3^{-(p-1)/2}((p'-1)/p'-1)^{1/p'(2^{p-1} + 1)^{1/p}(1, 1, 2^{p'-1})}
\]
\[
= -K(1, 1, 2^{p'-1})
\]
\[
\{E^*\{h_2\}^{-1}\}^{-1} = K(1, 2^{p'-1}, 1)
\]
\[
\{E^*\{h_3\}^{-1}\}^{-1} = K(2^{p'-1}, 1, -1).
\]

(c) By direct calculation, we obtain \( ||E^*\{h_1\}^{-1}|| = ||E^*\{h_2\}^{-1}|| = ||E^*\{h_3\}^{-1}|| = C \) using the above. For \( \rho \neq 2 \), \( C \) is \( >1 \) since by the Hölder inequality,
\[
6C = (2^{p} + 2)^{1/p}(2^{p'} + 2)^{1/p'} \geq 2^{1/p} 2^{1/p} = 6,
\]
where the equality sign holds only for \( \rho = 2 \).

(d) Now if \( f \in \mathcal{M}^2 \), then \( f = k\phi \), where \( \phi = (1, -1, 1) \). Thus
We can now verify by a direct calculation that (d) holds.

Conclusion. Our results permit us to conclude that

There exists a manifold $\mathcal{M}_0$ in $L_p$ and $L_p$ such that there exists no biorthogonal set $\{\phi_i, \psi_i\}$ where $\{\phi_i\}$ is a basis for $\mathcal{M}_0$ (cf. (B), Chapter VII, p. 110, §3), while the expansion

\[ (*) \quad \sum_{i=1}^{\infty} a_i \phi_i, \quad a_i = (f, \psi_i), \]

converges for each $f \in L_p$ or $L_p$.

Let us suppose that (*) converges for every $f$. Let $\mathcal{M}$ be the manifold determined by the $\phi_i$'s. The $\phi_i$'s are a basis for $\mathcal{M}$ (cf. (B), loc. cit.) for if $f \in \mathcal{M}$, then

\[ f = \sum_{i=1}^{\infty} a_i \phi_i, \quad a_i = (f, \psi_i) \]

by (B), Chapter VII, Theorem 2, p. 107.

We will show that under these circumstances $C(\mathcal{M})$ is $< \infty$. For let $E$ be the transformation defined by the equation

\[ Ef = \sum_{i=1}^{\infty} (f, \psi_i) \phi_i. \]

$Ef$ is defined for every $f$ since we assume that the series is convergent for every $f$. The same assumption implies that $E$ is limited for the partial sums are uniformly limited (cf. (B), Chapter VII, Theorem 2 and Theorem 5). $E$ is obviously additive and homogeneous. If $f \in \mathcal{M}$, $Ef = f$ by the above and the range of $E$ is included in $\mathcal{M}$, $E\Lambda = \mathcal{M}$ and $E^2 = E$. Thus $E$ is a projection of $\Lambda$ on $\mathcal{M}$. Hence $C(\mathcal{M})$ is $< \infty$.

Our construction also permits us to show that no statement concerning the norms of the $\phi_i$ and $\psi_i$ will insure convergence by itself. We can assume that $\|\phi_i\| = 1$ for every $i$. The least possible value for $\|\psi_i\|$ is then 1 since $(\phi_i, \psi_i) = 1$ and $|\langle \phi_i, \psi_i \rangle| \leq \|\phi_i\| \cdot \|\psi_i\|$. We will show that there exists in both $l_p$ and $L_p$ a biorthogonal set $\{\phi_i, \psi_i\}$ for which $\|\phi_i\| = \|\psi_i\| = 1$ and for which the associated expansion (*) does not always converge.

It is a consequence of the proof of Lemma 1.3.1 that if such a set exists in $l_p$, there must be a similar one in $L_p$. So we need only consider $l_p$. Owing to the nature of biorthogonal sets in $l_p$, we need only consider the case $1 < p < 2$. 

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Let $\mathcal{M}$ be the manifold of §3.6, and let $f_1 = (2^{-1/p}, 2^{-1/p}, 0) = \phi_1, f_2 = (\alpha, -\alpha, -2\alpha) = h_1$. We have $(f_1, \{f_2\}^{p-1}) = (\|f_1\|^{p-1}, f_2) = 0; (f_1, \{f_2\}^{p-1}) = (f_2, \{f_1\}^{p-1}) = 1, \|f_1\| = \|f_2\| = \|f_1\|^{p-1} = \|f_2\|^{p-1} = 1$. Of course $f_1$ and $f_2 \in \ell_{p,3}, \{f_1\}^{p-1}$ and $\{f_2\}^{p-1} \in \ell_{p',3}$.

We define $f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_n}, i_i = 1, 2; j = 1, \ldots, n$, as an element of $\ell_{p,3^n}$ as follows: $f_{i_1} \otimes f_{i_2}$ in $\ell_{p,3}$ has already been defined (cf. §3.1). Let us suppose that $f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_{n-1}}$ in $\ell_{p,3^{n-1}}$ has been defined. We define $f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_{n-1}} \otimes f_{i_n}$ as $(f_{i_1} \otimes \cdots \otimes f_{i_{n-1}}) \otimes f_{i_n}$ in $\ell_{p,3^n}$ using §3.1. Let $\mathcal{M}_1 = \mathcal{M}, \mathcal{M}_n$ in $\ell_{p,3^n}$ be $\mathcal{M}_{n-1} \otimes \mathcal{M}$. Then by successive applications of Lemma 3.2.1, we see that the set of elements $f_{i_1} \otimes \cdots \otimes f_{i_n}$ determine $\mathcal{M}_n$.

By Lemma 1.1.2, $\mathcal{M} = \sum_{-\infty}^{\infty} \otimes \ell_{p,3^n}$. Let $\Psi$ be the closed linear manifold in $\sum_{-\infty}^{\infty} \otimes \ell_{p,3^n}$ consisting of those elements $\{g_1, g_2, \cdots \}$ for which $g_\alpha$ is $\in \mathcal{M}_\alpha$ for every $\alpha$. Let $S$ consist of those elements which are such that every $g_\alpha = 0$ except for one $g_\alpha$ and $g_\alpha = f_{i_1} \otimes \cdots \otimes f_{i_n}$. Let $S'$ consist of elements of $\ell_{p'}$ in the form $\{g\}^{p-1}, g \in S$. Since as we have seen above the $f_{i_1} \otimes \cdots \otimes f_{i_n}$ determine $\mathcal{M}_n, S$ determines $\Psi$.

Now the sets $S$ and $S'$ are denumerable and it is easily seen by using Lemma 3.1.1 that with a suitable enumeration they form a biorthogonal set with $\|\phi_\alpha\| = \|\psi_\alpha\| = 1$. Since $S$ determines $\Psi$, we see from the above that this series (*) cannot converge always if $C(\mathcal{M}) = \infty$.

Let $C$ be as in §3.6, (c). By repeated applications of Lemma 3.5.1 and then using Lemma 2.4.1, one may prove that $C(\mathcal{M}_n) \geq C_\alpha$. It follows from the proof of Lemma 1.2.3 that $C(\Psi)$ is $\geq C(\mathcal{M}_n) \geq C_\alpha$ for every $n$ and since $C$ is $>1, p \neq 2$ this implies that $C(\Psi) = \infty$. As we have remarked above, this proves our statement.

Incidently we have explicitly constructed a manifold $\Psi$ in $\ell_p$, for which there exists no projection. Lemma 1.3.1 indicates how we can find a $\Psi$ in $\ell_p$ with the corresponding property.

In $\ell_p$, the space of complex-valued functions whose $p$th power is summable, the situation is the same. As pointed out in a previous paper by the writer,† the theorems given in (B) and used here can be generalized to the complex case. Chapter 1 of this paper also falls into this category. Some variations are necessary in Chapters 2 and 3 but they are not basic.

Finally it should be pointed out that the negative answer to (a) and (b) precludes the possibility of a spectral theory in $\ell_p$ and $L_p$ similar to the theory of self-adjoint operators in Hilbert space.

† These Transactions, vol. 39 (1936), pp. 83–100.

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