A CORRECTION†

BY

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I have discovered an error (and some minor misstatements) in my paper A local solution of the difference equation $\Delta y(x) = F(x)$ and of related equations (these Transactions, vol. 39 (1936), pp. 345–379). It is my purpose here to rectify that error. The vulnerable point occurs from relation (7.1) to Theorem 7.1. In order to secure convergence for series (7.2), it was asserted that the Mittag-Leffler theorem could be applied to (7.6), giving a meromorphic function $Z(x; x^*)$. Now the “poles” of this function are at the points

$$x = x^* + \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j),$$

where $n_2, \ldots, n_k$ range independently from $-\infty$ to $+\infty$. But there is no reason to suppose (as I did) that these “poles” have no limit points in the finite plane.† If there are finite limit points, then we cannot conclude that $L[Z]$ is an entire function, and therefore we cannot apply Theorem 6.6 (Carmichael’s theorem). Consequently, Theorem 7.1 will not follow from the argument given.

It becomes necessary to rewrite §7. Fortunately it is possible to give a rigorous treatment which is simpler than the old. We now indicate in what way the old paper is to be revised.

(1) In Theorem 4.4, in place of “\ldots and in this circle \(y(x)\) \ldots” read “\ldots and in the lens-region \(y(x)\) \ldots.”

(2) In Theorem 4.6, in place of “\ldots which in this circle satisfies \ldots” read “\ldots which in a lens-region about \(x = \alpha\) satisfies \ldots.”

(3) §5 should have as heading: “5. A geometric lemma.”

(4) Omit all of §5 beginning with the following line (just preceding Lemma 5.2: “We turn now to two lemmas \ldots” (This portion is now unnecessary.)

(5) In §6 omit the two paragraphs shortly after (6.17), beginning with “To treat the general case, in which \(F(x)\) \ldots,” and ending with “This implies no loss of generality of equation (6.3).” (This portion is now unnecessary.)

(6) §7 is to be replaced by the following:

† Received by the editors, June 27, 1936.

‡ For certain values of $n_2, \ldots, n_k$, the coefficient $b_{n_2, \ldots, n_k}$ is zero, so that the corresponding “pole” does not actually occur. Conceivably, enough “poles” may be absent so that there are no finite limit points, but there seems to be no way of enumerating the “poles” that are present.
7. The general case. To treat the general case where \( F(x) \) is merely analytic, we modify the method of §4. Instead of directly finding a solution of the homogeneous equation \( L[y] = 0 \), we shall obtain more than one meromorphic solution of equation

\[
(7.1) \quad L[y] = \frac{1}{x - \alpha}.
\]

Let \( C \) be the unique circle, center at \( x^* \) and radius \( \rho^* \), assured us by Lemma 5.1. As a consequence of uniqueness there are seen to be precisely two possibilities: Either

Case I. There are at least two points \( P_i \) on \( C \), and of these at least one pair, say \( P_1, P_2 \), are diametrically opposite. Or,

Case II. There are at least three points \( P_i \) on \( C \); of these no two are diametrically opposite, but at least one triad of them (say \( P_1, P_2, P_3 \)) forms an acute-angled triangle.

In Case I draw circles of radius \( \sigma \) (any number greater than \( \rho^* \)) with centers at \( P_1, P_2 \). These circles will form a lens-region enclosing \( x^* \), and as \( \sigma \to \rho^* \) the lens-region (including boundary) will close down on \( x^* \) as a unique limit point. That is, given any neighborhood \( \mathcal{R} \) of \( x^* \), there exists a \( \sigma \) such that the lens-region of radius \( \sigma \) will lie with its boundary wholly interior to \( \mathcal{R} \).

Now consider Case II. A simple geometric argument will show that of \( P_1, P_2, P_3 \), there will be one, say \( P_1 \), such that: (i) \( P_2 \) and \( P_3 \) are on opposite sides of the diameter \( d_i \) through \( P_1 \); and (ii) if \( d_i' \) is the diameter perpendicular to \( d_i \), then \( P_2 \) and \( P_3 \) are on that side of \( d_i' \) opposite \( P_1 \). Now let \( \sigma \) be any number greater than \( \rho^* \), and draw circles of radius \( \sigma \) and centers \( P_1, P_2, P_3 \). These circles form a curvilinear triangle and another simple geometric argument will show that as \( \sigma \to \rho^* \) this triangle closes down on the single point \( x^* \). Hence again, if \( \mathcal{R} \) is any neighborhood of \( x^* \), there is a \( \sigma \) such that the curvilinear triangle of radius \( \sigma \) lies with its boundary wholly inside \( \mathcal{R} \).

As a solution of (7.1) assume the series

\[
(7.2) \quad Y_1(x; \alpha) = \sum_{n_2, \ldots, n_k=0}^{+\infty} \frac{b_{n_2, \ldots, n_k}}{x - (\alpha + \omega_1) - \sum_{j=2}^{k} n_j(\omega_1 - \omega_j)}.\]

On substituting into (7.1) we get

\[
\frac{1}{x - \alpha} = \sum_{n_2, \ldots, n_k=0}^{+\infty} \alpha_1 b_{n_2, \ldots, n_k} + \sum_{j=2}^{k} \alpha_j b_{n_2, \ldots, n_{j-1}, n_j-1, n_{j+1}, \ldots, n_k}
\frac{\alpha_j b_{n_2, \ldots, n_k}}{x - \alpha - \sum_{j=2}^{k} n_j(\omega_1 - \omega_j)}.
\]
where all b's with a negative subscript are zero. This condition is formally fulfilled if we choose the b's to satisfy

\[
\alpha_1 b_{n_2 \ldots n_k} + \sum_{j=2}^{k} \alpha_j b_{n_2, \ldots, n_{j-1}, n_{j-1}, n_{j+1}, \ldots, n_k} = \begin{cases} 1, & \text{for } n_2 = \cdots = n_k = 0; \\ 0, & \text{for } n_2, \ldots, n_k = 0, 1, 2, \ldots \text{ (but not all zero)}. \end{cases}
\] (7.3)

We find from \((n_2, \ldots, n_k) = (0, \ldots, 0)\) that \(b_0, \ldots, 0 = 1/a_1\). Then on choosing \(n_2 + \cdots + n_k = 1\) in all possible ways we find that the \(b_{n_2 \ldots n_k}\)'s \((n_2 + \cdots + n_k = 1)\) are uniquely determined; then the \(b_{n_2 \ldots n_k}\)'s for \(n_2 + \cdots + n_k = 2\); etc. That is, there is a unique set of b's for which relations (7.3) hold. These values we choose for the coefficients in (7.2).

(7.2) is a formal series; there is no reason to suppose that it converges. But if we examine its formal “poles,” namely, the points

\[ x = \alpha + \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j), \]

we find that they have no limit point in the finite plane.† The classic theorem of Mittag-Leffler is therefore applicable. Set \(u = x - \alpha\), so that

\[
Y_1(x; \alpha) = \sum_{n_2, \ldots, n_k = 0}^{+\infty} \frac{b_{n_2 \ldots n_k}}{u - \omega_1 - \sum_{j=2}^{k} n_j(\omega_1 - \omega_j)}.
\]

Then there exist polynomials \(P_{n_2 \ldots n_k}(u)\) such that

\[
(7.4) \quad Z_1(u) = \sum_{n_2, \ldots, n_k = 0}^{+\infty} \left( \frac{b_{n_2 \ldots n_k}}{u - \omega_1 - \sum_{j=2}^{k} n_j(\omega_1 - \omega_j)} + P_{n_2 \ldots n_k}(u) \right)
\]

defines a meromorphic function whose only poles are at

\[ u = \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j) \]

(with corresponding residues \(b_{n_2 \ldots n_k}\)), the series converging uniformly and

† For consider the point \(P_1\) (i.e., \(-\omega_1\)), which lies on \(C\). If \(L_1\) is the tangent line to \(C\) at \(P_1\), then \(P_2, \ldots, P_k\) all lie on the same side of \(L_1\). Hence if we start at \(P_1\) and lay off vectors \(\sum_{j=2}^{k} n_j(\omega_1 - \omega_j)\), we see (since \(n_j \geq 0\)) that the ends of the vectors all lie on this same side of \(L_1\) (save for \(n_k = \cdots = n_k = 0\)). If we take components of these vectors in the direction perpendicular to the line \(L_1\), we see that the ends of the vectors go off to infinity as any \(n_j\) becomes infinite, so that no finite limit point is possible.
absolutely in every bounded region (the poles in this region being deleted).

On applying the operator $L$ term-wise to the series for $Z_1$ (as is permissible) we obtain

$$L[Z_1(x - \alpha)] = \sum_{n_2 \cdots n_k=0}^{+\infty} \left( \frac{A_{n_2 \cdots n_k}}{x - \alpha - \sum_{i=2}^{k} n_i(\omega_1 - \omega_i)} + L[P_{n_2 \cdots n_k}(x - \alpha)] \right)$$

where $A_{n_2 \cdots n_k}$ is the left member of (7.3), so that

$$(7.5) \quad L[Z_1(x - \alpha)] = \frac{1}{x - \alpha} + \sum_{n_2 \cdots n_k=0}^{+\infty} L[P_{n_2 \cdots n_k}(x - \alpha)].$$

The right-hand side converges for all $x$ (the point $x = \alpha$ is singular only for the term $1/(x - \alpha)$). That is, the series on the right is an entire function. Now by Theorem 6.6 the equation

$$(7.6) \quad L[g_1(u)] = \sum_{n_2 \cdots n_k=0}^{+\infty} L[P_{n_2 \cdots n_k}(u)]$$

has an entire function solution $g_1(u)$. Consequently

$$(7.7) \quad W_1(u) = Z_1(u) - g_1(u)$$

is a meromorphic function satisfying the equation

$$(7.8) \quad L[W_1(x - \alpha)] = \frac{1}{x - \alpha};$$

its only poles are simple poles at the points

$$x = \alpha + \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j) \quad (n_2, \ldots, n_k = 0, 1, 2, \ldots),$$

and the corresponding residues are $b_{n_2 \cdots n_k}$ as given by (7.3).

In the above work concerning $Y_1$, $Z_1$, and $W_1$, the point $P_1$ (i.e., the number $-\omega_1$) was preferred over all the others. But in Case I point $P_2$ is also on $C$, and in Case II points $P_2$, $P_3$ are on $C$; and these points may equally well be used as was $P_1$. We thus get, according to the case, one or two unique formal series

$$Y_2(x; \alpha) = \sum_{n_1, n_2, \ldots, n_k=0}^{+\infty} \frac{c_{n_1 n_2 \cdots n_k}}{x - (\alpha + \omega_2) - \sum_{j=1,3,\ldots,k}^{k} n_j(\omega_2 - \omega_j)};$$

† The $c$'s and $d$'s satisfy recurrence relations similar to (7.3).
\[ Y_3(x; \alpha) = \sum_{n_1, n_2, n_4, \ldots, n_k=0}^{+\infty} \frac{d_{n_1n_2n_4\ldots n_k}}{x - (\alpha + \omega_3) - \sum_{j=1,2,4,\ldots,k} n_j(\omega_3 - \omega_j)}; \]

one or two functions \(Z_2(x - \alpha), Z_3(x - \alpha)\); and one or two meromorphic functions \(W_2(x - \alpha), W_3(x - \alpha)\) satisfying (respectively)

\[ L\left[W_2(x - \alpha)\right] = \frac{1}{x - \alpha}, \quad L\left[W_3(x - \alpha)\right] = \frac{1}{x - \alpha}, \]

with respective simple poles at

\[ x = \alpha + \omega_2 + \sum_{j=1,2,\ldots,k} n_j(\omega_2 - \omega_j), \quad x = \alpha + \omega_3 + \sum_{j=1,2,4,\ldots,k} n_j(\omega_3 - \omega_j), \]

and residues \(c_{n_1n_2\ldots n_k}d_{n_1n_2n_4\ldots n_k}\).

Case I. Here \(P_1, P_2\) are diametrically opposite on \(C\). Let \(\sigma\) be only slightly larger than \(\rho^*\), and with radius \(\sigma\) and centers \(P_1, P_2\), draw a lens-region \(\mathcal{L}\) around \(x^*\). Let the bounding arcs of \(\mathcal{L}\) be \(C_1, C_2\) (\(C_i\) being that arc with center at \(P_i\)). Let \(\alpha\) remain on \(C_1\). Since the point \(x^* + \omega_1\) is where the point \(P_2\) would be if \(C\) were translated so that \(x^*\) falls at the origin, it is seen that as \(\alpha\) traverses \(C_1\), \(\alpha + \omega_1\) will trace a small arc (in the neighborhood of \(x^* + \omega_1\)) of a circle of radius \(\sigma\) and center the origin. Hence from our knowledge of the position of the poles

\[ x = \alpha + \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j), \]

we can say, if \(\sigma\) is sufficiently close to \(\rho^*\), that \(W_1(x - \alpha)\) is analytic about the origin in a circle of radius exceeding \(\rho^*\). That is,

\[ W_1(x - \alpha) = \sum_{n=0}^{\infty} B_{1n}(\alpha)x^n, \quad (7.9) \]

where the \(B_{1n}(\alpha)\) are analytic functions in the neighborhood of \(\alpha = x^*\) (and in particular for \(\alpha\) on \(C_1\)), and where there is a number \(\sigma_1 > \rho^*, \text{ independent of} \alpha\) on \(C_1\), such that (7.9) converges uniformly for \(\alpha\) on \(C_1\) and \(x\) in \(|x| \leq \sigma_1\).

An analogous statement applies to \(W_2(x - \alpha)\) for \(\alpha\) on \(C_2\):

\[ W_2(x - \alpha) = \sum_{n=0}^{\infty} B_{2n}(\alpha)x^n, \quad (7.10) \]

uniformly convergent for \(\alpha\) on \(C_2\) and \(x\) in \(|x| \leq \sigma_2\), where \(\sigma_2\) is some number exceeding \(\rho^*\). (The \(B_{2n}(\alpha)\) are analytic functions in a neighborhood of \(x^*\) containing \(C_2\).)
Case II. $P_1, P_2, P_3$ are on $C$. Again choosing $\sigma$ only slightly larger than $\rho^*$, we obtain a curvilinear triangle $\mathcal{C}$ of radius $\sigma$ by drawing arcs $C_1, C_2, C_3$ with centers $P_1, P_2, P_3$. For $\alpha$ on $C_i$ the argument used in Case I applies, giving us (7.9), (7.10) or

\begin{equation}
W_3(x - \alpha) = \sum_{n=0}^{\infty} B_{3n}(\alpha)x^n,
\end{equation}

uniformly convergent for $\alpha$ on $C_3$ and $x$ in $|x| \leq \sigma_3$, where $\sigma_3$ is some number greater than $\rho^*$. (The $B_{3n}(\alpha)$ are analytic in a neighborhood of $x^*$ containing $C_3$.)

For any value of $i$ ($i=1$ or 2 in Case I and 1, 2, or 3 in Case II),

\[ \lim \sup |B_{in}(\alpha)|^{1/n} \leq 1/\sigma_i \leq 1/\sigma < 1/\rho^* \]

where $\sigma$ = smallest of $\sigma_1, \sigma_2, \sigma_3$ and $\alpha$ is on $C_i$. It follows from Theorem 6.2 and Corollary 6.1 that the series

\[ \sum_{n=0}^{\infty} B_{in}(\alpha)A_n(x) \]

converges uniformly for $x$ in some curvilinear polygon $\mathcal{P}$ about $x^*$ and $\alpha$ on $C_i$, where $\mathcal{P}$ can be chosen independent of $i$. But this series is what we get when we apply $L$ term-wise to the $W_i(x - \alpha)$ series. From this follows

**Theorem 7.1.** According to the case, if a lens-region $\mathcal{R}$ or a curvilinear triangle $\mathcal{C}$ be drawn† about $x^*$, with radius $\sigma$ sufficiently near to $\rho^*$, then

\begin{equation}
\frac{1}{x - \alpha} = \sum_{n=0}^{\infty} B_{jn}(\alpha)A_n(x), \quad \begin{cases} j = 1, 2 & \text{in Case I}, \\ j = 1, 2, 3 & \text{in Case II}, \end{cases}
\end{equation}

the convergence being uniform in $x$ and $\alpha$ for $\alpha$ on $C_i$ and $x$ in some neighborhood $\mathcal{R}$ of $x^*$. ($\mathcal{R}$ can be chosen independent of $i$.)

Now let $F(x)$ be analytic about $x = x^*$. Then there exists a lens or triangle $\mathcal{R}$ around $x^*$ lying (together with its boundary) wholly in the region of analyticity of $F(x)$, and with radius so close to $\rho^*$ that Theorem 7.1 applies. Let $x$ be in the region $\mathcal{R}$ of Theorem 7.1. Multiply (7.11) by $F(\alpha)$ and integrate over $\mathcal{J}$. This gives

\begin{equation}
F(x) = \sum_{n=0}^{\infty} f_nA_n(x),
\end{equation}

where

\begin{equation}
f_n = -\sum_{j=1,2} \frac{1}{2\pi i} \int_{C_j} F(\alpha)B_{jn}(\alpha)d\alpha; \quad \text{or} \quad f_{j=1,2,3}
\end{equation}

and (7.12) converges uniformly in $\mathcal{R}$. We thus have

† The points $P_i$ will of course be the centers of the arcs forming $\mathcal{R}$.
Theorem 7.2. If $F(x)$ is analytic about $x=x^*$, it has a convergent $A_n$-expansion, given by (7.12).

Combining this with Theorem 6.1:

Theorem 7.3. A necessary and sufficient condition that a function $F(x)$ have an (convergent) $A_n$-expansion is that it be analytic at $x=x^*$.

By Theorem 6.2, $\lim \sup |f_n|^{1/n} < 1/\rho^*$, so that the series

$$y(x) = \sum_{n=0}^{\infty} f_n x^n$$

converges in $|x| < \rho^* + \epsilon$, for some $\epsilon > 0$. On applying $L$ to (7.17) we get

$$L[y(x)] = \sum f_n L[x^n] = \sum f_n A_n(x) = F(x),$$

so that we have

Theorem 7.4. If $F(x)$ is analytic about $x=x^*$, then the function $y(x)$ of (7.14) is analytic in a circle (about $x=0$) of radius greater than $\rho^*$, and for all $x$ in a sufficiently small curvilinear polygon (about $x=x^*$) $y(x)$ satisfies the equation

$$(6.3) \quad L[y(x)] = F(x).$$

The point $x^*$ is of course significant for $A_n$-expansions, but not for equation (6.3). For let $F(x)$ be analytic about $x=c$, and define $G(x) = F(x+c-x^*)$. $G(x)$ is analytic about $x=x^*$ and therefore there exists a function $z(x)$, analytic at $x^*$, such that $L[z(x)] = G(x)$. Consequently, the function $y(x) = z(x-c+x^*)$ satisfies $L[y(x)] = F(x)$, and we have the final

Theorem 7.5. If $F(x)$ is analytic about $x=c$, there exists a function $y(x)$, analytic about $x=c-x^*$ in a circle of radius exceeding $\rho^*$, such that for all $x$ in a sufficiently small neighborhood (curvilinear polygon) of $x=c$, $y(x)$ satisfies equation (6.3).

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