THE BINET OF QUADRICS IN S_3*

BY

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1. **Introduction.** The concept of a linear system of quadrics was introduced by Dupin in 1808.† He considered the pencil determined by two concentric central quadrics. Ten years later, Lamé‡ defined the net of quadrics. The web of quadrics was defined and considered first by de Jonquières§ in 1862. These systems of quadrics have since been treated by many mathematicians.‖

A binet of quadrics is an ∞^4 linear system of quadric surfaces in S_3. The term binet is suggested for this system merely because it contains twice the number of essential parameters contained in a net. The system is not in any other sense a "double net."

The binet has not been treated as such, but only as implied in certain polar and apolar relations associated with a general linear system of surfaces.

The purpose of this paper is to determine the characteristics of a binet of quadrics in S_3.

2. **The binet.** The equation of a binet of quadrics in S_3 is

$$\sum \lambda_i f_i = 0, \qquad i = 1, 2, 3, 4, 5,$$

wherein the $f_i = 0$ are the equations of five quadrics in S_3. These five quadrics are not restricted or related in any way.

The binet contains ∞^3, ∞^2, ∞^1, finite systems of surfaces, both linear and non-linear, satisfying one, two, three, four conditions respectively. Linear systems will be discussed in §4 and non-linear systems in §8 and §9.

The two non-linear ∞^3 systems, however, will be defined now because their locus is to be used in developing the theory. One is the system of quadric cones and the other, the system of pencils of quadrics with one contact.

Since it is one condition for a quadric to be a cone, the binet contains ∞^3 cones; that is, any point in S_3 is the vertex of a finite number of cones of the

* Presented to the Society, September 3, 1936; received by the editors July 16, 1936.

† Dupin, Journal de l'École Polytechnique, vol. 14 (1808), p. 80.

‡ Lamé, Examen (1818), pp. 29 and 37. On pp. 28 and 35 of the same paper, Lamé also wrote the equation of a general pencil of quadrics in the form $f+\lambda g=0$.

§ de Jonquières, Journal de Mathématiques, (2), vol. 7 (1862), p. 412.

‖ Encyklopädie der Mathematischen Wissenschaften, vol. III_2, pp. 212–223, 246–250, 250–254. Pascal, Repertorium der höhere Mathematik, vol. II_2 (1922), pp. 616–626, 626–628, 629–631. (In both references, the three sets of pages refer to pencils, nets, webs, respectively.)

binet. Also, since it is one condition for two quadrics to have one contact and since these two quadrics determine a pencil of quadrics, all of which have contact with each other at that point, the binet contains ∞^3 such pencils and any point of S_3 is the point of contact of the quadrics in a finite number of pencils. The jacobian of a binet of quadrics in S_3, i.e., the locus of the vertices of cones and the locus of simple contacts of the binet, is, then, the space S_3 containing the binet.

3. **The associated correspondence.** The (1, 1) correspondence between the primes of S_4 and the quadrics of the binet is defined by the equations,

$$\rho y_i = f_i, \qquad i = 1, 2, 3, 4, 5,$$

in which $y_i=0$ are the equations of the primes of the coordinate pentaprime in S_4 and $f_i=0$ are any five quadrics of the binet.

To a plane of S_4 corresponds an elliptic quartic curve, the intersection of two quadrics of the binet and the basis curve of a pencil of quadrics of the binet. To a line of S_4 corresponds 8 points of intersection of three quadrics or the basis points of a net of quadrics of the binet.

In the transformation from S_3 to S_4, the image of a plane of S_3 is a rational quartic surface of S_4 and the image of a line of S_3 is a conic of S_4.

4. **Linear systems of the binet.** To the ∞^6 planes of S_4, each considered as bearing a pencil of primes, correspond ∞^6 pencils of quadrics of the binet. Each pencil has an elliptic quartic as basis curve and contains four quadric cones of the binet.

To the ∞^6 lines of S_4, each considered as bearing a bundle of primes, correspond ∞^6 nets of quadrics of the binet. Each net of quadrics has 8 basis points. Each net also has a jacobian curve γ_i of order 6, rank 16 and genus 3 with only apparent double points. This jacobian curve is the locus of nodes and contacts of quadrics of the net.

Each line of S_4 also bears a bundle of planes, the intersections of the primes of the bundle. To the ∞^6 lines of S_4, therefore, considered as bearing bundles of planes, correspond ∞^6 nets of elliptic quartic space curves, each net of curves being the intersections of pairs of quadrics of the associated net of quadrics.

If two quadrics of a net F_1 are tangent at a point P_1, their curve of intersection has a node at P_1. The locus of P_1, considered as contacts of pencils of quadrics of F_1, is the jacobian curve γ_1 of F_1. The locus of the same point P_1, considered as a node on the quartic curves associated with F_1, is the jacobian curve of the net of quartic curves. Therefore each γ_i is the jacobian, both of a net of quadric surfaces and of the net of associated quartic curves.

To the ∞^4 points of S_4, each point considered as bearing ∞^3 primes, correspond ∞^4 webs of quadrics of the binet. Each web of quadrics has a jacobian surface J of order 4, genus 1 and containing ten lines, the axes of the ten pairs of planes of the web.

Each point of S_4 also bears ∞^4 planes and ∞^3 lines, the intersections of the ∞^3 primes through that point. To the ∞^4 planes corresponds a binet of elliptic quartic curves and to the ∞^3 lines correspond ∞^3 sets of 8 points, belonging to ∞^3 nets of quadrics of the binet. Thus there are ∞^4 binets of quartic curves contained in the binet of quadrics. However, the jacobians of all the webs of quartic curves in the binet of curves associated with the point P_1 are one and the same jacobian J_1, the jacobian of the web W_1 of quadrics associated with P_1, since J_1 is the locus of contacts of quadrics of W_1, and a curve of the associated binet can have a node when and only when two quadrics of W_1 have a contact.

Since two points of S_4 have a line in common, any two webs of primes in S_4 have a bundle of primes in common and, correspondingly, in S_3, any two webs of quadrics have, in common, a net of quadrics. The Jacobian surfaces of two webs intersect in a curve of order 16 consisting of the curve γ of order 6 and genus 3, the jacobian of the net common to the two webs, and a curve ϕ of order 10 which is common to the ∞^4 jacobians J_i of the webs of the binet. The jacobian quartic surfaces J_i, therefore, form a binet with ϕ as basis curve. This basis curve ϕ is of order 10, genus 11, rank 40 and has 25 apparent double points. Each jacobian curve γ intersects ϕ in 20 points.

Since three points of S_4 determine a plane, any three webs of quadrics in S_3 have a pencil of quadrics in common. The three J_i of the three webs intersect in 64 points. Of these points, 4 are the vertices of the four cones of the common pencil and the other 60 are basis points common to all the J_i of the binet of jacobians. It has been shown that these J_i have a basis curve ϕ of order 10 and rank 40. The equivalence of ϕ on three J_i is 60. Therefore the basis curve ϕ absorbs the 60 basis points and the binet of quartic surfaces J_i has the curve ϕ as its only basis element.

5. **The locus of axes of composite quadrics.** Let f_0 be a quartic of the binet that consists of two planes intersecting in the line l. Through l passes a web of jacobian quartics J_i, the jacobians of the ∞^3 webs of quadrics which have f_0 in common. Since this web of jacobian quartics J_i has a net in common with any other web of jacobian quartics, it follows that the locus of the axes of composite quadrics of the binet is a ruled surface which is a fixed component of the jacobians of all the webs of J_i.

The jacobian of a web of quartic surfaces J_i with ϕ as basis curve, is a surface of order 12 containing ϕ as a triple curve. This surface consists of a quadric of the binet and a ruled surface R of order 10 containing ϕ as a triple curve. R is the locus of the ∞^1 lines which are axes of composite quadrics of the binet. The genus of a plane section of R is 6.

Each jacobian J intersects R in a curve of order 40 consisting of ϕ counted three times and the 10 lines lying on that jacobian.

The two systems of surfaces, the ∞^4 webs of quadrics with no basis elements and the ∞^4 webs of J_i with the basis curve ϕ, are thus so related that, aside from the common surface R, the jacobian of a web of either system is a surface of the other system.

6. **The branch-point primal L in S_4.** In nets and webs of quadrics, the branch-point primal is the image of the jacobian of the system of quadrics. This is also true in the case of a binet. The jacobian of the binet is the linear space S_3 containing the binet. L is, therefore, in (1, 1) correspondence with S_3 and is rational. L is of order 8, the number of intersections of three quadrics of S_3.

The complete image of L is S_3. Since the primes of S_4 are in (1, 1) correspondence with the quadrics of the binet, the birational image of a prime section of L is a quadric of the binet. Then the prime sections of L are also rational.

To a tangent prime of L corresponds a cone of the binet, the point of contact in S_4 corresponding to the vertex of the quadric cone in S_3. To any point P_0 in S_4, considered as bearing a web of primes, corresponds a web W_0 of quadrics in S_3. The locus of the vertices of the cones of W_0 is J_0. The locus of contacts of the tangent primes to L from P_0 is the contour surface Λ_0 on L of the tangent cone from P_0. Λ_0 is in (1, 1) correspondence with J_0 and is, therefore, the image of J_0 on L. Since J_0 is of order 4, Λ_0 is of order 16 and therefore the order of the tangent cone to L from a point of S_4 is 16. Also, since the J_i form a binet in S_3, the image surfaces Λ_i form a binet on L. Each J_i is of genus 1. Then each Λ_i is of genus 1 and the tangent cones to L from a point are of genus 1.

A section by any prime π_0 of the tangent cone to L from P_0 is a surface L_0, the branch-point surface of the web of quadrics corresponding to the web of primes on P_0. The tangent planes to L_0 are sections by π_0 of the tangent primes to L from P_0.

The following symbols will be used for the characteristics of a surface, the subscripts 3 and 0 referring to the prime section of L and of the tangent cone to L respectively: n, order; n', class; a', class of plane section; a, order of tangent cone; b, order of nodal curve; c, order of cuspidal curve; δ', number of bitangents of plane section; κ', number of inflections of plane section; δ, number of nodal lines of tangent cone; κ, number of cuspidal lines of tangent cone; q, rank of nodal curve; r, rank of cuspidal curve; b', class of bitangential developable; c', class of parabolic developable; γ, number of intersections of nodal and cuspidal curve, cusps on nodal curve; β, number of intersections of nodal and cuspidal curve, cusps on cuspidal curve; t, number of triple points of nodal curve; β', number of tacnodal tangent planes; γ', number of nodo-cuspidal tangent planes; t', number of triple tangent planes; ρ, class of nodal developable; σ, class of cuspidal developable; ρ', order of bitangential curve; σ', order of parabolic curve; C', number of conic tropes; D, genus.

Since the prime section L_0 of the tangent cone to L from any point P_0 of S_4 is the branch-point surface associated with the web W_0, L_0 has the following characteristics:*

$$n_0 = 16, \quad n_0' = 4, \quad a_0' = a_0 = 12, \quad D_0 = 1, \quad C_0' = 10, \quad \delta_0' = 22, \quad \kappa_0' = 24,$$
$$b_0 = 60, \quad q_0 = 40, \quad \gamma_0 = 120, \quad t_0 = 80,$$
$$c_0 = 36, \quad r_0 = 68, \quad \beta_0 = 80,$$
$$\delta_0 = 28, \quad \kappa_0 = 24, \quad \sigma_0 = 32, \quad \rho_0 = 80,$$
$$b_0' = c_0' = \beta_0' = \gamma_0' = t_0' = \sigma_0' = \rho_0' = 0.$$

Each 4-space surface Λ_0 is the projection on L from P_0 of the 3-space surface L_0.

To the bundle of primes through any line l_1 of S_4 corresponds a net of quadrics F_1 of the binet. The locus of the vertices of cones in F_1 is the curve γ_1. Since the contacts of the ∞^1 tangent primes to L from l_1 are in (1, 1) correspondence with the vertices on γ_1 of the cones of F_1, the contour curve Γ_1 of the cone (of the second species) from l_1 to L is in (1, 1) correspondence with γ_1. These curves Γ_i form an ∞^6 linear system (or biweb) on L.

The plane section of the tangent cone to L from l_1 made by any plane of S_4 is the branch-point curve G_1 in that plane, corresponding birationally to the jacobian curve γ_1 of F_1. The plane curve G_1 has the characteristics of the plane section of the 3-space tangent cone to L_0, the branch-point surface of a web. Then G_1 has the characteristics:

order $n_1=12$,	nodes $\delta_1=28$,	cusps $\kappa_1=24$,
class $m_1=4$,	bitangents $\tau_1=0$,	inflections $\iota_1=0$.

Each 4-space curve Γ_i on L is the projection on L from l_i of the plane curve G_i.

The plane section of L made by any plane of S_4 is the birational image of an elliptic quartic curve, the intersection of two quadrics of the binet. This plane section is of order 8 and genus 1. Its class equals the order of the tan-

* T. R. Hollcroft, *The general web of algebraic surfaces of order n and the involution defined by it*, these Transactions, vol. 35 (1933), p. 868.

gent cone to L from a point, which is 16. The remaining characteristics of the plane section are obtained by Plücker's equations. The characteristics are:

order $n_2=8$, class $m_2=16$, genus $p_2=1$, nodes $b_2=20$,
cusps $c_2=0$, bitangents $\delta_2=80$, inflections $\kappa_2=24$.

The prime section of L made by any prime of S_4 is the birational image of a quadric of the binet and is, therefore, of order 8 and genus 0. The class of an S_3 section equals the order of the tangent cone to L from a line, which is 12. The characteristics of a plane section of the S_3 section are the same as the characteristics of a plane section of L.

Since the nodal curve of the prime section is of order 20, the nodal surface b of L is of order 20. This nodal surface has a pinch curve C_j whose image in S_3 is the basis curve ϕ of the jacobian binet. The curves C_j and ϕ are in (1, 1) correspondence. Since ϕ is of order 10 and genus 11, C_j is of order 20 and genus 11. The curve C_j is cut by an S_3 in 20 points which are j_3 pinch points of the nodal curve of the prime section.

The remaining characteristics of the prime section of L are obtained from the above characteristics by means of the Cayley-Zeuthen equations.

The characteristics of the prime section of L are:

$$
\begin{array}{lllll}
n_3 = 8, & n_3' = 12, & a_3 = a_3' = 16, & D_3 = 0, & \\
\delta_3' = 80, & \kappa_3' = 24, & c_3 = r_3 = \beta_3 = \gamma_3 = 0, & & \\
b_3 = 20, & q_3 = 50, & k_3 = 105, & t_3 = 20, & j_3 = 20, \\
b_3' = 22, & c_3' = 24, & \delta_3 = 60, & \kappa_3 = 36, & \\
\sigma_3' = 32, & \sigma_3 = 0, & \rho_3' = 72, & \rho_3 = 60, & \\
\gamma_3' = 0, & \beta_3' = 44, & t_3' = 24, & & \\
q_3' = 50, & r_3' = 60, & k_3' = 134. & &
\end{array}
$$

In the above, k_3 is the number of apparent double points of the nodal curve and q_3', r_3', k_3' are the reciprocals of the respective unaccented symbols already defined.

The remaining characteristics of the primal L will now be derived.

Since a quadric cannot have two or more distinct nodes or a singular point of higher order than a node, L does not have tangent primes with contact at more than one distinct point or with higher contact at one point. This is also revealed by the characteristics of the tangent cones to L in that there are no stationary or bitangent primes to L from a point or from a line. Thus, for L, the orders of the bitangential surface, the parabolic surface and their associated curves are all zero as are also the numbers of tangent primes satisfying four conditions whose contacts would occur at intersections of these surfaces and curves.

Also, since $c_3=0$ for the prime section, L has no cuspidal curve.

A prime section L_0 of the tangent cone to L from any point P_0 has $C_0' = 10$ conic tropes, i.e., ten planes, each tangent to L_0 along a conic. Each conic trope corresponds uniquely to a composite quadric of the web W_0. The image of the plane α_0 of a conic trope of L_0 is a composite quadric f_0 of W_0. The image of the axis of f_0, which is also a line of J_0, is the contact conic c_0' of the conic trope.

The prime section of L by the prime π_1 determined by P_0 and α_0 corresponds to the composite quadric f_0 considered as a quadric of the binet. The prime π_1 is tangent to L along a conic c_0 and intersects L in two rational quartic surfaces which are tangent to each other along c_0. The two quartic surfaces are the images respectively of the two planes of f_0 and c_0 is the image of the axis of f_0.

The image of R, the locus of axes of composite quadrics of the binet, is the surface T on L. T is of order 20 and has the pinch curve C_j of order 20 as a triple curve. T is a surface generated by the singly infinite non-linear system of conics which are in $(1, 1)$ correspondence with the axes of composite quadrics of the binet. Through each point of S_4 pass ten primes, each of the nature of π_1 described in the preceding paragraph. T counts doubly in the intersection of L and its hessian.

The jacobian of the web of prime sections of L from a point P_0 is the contour surface Λ_0 of order 16 of the tangent cone from P_0 to L. Λ_0 contains C_j as a simple curve. Since S_4 contains ∞^4 points, the contour surfaces Λ_i form a binet on L for which C_j is a simple curve. The jacobian of the web of Λ_i associated with the points of any prime of S_4 consists of the surface T and the prime section of L made by that same prime. T is, then, common to the jacobians of all the ∞^4 webs of Λ_i of the binet of Λ_i on L. T is the locus of contacts of the contour surfaces Λ_i.

From the characteristics of L already found, the others are obtained by means of formulas for primals in four dimensions derived by Roth.* The characteristics of L, other than those of its cones and sections, are, therefore, as follows:

Order $N=8$; rank $a=16$; first class $m=12$, class $w=4$; order of nodal surface b, $\mu_0=20$; order of tangent cone of b, $\mu_1=50$; class of b, $\mu_2=20$; order apparent double curve† of b, $k=105$; number of apparent triple points of b

* L. Roth, *Some formulae for primals in four dimensions*, Proceedings of the London Mathematical Society, (2), vol. 35 (1933), pp. 540–552, and vol. 39 (1935), pp. 334–338.

† The order of the apparent double curve of a surface in S_4 is here defined as the order of the double curve of the projection of this surface on S_3.

and of its apparent double curve, $T_1=140$; order of triple curve C_t of b, $t=20$; rank of C_t, $r_t=60$; class of immersion of C_t in L, $t_1=80$; class of immersion of C_t in b, $t_0=140$; number of quadruple points of C_t and L (6-fold on b), $\xi=5$; order of pinch curve C_j, $j=20$; rank of C_j, $r_j=60$; class of immersion of C_j in $L, j_1=40$; class of immersion of C_j in $b, j_0=80$; number of pinch points of b (actual intersections of C_t and C_j), $\tau=40$; number of apparent pinch points of b (ceto of L), $\nu_2=20$; order of curve C_ρ, the locus of contacts of tangent primes from any point whose contacts lie on the nodal surface b, $\rho=60$; order of surface T on L (T is the image of R in S_3), $T=20$.

7. **Images in S_3 of L and its singularities.** The image of L is S_3. The images of the prime sections of L are the quadrics of the binet in S_3.

The birational image of the pinch curve C_j (of order 20 and genus 11) of the nodal surface b of L is the basis curve ϕ (of order 10 and genus 11) of the binet of jacobian surfaces J_i in S_3.

The image of the ruled surface R of order 10, the locus of axes of composite quadrics of the binet, is T, a surface generated by conics of order 20 on L. R contains ϕ as a triple curve and T contains C_j as a triple curve.

The image of the nodal surface b (of order 20) of L is a surface B of order 10. B contains ϕ as a simple curve. B also contains the double curve C_t', a curve of order 30 and the image of the triple curve C_t. The surfaces b and B are in (1, 2) point correspondence.

The curves ϕ and C_t' in S_3 intersect in $\tau=40$ points, images of the 40 intersections of C_j and C_t on b, which are actual pinch points of b.

The surfaces B and R intersect in a curve of order 100 consisting of ϕ counted three times and a C_{70}, the image of the curve of intersection of order 140 (in addition to C_j) of b and T on L.

R and any J_i intersect in a curve of order 40 consisting of ϕ counted three times and the ten lines of J_i. Correspondingly, on L, T and any Λ_i intersect in a curve of order 80, consisting of C_j counted three times and ten conics, images of the ten lines of J_i.

B and any J_i intersect in a curve of order 40 consisting of ϕ and a C_{30}, corresponding respectively to C_j and C_ρ (of order $\rho=60$) and comprising the total intersection of b and the associated Λ_i on L. C_ρ is the locus of points on b at which primes through the point associated with the given J_i are tangent to L.

8. **Singular quadrics of the binet.** The only singularities a quadric may have are a node and a binode. One condition is necessary for the quadric to have a node, in which case the quadric becomes the nodal cone. Three conditions are necessary for the quadric to break up into two planes, i.e., be the planes and axis of a binode.

The binet, therefore, contains ∞^3 quadric cones, the locus of whose vertices (nodes) is S_3; and ∞^1 composite quadrics, the locus of whose axes (line of nodes) is the ruled surface R of order 10 containing ϕ as a triple curve.

9. **Loci of contacts of quadrics of the binet.** Every point of S_3 is a simple contact of the quadrics of a pencil of the binet.

The images in S_3 of singular surfaces, curves and points of L are defined in §7. With the exception of R, these images are contact loci of the binet.

The surface B is the locus of contacts of pencils of quadrics of the binet in each of which the quadrics have contact at two distinct points. B, the image of b, is of order 10 and contains ϕ as a simple curve and C_t' as a double curve. The curve ϕ is the locus of contacts of pencils of quadrics such that in each pencil the quadrics have four-point contact at one point. C_t' is the locus of contacts of pencils of quadrics in each of which the quadrics have contact at three distinct points.

C_t' and ϕ intersect in 40 points at each of which two of the three contacts of a pencil of quadrics coincide. The third contact associated with each such coincidence is on C_t' but not on ϕ. Then, in the binet, there are 40 pencils of quadrics in each of which the quadrics have one simple and one four-point contact.

There are $\xi=5$ quadruple points of C_t and of L. To each of these correspond four distinct points on C_t', which are nodes of C_t', such that the quadrics of a pencil have simple contact at each of these four points. Therefore, the binet of quadrics contains five pencils in each of which the quadrics have four distinct contacts.

R and B intersect in ϕ counted three times and in a C_{70}, the locus of contacts of pencils of quadrics with two contacts and such that one quadric of each pencil is composite and also such that the two planes of the composite quadric are the common tangent planes respectively of the quadrics at the two contacts.

The binet has no locus of stationary contacts of quadrics and no locus (surface, curve, or set of points) of contacts that involve stationary contacts.* Every point of S_3, however, is a stationary contact of a pencil of quadrics belonging to a web of quadrics of the binet.

* A somewhat similar situation occurs in a net of conics. The net of conics contains nine pencils in each of which the conics osculate each other, but no pencils with pairs of contacts.

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