

# THE BINET OF QUADRICS IN $S_3$ \*

BY

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1. **Introduction.** The concept of a linear system of quadrics was introduced by Dupin in 1808.† He considered the pencil determined by two concentric central quadrics. Ten years later, Lamé‡ defined the net of quadrics. The web of quadrics was defined and considered first by de Jonquières§ in 1862. These systems of quadrics have since been treated by many mathematicians.||

A binet of quadrics is an  $\infty^4$  linear system of quadric surfaces in  $S_3$ . The term binet is suggested for this system merely because it contains twice the number of essential parameters contained in a net. The system is not in any other sense a “double net.”

The binet has not been treated as such, but only as implied in certain polar and apolar relations associated with a general linear system of surfaces.

The purpose of this paper is to determine the characteristics of a binet of quadrics in  $S_3$ .

2. **The binet.** The equation of a binet of quadrics in  $S_3$  is

$$\sum \lambda_i f_i = 0, \quad i = 1, 2, 3, 4, 5,$$

wherein the  $f_i = 0$  are the equations of five quadrics in  $S_3$ . These five quadrics are not restricted or related in any way.

The binet contains  $\infty^3$ ,  $\infty^2$ ,  $\infty^1$ , finite systems of surfaces, both linear and non-linear, satisfying one, two, three, four conditions respectively. Linear systems will be discussed in §4 and non-linear systems in §8 and §9.

The two non-linear  $\infty^3$  systems, however, will be defined now because their locus is to be used in developing the theory. One is the system of quadric cones and the other, the system of pencils of quadrics with one contact.

Since it is one condition for a quadric to be a cone, the binet contains  $\infty^3$  cones; that is, any point in  $S_3$  is the vertex of a finite number of cones of the

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† Dupin, *Journal de l'École Polytechnique*, vol. 14 (1808), p. 80.

‡ Lamé, *Examen* (1818), pp. 29 and 37. On pp. 28 and 35 of the same paper, Lamé also wrote the equation of a general pencil of quadrics in the form  $f + \lambda g = 0$ .

§ de Jonquières, *Journal de Mathématiques*, (2), vol. 7 (1862), p. 412.

|| *Encyklopädie der Mathematischen Wissenschaften*, vol. III<sub>2</sub>, pp. 212–223, 246–250, 250–254.

Pascal, *Repertorium der höhere Mathematik*, vol. II<sub>2</sub> (1922), pp. 616–626, 626–628, 629–631.

(In both references, the three sets of pages refer to pencils, nets, webs, respectively.)

binet. Also, since it is one condition for two quadrics to have one contact and since these two quadrics determine a pencil of quadrics, all of which have contact with each other at that point, the binet contains  $\infty^3$  such pencils and any point of  $S_3$  is the point of contact of the quadrics in a finite number of pencils. The jacobian of a binet of quadrics in  $S_3$ , i.e., the locus of the vertices of cones and the locus of simple contacts of the binet, is, then, the space  $S_3$  containing the binet.

3. **The associated correspondence.** The (1, 1) correspondence between the primes of  $S_4$  and the quadrics of the binet is defined by the equations,

$$\rho y_i = f_i, \quad i = 1, 2, 3, 4, 5,$$

in which  $y_i=0$  are the equations of the primes of the coordinate pentaprime in  $S_4$  and  $f_i=0$  are any five quadrics of the binet.

To a plane of  $S_4$  corresponds an elliptic quartic curve, the intersection of two quadrics of the binet and the basis curve of a pencil of quadrics of the binet. To a line of  $S_4$  corresponds 8 points of intersection of three quadrics or the basis points of a net of quadrics of the binet.

In the transformation from  $S_3$  to  $S_4$ , the image of a plane of  $S_3$  is a rational quartic surface of  $S_4$  and the image of a line of  $S_3$  is a conic of  $S_4$ .

4. **Linear systems of the binet.** To the  $\infty^6$  planes of  $S_4$ , each considered as bearing a pencil of primes, correspond  $\infty^6$  pencils of quadrics of the binet. Each pencil has an elliptic quartic as basis curve and contains four quadric cones of the binet.

To the  $\infty^6$  lines of  $S_4$ , each considered as bearing a bundle of primes, correspond  $\infty^6$  nets of quadrics of the binet. Each net of quadrics has 8 basis points. Each net also has a jacobian curve  $\gamma_i$  of order 6, rank 16 and genus 3 with only apparent double points. This jacobian curve is the locus of nodes and contacts of quadrics of the net.

Each line of  $S_4$  also bears a bundle of planes, the intersections of the primes of the bundle. To the  $\infty^6$  lines of  $S_4$ , therefore, considered as bearing bundles of planes, correspond  $\infty^6$  nets of elliptic quartic space curves, each net of curves being the intersections of pairs of quadrics of the associated net of quadrics.

If two quadrics of a net  $F_1$  are tangent at a point  $P_1$ , their curve of intersection has a node at  $P_1$ . The locus of  $P_1$ , considered as contacts of pencils of quadrics of  $F_1$ , is the jacobian curve  $\gamma_1$  of  $F_1$ . The locus of the same point  $P_1$ , considered as a node on the quartic curves associated with  $F_1$ , is the jacobian curve of the net of quartic curves. Therefore each  $\gamma_i$  is the jacobian, both of a net of quadric surfaces and of the net of associated quartic curves.

To the  $\infty^4$  points of  $S_4$ , each point considered as bearing  $\infty^3$  primes, cor-

respond  $\infty^4$  webs of quadrics of the binet. Each web of quadrics has a jacobian surface  $J$  of order 4, genus 1 and containing ten lines, the axes of the ten pairs of planes of the web.

Each point of  $S_4$  also bears  $\infty^4$  planes and  $\infty^3$  lines, the intersections of the  $\infty^3$  primes through that point. To the  $\infty^4$  planes corresponds a binet of elliptic quartic curves and to the  $\infty^3$  lines correspond  $\infty^3$  sets of 8 points, belonging to  $\infty^3$  nets of quadrics of the binet. Thus there are  $\infty^4$  binets of quartic curves contained in the binet of quadrics. However, the jacobians of all the webs of quartic curves in the binet of curves associated with the point  $P_1$  are one and the same jacobian  $J_1$ , the jacobian of the web  $W_1$  of quadrics associated with  $P_1$ , since  $J_1$  is the locus of contacts of quadrics of  $W_1$ , and a curve of the associated binet can have a node when and only when two quadrics of  $W_1$  have a contact.

Since two points of  $S_4$  have a line in common, any two webs of primes in  $S_4$  have a bundle of primes in common and, correspondingly, in  $S_3$ , any two webs of quadrics have, in common, a net of quadrics. The Jacobian surfaces of two webs intersect in a curve of order 16 consisting of the curve  $\gamma$  of order 6 and genus 3, the jacobian of the net common to the two webs, and a curve  $\phi$  of order 10 which is common to the  $\infty^4$  jacobians  $J_i$  of the webs of the binet. The jacobian quartic surfaces  $J_i$ , therefore, form a binet with  $\phi$  as basis curve. This basis curve  $\phi$  is of order 10, genus 11, rank 40 and has 25 apparent double points. Each jacobian curve  $\gamma$  intersects  $\phi$  in 20 points.

Since three points of  $S_4$  determine a plane, any three webs of quadrics in  $S_3$  have a pencil of quadrics in common. The three  $J_i$  of the three webs intersect in 64 points. Of these points, 4 are the vertices of the four cones of the common pencil and the other 60 are basis points common to all the  $J_i$  of the binet of jacobians. It has been shown that these  $J_i$  have a basis curve  $\phi$  of order 10 and rank 40. The equivalence of  $\phi$  on three  $J_i$  is 60. Therefore the basis curve  $\phi$  absorbs the 60 basis points and the binet of quartic surfaces  $J_i$  has the curve  $\phi$  as its only basis element.

5. **The locus of axes of composite quadrics.** Let  $f_0$  be a quartic of the binet that consists of two planes intersecting in the line  $l$ . Through  $l$  passes a web of jacobian quartics  $J_i$ , the jacobians of the  $\infty^3$  webs of quadrics which have  $f_0$  in common. Since this web of jacobian quartics  $J_i$  has a net in common with any other web of jacobian quartics, it follows that the locus of the axes of composite quadrics of the binet is a ruled surface which is a fixed component of the jacobians of all the webs of  $J_i$ .

The jacobian of a web of quartic surfaces  $J_i$  with  $\phi$  as basis curve, is a surface of order 12 containing  $\phi$  as a triple curve. This surface consists of a quadric of the binet and a ruled surface  $R$  of order 10 containing  $\phi$  as a triple

curve.  $R$  is the locus of the  $\infty^1$  lines which are axes of composite quadrics of the binet. The genus of a plane section of  $R$  is 6.

Each jacobian  $J$  intersects  $R$  in a curve of order 40 consisting of  $\phi$  counted three times and the 10 lines lying on that jacobian.

The two systems of surfaces, the  $\infty^4$  webs of quadrics with no basis elements and the  $\infty^4$  webs of  $J_i$  with the basis curve  $\phi$ , are thus so related that, aside from the common surface  $R$ , the jacobian of a web of either system is a surface of the other system.

6. The branch-point primal  $L$  in  $S_4$ . In nets and webs of quadrics, the branch-point primal is the image of the jacobian of the system of quadrics. This is also true in the case of a binet. The jacobian of the binet is the linear space  $S_3$  containing the binet.  $L$  is, therefore, in (1, 1) correspondence with  $S_3$  and is rational.  $L$  is of order 8, the number of intersections of three quadrics of  $S_3$ .

The complete image of  $L$  is  $S_3$ . Since the primes of  $S_4$  are in (1, 1) correspondence with the quadrics of the binet, the birational image of a prime section of  $L$  is a quadric of the binet. Then the prime sections of  $L$  are also rational.

To a tangent prime of  $L$  corresponds a cone of the binet, the point of contact in  $S_4$  corresponding to the vertex of the quadric cone in  $S_3$ . To any point  $P_0$  in  $S_4$ , considered as bearing a web of primes, corresponds a web  $W_0$  of quadrics in  $S_3$ . The locus of the vertices of the cones of  $W_0$  is  $J_0$ . The locus of contacts of the tangent primes to  $L$  from  $P_0$  is the contour surface  $\Lambda_0$  on  $L$  of the tangent cone from  $P_0$ .  $\Lambda_0$  is in (1, 1) correspondence with  $J_0$  and is, therefore, the image of  $J_0$  on  $L$ . Since  $J_0$  is of order 4,  $\Lambda_0$  is of order 16 and therefore the order of the tangent cone to  $L$  from a point of  $S_4$  is 16. Also, since the  $J_i$  form a binet in  $S_3$ , the image surfaces  $\Lambda_i$  form a binet on  $L$ . Each  $J_i$  is of genus 1. Then each  $\Lambda_i$  is of genus 1 and the tangent cones to  $L$  from a point are of genus 1.

A section by any prime  $\pi_0$  of the tangent cone to  $L$  from  $P_0$  is a surface  $L_0$ , the branch-point surface of the web of quadrics corresponding to the web of primes on  $P_0$ . The tangent planes to  $L_0$  are sections by  $\pi_0$  of the tangent primes to  $L$  from  $P_0$ .

The following symbols will be used for the characteristics of a surface, the subscripts 3 and 0 referring to the prime section of  $L$  and of the tangent cone to  $L$  respectively:  $n$ , order;  $n'$ , class;  $a'$ , class of plane section;  $a$ , order of tangent cone;  $b$ , order of nodal curve;  $c$ , order of cuspidal curve;  $\delta'$ , number of bitangents of plane section;  $\kappa'$ , number of inflections of plane section;  $\delta$ , number of nodal lines of tangent cone;  $\kappa$ , number of cuspidal lines of tangent cone;  $q$ , rank of nodal curve;  $r$ , rank of cuspidal curve;  $b'$ , class of

bitangential developable;  $c'$ , class of parabolic developable;  $\gamma$ , number of intersections of nodal and cuspidal curve, cusps on nodal curve;  $\beta$ , number of intersections of nodal and cuspidal curve, cusps on cuspidal curve;  $t$ , number of triple points of nodal curve;  $\beta'$ , number of tacnodal tangent planes;  $\gamma'$ , number of nodo-cuspidal tangent planes;  $t'$ , number of triple tangent planes;  $\rho$ , class of nodal developable;  $\sigma$ , class of cuspidal developable;  $\rho'$ , order of bitangential curve;  $\sigma'$ , order of parabolic curve;  $C'$ , number of conic tropes;  $D$ , genus.

Since the prime section  $L_0$  of the tangent cone to  $L$  from any point  $P_0$  of  $S_4$  is the branch-point surface associated with the web  $W_0$ ,  $L_0$  has the following characteristics:\*

$$\begin{aligned} n_0 &= 16, & n'_0 &= 4, & a'_0 &= a_0 = 12, & D_0 &= 1, & C'_0 &= 10, & \delta'_0 &= 22, & \kappa'_0 &= 24, \\ b_0 &= 60, & q_0 &= 40, & \gamma_0 &= 120, & t_0 &= 80, \\ c_0 &= 36, & r_0 &= 68, & \beta_0 &= 80, \\ \delta_0 &= 28, & \kappa_0 &= 24, & \sigma_0 &= 32, & \rho_0 &= 80, \\ b'_0 &= c'_0 = \beta'_0 = \gamma'_0 = t'_0 = \sigma'_0 = \rho'_0 = 0. \end{aligned}$$

Each 4-space surface  $\Lambda_0$  is the projection on  $L$  from  $P_0$  of the 3-space surface  $L_0$ .

To the bundle of primes through any line  $l_1$  of  $S_4$  corresponds a net of quadrics  $F_1$  of the binet. The locus of the vertices of cones in  $F_1$  is the curve  $\gamma_1$ . Since the contacts of the  $\infty^1$  tangent primes to  $L$  from  $l_1$  are in (1, 1) correspondence with the vertices on  $\gamma_1$  of the cones of  $F_1$ , the contour curve  $\Gamma_1$  of the cone (of the second species) from  $l_1$  to  $L$  is in (1, 1) correspondence with  $\gamma_1$ . These curves  $\Gamma_i$  form an  $\infty^6$  linear system (or biweb) on  $L$ .

The plane section of the tangent cone to  $L$  from  $l_1$  made by any plane of  $S_4$  is the branch-point curve  $G_1$  in that plane, corresponding birationally to the jacobian curve  $\gamma_1$  of  $F_1$ . The plane curve  $G_1$  has the characteristics of the plane section of the 3-space tangent cone to  $L_0$ , the branch-point surface of a web. Then  $G_1$  has the characteristics:

$$\begin{array}{lll} \text{order } n_1 = 12, & \text{nodes } \delta_1 = 28, & \text{cusps } \kappa_1 = 24, \\ \text{class } m_1 = 4, & \text{bitangents } \tau_1 = 0, & \text{inflections } \iota_1 = 0. \end{array}$$

Each 4-space curve  $\Gamma_i$  on  $L$  is the projection on  $L$  from  $l_i$  of the plane curve  $G_i$ .

The plane section of  $L$  made by any plane of  $S_4$  is the birational image of an elliptic quartic curve, the intersection of two quadrics of the binet. This plane section is of order 8 and genus 1. Its class equals the order of the tan-

\* T. R. Hollcroft, *The general web of algebraic surfaces of order  $n$  and the involution defined by it*, these Transactions, vol. 35 (1933), p. 868.

gent cone to  $L$  from a point, which is 16. The remaining characteristics of the plane section are obtained by Plücker's equations. The characteristics are:

$$\begin{array}{llll} \text{order } n_2 = 8, & \text{class } m_2 = 16, & \text{genus } p_2 = 1, & \text{nodes } b_2 = 20, \\ \text{cusps } c_2 = 0, & \text{bitangents } \delta_2 = 80, & \text{inflections } \kappa_2 = 24. & \end{array}$$

The prime section of  $L$  made by any prime of  $S_4$  is the birational image of a quadric of the binet and is, therefore, of order 8 and genus 0. The class of an  $S_3$  section equals the order of the tangent cone to  $L$  from a line, which is 12. The characteristics of a plane section of the  $S_3$  section are the same as the characteristics of a plane section of  $L$ .

Since the nodal curve of the prime section is of order 20, the nodal surface  $b$  of  $L$  is of order 20. This nodal surface has a pinch curve  $C_j$  whose image in  $S_3$  is the basis curve  $\phi$  of the jacobian binet. The curves  $C_j$  and  $\phi$  are in (1, 1) correspondence. Since  $\phi$  is of order 10 and genus 11,  $C_j$  is of order 20 and genus 11. The curve  $C_j$  is cut by an  $S_3$  in 20 points which are  $j_3$  pinch points of the nodal curve of the prime section.

The remaining characteristics of the prime section of  $L$  are obtained from the above characteristics by means of the Cayley-Zeuthen equations.

The characteristics of the prime section of  $L$  are:

$$\begin{array}{llll} n_3 = 8, & n'_3 = 12, & a_3 = a'_3 = 16, & D_3 = 0, \\ \delta'_3 = 80, & \kappa'_3 = 24, & c_3 = r_3 = \beta_3 = \gamma_3 = 0, & \\ b_3 = 20, & q_3 = 50, & k_3 = 105, & t_3 = 20, \quad j_3 = 20, \\ b'_3 = 22, & c'_3 = 24, & \delta_3 = 60, & \kappa_3 = 36, \\ \sigma'_3 = 32, & \sigma_3 = 0, & \rho'_3 = 72, & \rho_3 = 60, \\ \gamma'_3 = 0, & \beta'_3 = 44, & t'_3 = 24, & \\ q'_3 = 50, & r'_3 = 60, & k'_3 = 134. & \end{array}$$

In the above,  $k_3$  is the number of apparent double points of the nodal curve and  $q'_3, r'_3, k'_3$  are the reciprocals of the respective unaccented symbols already defined.

The remaining characteristics of the primal  $L$  will now be derived.

Since a quadric cannot have two or more distinct nodes or a singular point of higher order than a node,  $L$  does not have tangent primes with contact at more than one distinct point or with higher contact at one point. This is also revealed by the characteristics of the tangent cones to  $L$  in that there are no stationary or bitangent primes to  $L$  from a point or from a line. Thus, for  $L$ , the orders of the bitangential surface, the parabolic surface and their associated curves are all zero as are also the numbers of tangent primes

satisfying four conditions whose contacts would occur at intersections of these surfaces and curves.

Also, since  $c_3 = 0$  for the prime section,  $L$  has no cuspidal curve.

A prime section  $L_0$  of the tangent cone to  $L$  from any point  $P_0$  has  $C'_0 = 10$  conic tropes, i.e., ten planes, each tangent to  $L_0$  along a conic. Each conic trope corresponds uniquely to a composite quadric of the web  $W_0$ . The image of the plane  $\alpha_0$  of a conic trope of  $L_0$  is a composite quadric  $f_0$  of  $W_0$ . The image of the axis of  $f_0$ , which is also a line of  $J_0$ , is the contact conic  $c'_0$  of the conic trope.

The prime section of  $L$  by the prime  $\pi_1$  determined by  $P_0$  and  $\alpha_0$  corresponds to the composite quadric  $f_0$  considered as a quadric of the binet. The prime  $\pi_1$  is tangent to  $L$  along a conic  $c_0$  and intersects  $L$  in two rational quartic surfaces which are tangent to each other along  $c_0$ . The two quartic surfaces are the images respectively of the two planes of  $f_0$  and  $c_0$  is the image of the axis of  $f_0$ .

The image of  $R$ , the locus of axes of composite quadrics of the binet, is the surface  $T$  on  $L$ .  $T$  is of order 20 and has the pinch curve  $C_j$  of order 20 as a triple curve.  $T$  is a surface generated by the singly infinite non-linear system of conics which are in (1, 1) correspondence with the axes of composite quadrics of the binet. Through each point of  $S_4$  pass ten primes, each of the nature of  $\pi_1$  described in the preceding paragraph.  $T$  counts doubly in the intersection of  $L$  and its hessian.

The jacobian of the web of prime sections of  $L$  from a point  $P_0$  is the contour surface  $\Lambda_0$  of order 16 of the tangent cone from  $P_0$  to  $L$ .  $\Lambda_0$  contains  $C_j$  as a simple curve. Since  $S_4$  contains  $\infty^4$  points, the contour surfaces  $\Lambda_i$  form a binet on  $L$  for which  $C_j$  is a simple curve. The jacobian of the web of  $\Lambda_i$  associated with the points of any prime of  $S_4$  consists of the surface  $T$  and the prime section of  $L$  made by that same prime.  $T$  is, then, common to the jacobians of all the  $\infty^4$  webs of  $\Lambda_i$  of the binet of  $\Lambda_i$  on  $L$ .  $T$  is the locus of contacts of the contour surfaces  $\Lambda_i$ .

From the characteristics of  $L$  already found, the others are obtained by means of formulas for primals in four dimensions derived by Roth.\* The characteristics of  $L$ , other than those of its cones and sections, are, therefore, as follows:

Order  $N = 8$ ; rank  $a = 16$ ; first class  $m = 12$ , class  $w = 4$ ; order of nodal surface  $b$ ,  $\mu_0 = 20$ ; order of tangent cone of  $b$ ,  $\mu_1 = 50$ ; class of  $b$ ,  $\mu_2 = 20$ ; order apparent double curve† of  $b$ ,  $k = 105$ ; number of apparent triple points of  $b$

\* L. Roth, *Some formulae for primals in four dimensions*, Proceedings of the London Mathematical Society, (2), vol. 35 (1933), pp. 540–552, and vol. 39 (1935), pp. 334–338.

† The order of the apparent double curve of a surface in  $S_4$  is here defined as the order of the double curve of the projection of this surface on  $S_3$ .

and of its apparent double curve,  $T_1 = 140$ ; order of triple curve  $C_t$  of  $b$ ,  $t = 20$ ; rank of  $C_t$ ,  $r_t = 60$ ; class of immersion of  $C_t$  in  $L$ ,  $t_1 = 80$ ; class of immersion of  $C_t$  in  $b$ ,  $t_0 = 140$ ; number of quadruple points of  $C_t$  and  $L$  (6-fold on  $b$ ),  $\xi = 5$ ; order of pinch curve  $C_j$ ,  $j = 20$ ; rank of  $C_j$ ,  $r_j = 60$ ; class of immersion of  $C_j$  in  $L$ ,  $j_1 = 40$ ; class of immersion of  $C_j$  in  $b$ ,  $j_0 = 80$ ; number of pinch points of  $b$  (actual intersections of  $C_t$  and  $C_j$ ),  $\tau = 40$ ; number of apparent pinch points of  $b$  (ceto of  $L$ ),  $\nu_2 = 20$ ; order of curve  $C_\rho$ , the locus of contacts of tangent primes from any point whose contacts lie on the nodal surface  $b$ ,  $\rho = 60$ ; order of surface  $T$  on  $L$  ( $T$  is the image of  $R$  in  $S_3$ ),  $T = 20$ .

7. Images in  $S_3$  of  $L$  and its singularities. The image of  $L$  is  $S_3$ . The images of the prime sections of  $L$  are the quadrics of the binet in  $S_3$ .

The birational image of the pinch curve  $C_j$  (of order 20 and genus 11) of the nodal surface  $b$  of  $L$  is the basis curve  $\phi$  (of order 10 and genus 11) of the binet of jacobian surfaces  $J_i$  in  $S_3$ .

The image of the ruled surface  $R$  of order 10, the locus of axes of composite quadrics of the binet, is  $T$ , a surface generated by conics of order 20 on  $L$ .  $R$  contains  $\phi$  as a triple curve and  $T$  contains  $C_j$  as a triple curve.

The image of the nodal surface  $b$  (of order 20) of  $L$  is a surface  $B$  of order 10.  $B$  contains  $\phi$  as a simple curve.  $B$  also contains the double curve  $C'_t$ , a curve of order 30 and the image of the triple curve  $C_t$ . The surfaces  $b$  and  $B$  are in (1, 2) point correspondence.

The curves  $\phi$  and  $C'_t$  in  $S_3$  intersect in  $\tau = 40$  points, images of the 40 intersections of  $C_j$  and  $C_t$  on  $b$ , which are actual pinch points of  $b$ .

The surfaces  $B$  and  $R$  intersect in a curve of order 100 consisting of  $\phi$  counted three times and a  $C_{70}$ , the image of the curve of intersection of order 140 (in addition to  $C_j$ ) of  $b$  and  $T$  on  $L$ .

$R$  and any  $J_i$  intersect in a curve of order 40 consisting of  $\phi$  counted three times and the ten lines of  $J_i$ . Correspondingly, on  $L$ ,  $T$  and any  $\Lambda_i$  intersect in a curve of order 80, consisting of  $C_j$  counted three times and ten conics, images of the ten lines of  $J_i$ .

$B$  and any  $J_i$  intersect in a curve of order 40 consisting of  $\phi$  and a  $C_{30}$ , corresponding respectively to  $C_j$  and  $C_\rho$  (of order  $\rho = 60$ ) and comprising the total intersection of  $b$  and the associated  $\Lambda_i$  on  $L$ .  $C_\rho$  is the locus of points on  $b$  at which primes through the point associated with the given  $J_i$  are tangent to  $L$ .

8. Singular quadrics of the binet. The only singularities a quadric may have are a node and a binode. One condition is necessary for the quadric to have a node, in which case the quadric becomes the nodal cone. Three conditions are necessary for the quadric to break up into two planes, i.e., be the planes and axis of a binode.

The binet, therefore, contains  $\infty^3$  quadric cones, the locus of whose ver-

tices (nodes) is  $S_3$ ; and  $\infty^1$  composite quadrics, the locus of whose axes (line of nodes) is the ruled surface  $R$  of order 10 containing  $\phi$  as a triple curve.

9. **Loci of contacts of quadrics of the binet.** Every point of  $S_3$  is a simple contact of the quadrics of a pencil of the binet.

The images in  $S_3$  of singular surfaces, curves and points of  $L$  are defined in §7. With the exception of  $R$ , these images are contact loci of the binet.

The surface  $B$  is the locus of contacts of pencils of quadrics of the binet in each of which the quadrics have contact at two distinct points.  $B$ , the image of  $b$ , is of order 10 and contains  $\phi$  as a simple curve and  $C'_i$  as a double curve. The curve  $\phi$  is the locus of contacts of pencils of quadrics such that in each pencil the quadrics have four-point contact at one point.  $C'_i$  is the locus of contacts of pencils of quadrics in each of which the quadrics have contact at three distinct points.

$C'_i$  and  $\phi$  intersect in 40 points at each of which two of the three contacts of a pencil of quadrics coincide. The third contact associated with each such coincidence is on  $C'_i$  but not on  $\phi$ . Then, in the binet, there are 40 pencils of quadrics in each of which the quadrics have one simple and one four-point contact.

There are  $\xi=5$  quadruple points of  $C_i$  and of  $L$ . To each of these correspond four distinct points on  $C'_i$ , which are nodes of  $C'_i$ , such that the quadrics of a pencil have simple contact at each of these four points. Therefore, the binet of quadrics contains five pencils in each of which the quadrics have four distinct contacts.

$R$  and  $B$  intersect in  $\phi$  counted three times and in a  $C_{70}$ , the locus of contacts of pencils of quadrics with two contacts and such that one quadric of each pencil is composite and also such that the two planes of the composite quadric are the common tangent planes respectively of the quadrics at the two contacts.

The binet has no locus of stationary contacts of quadrics and no locus (surface, curve, or set of points) of contacts that involve stationary contacts.\* Every point of  $S_3$ , however, is a stationary contact of a pencil of quadrics belonging to a web of quadrics of the binet.

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\* A somewhat similar situation occurs in a net of conics. The net of conics contains nine pencils in each of which the conics osculate each other, but no pencils with pairs of contacts.