

CONCERNING CERTAIN TOPOLOGICALLY FLAT SPACES†

BY
F. BURTON JONES

In any space satisfying Axioms 0, 1, 2, 3, 4, and 5 of R. L. Moore's *Foundations of Point Set Theory*‡ a large body of Plane Analysis Situs theorems holds true. Nevertheless, Moore has an example§ of a completely separable space satisfying these axioms which because of its "hilly" character is not homeomorphic with a subset of a plane or a sphere. In this paper Axiom 5 is replaced by an axiom, Axiom 5_1^* , which is not satisfied by spaces of this "hilly" character. As a result, any completely separable (or metric) space satisfying Axioms 0–4 and 5_1^* is homeomorphic with a subset of a plane or a sphere. However, in the presence of Axioms 0–4 neither of the Axioms 5 and 5_1^* follows from the other.

Axiom 5_1 || of *Foundations* suggests the following two axioms:

AXIOM 5_1^1 . *If P is a point of region R , there exists in R a domain D containing P such that the boundary of D is compact.*¶

AXIOM 5_1^2 . *If P is a point of a region R , there exists in R a domain D containing P such that the boundary of D is connected.*††

There is, however, a certain amount of similarity between these last two axioms. For suppose that a space satisfies Moore's Axioms 0, 1, and 2 and

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‡ American Mathematical Society Colloquium Publications, volume XIII, New York, 1932. Hereafter, in this paper, this book will be referred to as *Foundations*.

§ This example of Moore's, although unpublished, has been well known among his students for some four or five years.

|| If P is a point of a region R , there exists a connected domain D containing P and bounded by a compact continuum T such that $D+T$ is a subset of R .

¶ A space satisfying Axiom 5_1^1 is said to be *locally peripherally compact*.

†† A space satisfying Axiom 5_1^2 is said to be *locally peripherally connected*.

Axiom 5₁[†]. † Then if P is a point of a region R , there exists a region R' containing P and lying together with its boundary in R . And by Axiom 5₁[†] there exists in R' a domain D' containing P whose boundary β is compact. But β may be covered by a finite collection $D_1, D_2, D_3, \dots, D_j$ of connected domains such that for each $n, n \leq j, \bar{D}_n$ is a subset of $R - P$. ‡ Let D denote the component of $R - \sum \bar{D}_n$ which contains P . Thus the following theorem holds true and will be used hereafter as

AXIOM 5₁^{*}. *If P is a point of a region R , there exists in R a domain D containing P whose boundary is a subset of the sum of a finite number of continua lying in $R - D$.*

In a sense Axiom 5₁^{*} is common to both Axioms 5₁[†] and 5₁[‡] in that it follows as a theorem from either of them in the presence of Axioms 0, 1, and 2.

Now let S denote a space satisfying Moore's Axioms 0, 1, and 2.

THEOREM A. *If Axiom 5₁^{*} holds true at each of two distinct points A and B of S and no point of S separates A from B , then there exists in S a simple closed curve J containing A and B . §*

Proof. Let AB denote an arc from A to B . || If R is a region containing A but not B , suppose that there exists no point O of $AB \cdot R$ such that no point X of the interval AO of AB separates A from O in R . Then there exists a sequence α of points of AB which converge to A such that if X and Y are points of α and X follows Y in α , then X separates A from Y in R . Hence, for each point X of α except the first, $R - X$ is the sum of two domains U_X and V_X such that V_X contains A and U_X contains every point of α preceding X in α . But no point separates A from B ; so for each point X of α , there exists

† Axioms referred to are stated either in the text of this paper or in *Foundations*.

AXIOM 0. *Every region is a point set.*

AXIOM 1. *There exists a sequence G_1, G_2, G_3, \dots such that (1) for each n, G_n is a collection of regions covering S ; (2) for each n, G_{n+1} is a subcollection of G_n ; (3) if R is any region whatsoever, X is a point of R , and Y is a point of R either identical with X or not, then there exists a natural number m such that if g is any region belonging to the collection G_m and containing X then \bar{g} is a subset of $(R - Y) + X$; (4) if M_1, M_2, M_3, \dots is a sequence of closed point sets such that, for each n, M_n contains M_{n+1} and, for each n , there exists a region g_n of the collection G_n such that M_n is a subset of \bar{g}_n , then there is at least one point common to all the point sets of the sequence M_1, M_2, M_3, \dots .*

AXIOM 2. *If P is a point of a region R there exists a non-degenerate connected domain containing P and lying wholly in R .*

‡ If M denotes a point set, \bar{M} denotes the point set consisting of M and all the limit points of M .

§ See Theorem 40 of Chapter II of *Foundations* and G. T. Whyburn, *On the cyclic connectivity theorem*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 429-433.

|| Theorem 1 of Chapter II of *Foundations*: *If A and B are distinct points of a connected domain D , there exists a simple continuous arc from A to B that lies wholly in D .*

an arc from A to B lying in $S - X$, and in this arc there exists a segment \widehat{PE} lying in R such that $AB \cdot PE = P$ and E belongs to the boundary of R .[†] Therefore, if for no X of α is P identical with A , there exist a subsequence X_1, X_2, X_3, \dots of α and a sequence $P_1E_1, P_2E_2, P_3E_3, \dots$ of arcs such that for each n (1) $AB \cdot P_nE_n = P_n$, (2) E_n belongs to the boundary of R , (3) $\widehat{P_nE_n}$ is a subset of R , and (4) X_{n+1} separates $\sum_{i=1}^n \widehat{P_iE_i}$ from A in R . But (1), (2), and (3) lead to a contradiction of (4), for since there exists in R a domain D containing A whose boundary is a subset of the sum of a finite collection Δ of continua lying in $R - D$, there exists a continuum H of Δ lying in $R - A$ such that H does not contain more than a finite number of the points X_1, X_2, X_3, \dots but for infinitely many values of n , H contains a point of $\widehat{P_nE_n}$. Therefore, either there exists an arc PE which is a subset of an arc from A to B such that $PE \cdot AB = P = A$, or if R is a region containing A but not B , there exists a point O of $AB \cdot R$ such that no point of the interval AO of AB separates A from O in R .

Suppose the latter is true. Let R_1, R_2, R_3, \dots be a monotonic sequence of regions closing down on A and O_1, O_2, O_3, \dots be a sequence of points of AB such that, for each n , (1) the interval AO_n of AB is a subset of R_n , (2) no point X of AO_n separates A from O_n in R_n . For each n and each point X of AO_n , there exists an arc T_{zn} lying in R_n and having only its end points in common with AB such that the segment S_{zn} of AB between these end points contains X and is a subset of R_n . For each n , there exists a finite collection of the segments S_{zn} covering the arc $O_{n+1}O_{n+2}$ of AB . But each $S_{zn} + T_{zn}$ is a simple closed curve lying in R_n . Hence there exists a sequence J_1, J_2, J_3, \dots of simple closed curves such that (1) for each n , J_n and J_{n+1} have at least two points in common, and (2) if R is a region containing A there exists a number δ such that if $n > \delta$, J_n is a subset of R . By Theorem 35 of Chapter II of *Foundations* there exists in $\sum J_n$ a simple closed curve J_A containing A and a segment of AB .

If, on the other hand, there exists an arc PE which is a subset of an arc PB from A to B such that $PE \cdot AB = P = A$, then there exists in PB an arc PX_1 such that P and X_1 are the only points PX_1 has in common with AB . But $PX_1 + AX_1$ (of AB) forms a simple closed curve containing A and some segment of AB .

So in either case there exists a simple closed curve J_A containing A and a segment \widehat{OF} of AB . Likewise, there exists a simple closed curve J_B containing B and a segment \widehat{YE} of AB . Now for each point X of the interval FY of AB , there exists an arc T_x such that T_x has only its end points in common with AB , and the segment S_x of AB lying between these end points contains X .

[†] If AB is an arc with end points A and B , \widehat{AB} is used to denote $AB - (A + B)$.

Some finite collection of these segments S_x covers the arc FY . Since every $S_x + T_x$ is a simple closed curve, there exists a sequence of simple closed curves $J_1 = J_A, J_2, \dots, J_k = J_B$ such that for each $n, n < k, J_n$ contains at least two points of J_{n+1} . By Theorem 35 of Chapter II of *Foundations* there exists in the sum of these simple closed curves a simple closed curve J containing A and B .†

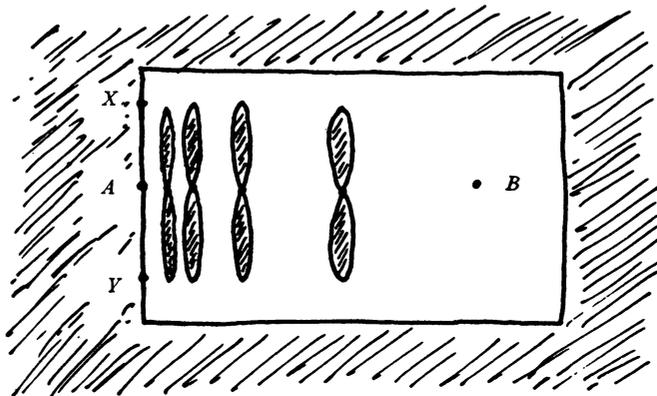


FIGURE 1

The example indicated in Figure 1 is a subset of a plane with the shaded portions removed. Except for those points of the arc XAY different from A , the boundary points of these shaded portions are not removed. It will be easily seen that the result is a connected, connected im kleinen inner limiting subset of the plane and hence satisfies Axioms 0, 1, and 2. Also Axiom 5_1^* is satisfied at every point except A . As a matter of fact, the space is locally compact at every point except A . It is also true in this space that if P is a point and R is a region containing A , there exists in R an arc (or a simple closed curve) separating A from P . The space contains no cut points. Yet there exists no simple closed curve containing A and B .

† This argument may be slightly modified to establish a proposition somewhat more general than Theorem A.

THEOREM A'. *If (1) S is a space satisfying R. L. Moore's Axioms 0, 1, and 2, (2) neither of the distinct points A and B of S is a cut point nor a local end point of S , and (3) no point separates A from B in S , then A and B lie on a simple closed curve in S . (See G. T. Whyburn, *The cyclic and higher connectivity of locally connected spaces*, American Journal of Mathematics, vol. 53 (1931), pp. 427-442.)*

The modification may be outlined as follows: In the first paragraph of the proof, if A is not a local end point, $\coprod V_X$ contains a point E' distinct from A which belongs to the component of R that contains A . Let AE' denote an arc from A to E' in R and let $E'B$ denote an arc from E' to B in $S - A$. Then $AE' + E'B$ contains an arc PE which is a subset of an arc from A to B such that $PE \cdot AB = P = A$. This takes us up to the last sentence of the first paragraph of the proof, which remains unchanged from there on.

Suppose now that, in addition to Axioms 0-2, S satisfies R. L. Moore's Axiom 4† and Axiom 5₁*.

THEOREM B. *The space S is either acyclic or contains no cut point.*

Proof. Suppose that S contains a simple closed curve J . Then it is evident from Axiom 4 that S is connected. Suppose that some point X separates S . The point X does not belong to J , for it is clear from Axiom 4 that no point of a simple closed curve separates S . Let AX denote an arc with end points A and X such that $AX \cdot J = A$. Let M denote the set of cut points of S that belong to AX and let B denote the first point of \overline{M} in the order from A to X . The point B is a cut point of S . For if it is not, then B is a sequential limit point of a sequence B_1, B_2, B_3, \dots of M and at the same time lies on a simple closed curve C containing a segment of AX which contains B ; hence, C contains points of the sequence B_1, B_2, B_3, \dots which contradicts Axiom 4. But since the segment AB of AX contains no point of M , no point separates A from B in X and by Theorem A there exists a simple closed curve containing A and B which again contradicts Axiom 4.

It is not the purpose of this paper to treat the acyclic case;‡ so we shall at this point assume R. L. Moore's Axiom 3.§

1. CONSEQUENCES OF AXIOMS 0-4 AND 5₁*.

THEOREM 1. *If A and B are distinct points, there exists a simple closed curve containing A and B .*

THEOREM 2. *If A and B are distinct points, there exists a simple closed curve separating A from B .*

Proof. There exists a simple closed curve J such that J is the sum of two arcs AXB and AYB . Since $S - J$ is the sum of two connected domains I and E each having J for its boundary, there exist two arcs $X_1O_1Y_1$ and $X_2O_2Y_2$ such that (1) the points X_1 and X_2 lie on the segment AXB and the points Y_1 and Y_2 lie on the segment AYB , and (2) the segment $X_1O_1Y_1$ is a subset of I and the segment $X_2O_2Y_2$ is a subset of E . The simple closed curve formed by the sum of the four arcs $X_1O_1Y_1$, $X_2O_2Y_2$, X_1X_2 (of AXB), and Y_1Y_2 (of AYB) separates A from B in S .

† If J is a simple closed curve, $S - J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.

‡ Such a space if connected would be a *Regular (Menger) Curve*, and these curves have been studied considerably already. See *Foundations* for the definition and references on *Regular (Menger) Curves*.

§ If O is a point, $S - O$ is connected.

THEOREM 3. *If P is a point not belonging to the closed and compact point set H , there exists a simple closed curve separating P from H .*

Proof. For each point X of H there exists a simple closed curve separating P from X . Since H is closed and compact, there exists a finite collection of simple closed curves J_1, J_2, \dots, J_n such that if X is any point of H some one of them separates P from X . For each integer $i, i \leq n$, let T_i denote an arc from J_1 to J_i lying in $S - P$ and let M denote the compact continuum $J_1 + J_2 + \dots + J_n + T_1 + T_2 + \dots + T_n$. The continuum M separates P from $H - H \cdot M$. Now for each point X of M there is a simple closed curve separating X from P . Let C_1, C_2, \dots, C_i denote a finite collection of these whose interiors with respect to P as the point at infinity[†] cover M . By Theorem 13 of Chapter III of *Foundations*, $C_1 + C_2 + \dots + C_i$ contains a simple closed curve separating P from M and hence, separating P from H .

With the help of Theorem 3, the arguments of R. L. Moore with slight modifications prove that a number of the theorems of Chapter IV of *Foundations* hold true in this space. Of these the following five theorems will be used.

THEOREM 4. *If the points A and B belong to different components of the closed and compact point set M , there exists a simple closed curve separating A from B and containing no point of M (Theorem 10 of Chapter IV of *Foundations*).*

THEOREM 5. *If α and β are two connected point sets and neither of the two mutually exclusive closed and compact point sets H and K separates α from β , then $H + K$ does not separate α from β (Theorem 16).*

THEOREM 6. *If the common part of the closed and compact point sets H and K is a continuum and neither H nor K separates the point A from the point B , then $H + K$ does not separate A from B (Theorem 18).*

THEOREM 7. *No arc separates S (Theorem 19).*

THEOREM 8. *If the points A and B are separated from each other by the closed and compact point set M then they are separated from each other by a continuum which is a subset of M and which contains no proper subset that separates A from B (Theorem 24).*

THEOREM 9. *If the compact continuum K does not separate the point A from the point B and G is a finite collection of compact continua such that (1) the common part of any two elements of G is a subset of K , and (2) if H is any element of G , H does not separate A from B and $H \cdot K$ is either connected or vacuous, then $K + G^*$ does not separate A from B .*

[†] For the definition of certain terms and phrases used without definition in this paper the reader is referred to *Foundations*.

Theorem 9 may be established by a finite number of applications of Theorems 5 and 6.

THEOREM 10. *If AXB is an arc and J is a simple closed curve separating A from B , then $J+AXB$ contains a simple closed curve J_1 separating A from B such that $J_1 \cdot AXB$ is connected.*

Proof. Let A_1 and B_1 denote the first and last points respectively that the arc AXB has in common with J . Let AYB denote an arc from A to B not containing any point of the interval A_1B_1 of AXB . Let G denote the collection of all simple closed curves C in $J+AXB$ such that $C \cdot AXB$ is connected and $C \cdot AYB$ is not vacuous. The collection G is finite. Let K denote $A_1B_1 + (J - J \cdot G^*)$.† The point set K is a continuum containing no point of AYB ; hence, K does not separate A from B . Since the common part of any two elements of G is a subset of K and the common part of K with an element of G is connected, if no simple closed curve of the collection G separated A from B , then by Theorem 9, $K+G^*$ would not separate A from B . But $K+G^*$ contains J and separates A from B . Therefore some element J_1 of G separates A from B .

DEFINITIONS. Suppose that P is a point of a domain Q , and Δ is a collection of continua whose sum separates P from the boundary of Q such that each continuum of Δ lies in a component of $Q-P$ whose boundary contains P but no component of $Q-P$ contains more than one element of Δ . Then (1) Δ is said to be *minimal*‡ with respect to Q and P , (2) Δ is said to *minimally separate* P from the boundary of Q , and (3) if D is a domain containing P whose boundary is a subset of Δ^* , Δ is said to *surround D minimally* with respect to Q and P .

THEOREM 11. *If Δ is a collection of continua which surrounds a connected domain D minimally with respect to a domain Q and a point P , then (a) each component of $Q-P$ whose boundary contains P , contains one and only one element of Δ , and (b) no component of $D-P$ has boundary points in more than one element of Δ .*

Proof. Consider (a). By definition, if C is a component of $Q-P$ whose boundary contains P , then C contains not more than one element of Δ . But since $S-P$ is connected, the boundary of C also contains a point of the boundary of Q ; hence C must contain at least one element of Δ . Therefore C contains one and only one element of Δ .

† If G denotes a collection of point sets, G^* will be used to denote the sum of the elements of G .

‡ The term *minimal* is used here because if Δ were finite no collection of fewer elements would separate P from the boundary of Q .

Next consider (b). Suppose that there exists a component I of $D - P$ which has boundary points in more than one element of Δ . Let C denote the component of $Q - P$ which contains I . Then C contains more than one element of Δ . But since D is connected, the boundary of C contains P . This is a contradiction of (a).

THEOREM 12. *If P is a point of a domain Q , not S , there exist in Q a finite collection Δ of continua and a connected domain D containing P such that Δ surrounds D minimally with respect to Q and P .*

Proof. Let D_1 denote a domain lying in Q and containing P whose boundary is a subset of the sum of the elements of a finite collection Δ_1 of continua lying in $Q - D_1$. Let N denote the sum of all the continua of Δ_1 which lie in components of $Q - P$ whose boundaries contain P . If H and K are components of N lying in the same component C of $Q - P$, let T denote an arc in C from a point of H to a point of K . Let Δ denote the collection of all components of the point set obtained by adding N to the sum of the arcs T . The collection Δ minimally separates P from the boundary of Q . Let D denote the component of $Q - \Delta^*$ which contains P . Then the boundary of D is a subset of Δ^* and hence Δ surrounds D minimally with respect to Q and P .

DEFINITION. If a domain is connected and contains one of the complementary domains of each simple closed curve lying in it, then it is said to be *simply connected*.

DEFINITION. If D is a connected domain and D contains one of the complementary domains of each simple closed curve lying in \bar{D} , then \bar{D} is said to be *simply connected* and D is said to be *strongly simply connected*.

THEOREM 13. *Space is simply connected.*

THEOREM 14. *If D is a connected domain and there exist a collection Δ of continua, a domain Q , and a point P of D , such that Δ surrounds D minimally with respect to Q and P , then D is simply connected.*

Proof. Suppose, on the contrary, that there exists in D a simple closed curve J such that neither of its complementary domains, I and E , is a subset of D . Since J lies in D , both I and E contain points of D and, therefore, points of the boundary β of D . Let T denote an arc lying in $S - P$ irreducible from $\beta \cdot I$ to $\beta \cdot E$. The segment T intersects J and, hence, is a subset of D . Therefore T lies in $Q - P$ and contains points of two different continua of Δ . This contradicts Theorem 11.

THEOREM 15. *If P is a point of a domain Q , there exists in Q a simply connected domain D containing P .*

Theorem 15 is a consequence of Theorems 12, 13, and 14.

THEOREM 16. *If D is a connected domain, every component of $S - \bar{D}$ is strongly simply connected.*

Proof. Let C denote a component of $S - \bar{D}$. Let J denote a simple closed curve lying in \bar{C} , and let I denote the interior of J with respect to a point ω of D . Suppose that X is a point of \bar{D} and belongs to I . Then $X + I$ is a connected point set containing ω and a point of I but no point of J . This is a contradiction.

THEOREM 17. *If D is a complementary domain of a locally compact continuous curve M and P is a point of the boundary of D , then $K = D + P$ is a connected, connected im kleinen inner limiting set.*

Proof. It is evident that K is a connected inner limiting set. It is furthermore evident that K is connected im kleinen at every one of its points with the possible exception of P . Suppose that K is not connected im kleinen at P . Then there exists a simply connected domain Q containing P such that (1) the component C of $Q \cdot K$ that contains P is not open with respect to K at the point P , and (2) $Q \cdot M$ is compact. Hence there exists a sequence α of points of K converging to P such that each point of α belongs to some component of $Q \cdot K$ but not to C and no two points of α belong to the same component of $Q \cdot K$. There exists a point O of K not belonging to Q , for otherwise $Q \cdot K = K$ would be connected and open with respect to K at P . For each point X of α let XO denote an arc from X to O lying in D , and let Y denote the first point in the order from X to O that XO has in common with the boundary of Q .

Now let Q_1 denote a domain lying in Q and containing P such that β_1 , the boundary of Q_1 , is a subset of the sum of a finite collection Δ of continua lying in $Q - Q_1$. For infinitely many points X of the sequence α the interval XY of the arc XO contains a point of some one of the continua of the finite collection Δ . Hence there exists a continuum H of Δ such that for infinitely many points X of α the arc XY contains a point of H . Now let D_H and D_P denote connected domains lying in Q and containing H and P respectively such that $\bar{D}_H \cdot \bar{D}_P = 0$.

For each point X of α such that X lies in D_P and the arc XY contains some point of H , let TX denote an arc lying in D_P such that T belongs to M but the segment TX is a subset of D , and let Z denote a point of $XY \cdot H$. Now since D_H lies in Q and contains points of more than one component of $Q \cdot K$, D_H contains some point of M . For each Z let ZW denote an arc lying in D_H such that the segment ZW is a subset of Q and W is a point of M . Thus for some infinite subsequence x_1, x_2, x_3, \dots of α there correspond sequences

of points T_1, T_2, T_3, \dots and W_1, W_2, W_3, \dots such that (1) the points T_1, T_2, T_3, \dots belong to $D_P \cdot M$, (2) the points W_1, W_2, W_3, \dots belong to $D_H \cdot M$, (3) for each n , $T_n X_n + X_n Z_n + X_n W_n$ contains an arc $T_n W_n$ lying in Q , and (4) if $m \neq n$, $T_m W_m$ and $T_n W_n$ are mutually exclusive and lie in different components of $Q \cdot K$. Since M is a continuous curve, $Q \cdot M$ is compact and $\overline{D_H} \cdot \overline{D_P} = 0$, there exist two mutually exclusive connected subsets d_P and d_H of $Q \cdot M$ which are open with respect to M such that for infinitely many values of n and in particular for the three different values a, b , and c of n , T_n and W_n lie in d_P and d_H respectively. Let $T_a T_b$ and $T_a T_c$ denote arcs lying in d_P and $W_a W_b$ and $W_a W_c$ denote arcs lying in d_H . There exist three arcs $AO_a B$, $AO_b B$, and $AO_c B$ lying in $T_a T_b + T_a W_a + W_a W_b$, $T_a T_b + T_b W_b + W_a W_b$, and $T_a T_c + T_c W_c + W_a W_c$ respectively and having only their end points A and B in common. By Theorem 5 of Chapter III of *Foundations*, the sum of one pair of these three arcs, say $AO_a B$ and $AO_c B$, forms a simple closed curve J lying in Q and whose interior I with respect to a point ω of the boundary of Q contains the other segment, $\overline{AO_b B}$. Hence I contains the component C of $Q \cdot K$ which contains $\overline{T_b W_b}$, for C contains no point of J . However, C has limit points in the boundary of Q . But since Q is simply connected, \overline{I} is a subset of Q . This is a contradiction, and K is connected im kleinen at P .

THEOREM 18. *If D is a complementary domain of a locally compact continuous curve, and P is a point of the boundary of D , then P is accessible from D .*

Theorem 18 follows immediately from Theorems 1 and 10 of *Foundations* and Theorem 17.

A number of the intuitive propositions of the plane concerning abutting and crossing arcs hold true. Although some of these will be used in arguments to follow, they will not be stated and the reader is referred to Chapter IV, Theorems 28–32, of *Foundations* for their precise statement and proof. Some of these proofs must be modified, however, to be valid for the set of axioms used here.

THEOREM 19. *An arc is accessible from both sides at any interior point.*

THEOREM 20. *A ray separates space into at most two connected domains.*

Proof. Suppose, on the contrary, that there exists a ray \overrightarrow{PB} such that $S - \overrightarrow{PB}$ contains three components. There exist three arcs $A_1 B_1$, $A_2 B_2$, and $A_3 B_3$ no two of which contain points of the same component of $S - \overrightarrow{PB}$ such that (1) for each i , ($i = 1, 2, 3$), $A_i B_i \cdot \overrightarrow{PB} = \overline{B_i}$, and (2) B_1 belongs to the interval $P B_2$, and B_2 to the interval $P B_3$, of \overrightarrow{PB} . By Theorems 7 and 9, $M = A_1 B_1 + A_2 B_2 + A_3 B_3 + P B_3$ does not separate space. By Theorem 18 the

points $A_1, A_2,$ and A_3 are accessible from $S-M$. Thus for each $i, i=1, 2, 3,$ there exists an arc A_iB_i' lying in $(S-M)+A_i$ and having only the point B_i' in common with \overrightarrow{PB} . For each $i, i=1, 2, 3,$ let J_i denote the simple closed curve composed of A_iB_i, A_iB_i' and the interval B_iB_i' of \overrightarrow{PB} , and let I_i denote that complementary domain of J_i whose common part (if any) with \overrightarrow{PB} is compact. Either (1) I_1 contains $A_2,$ or (2) I_3 contains $A_2,$ or (3) I_2 contains $A_1,$ or (4) I_2 contains $A_3.$ Any one of these four possibilities leads to a contradiction. For instance, suppose that I_1 contains $A_2.$ Since $I_1 \cdot \overrightarrow{PB}$ is a subset of $PB_1,$ it is clear, by Theorems 9 and 18, that I_1+A_1 contains an arc from A_1 to A_2 not intersecting $\overrightarrow{PB}.$

THEOREM 21. *If the arc AB is a subset of the connected domain D and $D-AB$ is connected, then D contains a simple closed curve separating A from $B.$*

Proof. Since the arc AB is accessible from both sides at an interior point $O,$ there exist two arcs EO and FO abutting on AB from different sides such that neither EO nor FO has any point except O in common with the arc $AB.$ The arcs EO and FO contain points E_1 and $F_1,$ respectively, such that the intervals E_1O and F_1O of EO and FO respectively lie in $D.$ In $D-AB$ there exists an arc $E_1F_1,$ and in $E_1O+E_1F_1+F_1O$ there exists a simple closed curve J which lies in D and separates A from $B.$

THEOREM 22. *If P is a point of a connected domain $Q,$ then there exists a region R containing P such that if X is a point of $R-P,$ X lies in a simple domain which together with its boundary is a subset of $Q.$*

Proof. If Q is $S,$ the theorem is evident from Theorem 2. Suppose that the boundary of Q is not vacuous and that the theorem is false.

Then there exists a sequence α of points X of $Q-P$ converging to P such that X does not lie in a simple domain which together with its boundary is a subset of $Q.$ Let ρ denote a monotonic sequence of simply connected domains D closing down on $P.$ Let D_1 denote an element of ρ lying in $Q.$ It is clear from Axioms 3 and 5_1^* that Q_1-P contains only a finite number of components. One of these components C_1 contains two points X_1 and X_2 of $\alpha.$ Let T_1 denote an arc from X_1 to X_2 in $C_1.$ Let D_2 denote the first element of ρ which follows D_1 and contains no point of $T_1.$ One of the components of D_2-P contains two points X_3 and X_4 of α and an arc T_2 from X_3 to $X_4.$ Let D_3 denote the first element of ρ not containing a point of $T_1+T_2.$ Then D_3-P contains an arc T_3 whose end points are points of $\alpha.$ This process may be continued. Thus there exists a sequence of mutually exclusive arcs T_1, T_2, T_3, \dots converging to P such that for each n, T_n is a subset of D_n-P and the end points of T_n are points of $\alpha.$ By the preceding theorem, for each n, T_n separates $D_1.$

For each n , one of the components of $D_1 - T_n$ does not contain P and in this component there exists a segment W_n having one end point in T_n and containing no point of $T_1 + T_2 + \cdots + T_{n-1}$. By Theorems 7 and 9, $P + T_1 + T_2 + \cdots + T_n$ does not separate space. Hence there exists a segment W'_n lying in $D_1 - (P + T_1 + T_2 + \cdots + T_n)$ having one end point in W_n and the other end point in the boundary of D_1 . For each n let L_n denote the component of $D_1 - (P + T_1 + T_2 + \cdots + T_n)$ which contains $W_n + W'_n$. Then for each n , T_n separates L_n from P in D_1 , and \bar{L}_n contains points of both T_n and the boundary of D_1 . By Axiom 5₁* there exists in D_1 a domain U which contains P and whose boundary is a subset of the sum of a finite number of continua lying in $D_1 - U$. There is a continuum H of this set which, for infinitely many values j of n , contains a point of L_j while U contains T_j . But if k is the smallest value of j , $P + H + \sum L_j$ is a connected point set in $D_1 - T_k$ containing P and L_k . This is a contradiction.

DEFINITIONS. If P is a point and there exists a sequence of simple domains closing down on P , then P is said to be a *simple point*. A non-simple point is said to be an *edge point*.

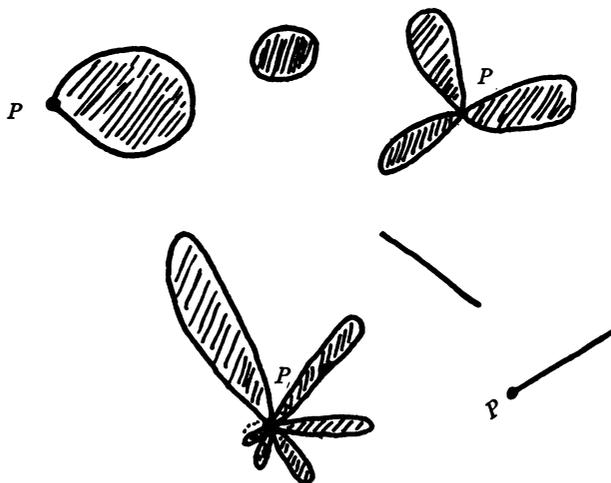


FIGURE 2

If from the plane of Figure 2 the shaded domains and their boundaries except for the points P are removed, the result is a space which satisfies Axioms 0-4 and 5₁*. The points P are *edge points* while all other points are *simple points*.

THEOREM 23. *The set of simple points is everywhere dense.*

Proof. Suppose that R is a region. With the help of repeated applications

of Theorem 22 it may be shown that there exists a sequence D_1, D_2, D_3, \dots of simple domains lying in R such that, for each n , (1) D_n contains \bar{D}_{n+1} , and (2) \bar{D}_n lies in some region of the collection G_n of Axiom 1. By (4) of Axiom 1, there is a point P common to $\bar{D}_1, \bar{D}_2, \bar{D}_3, \dots$ and hence, to D_1, D_2, D_3, \dots ; and by (3) of Axiom 1, P is the only common point. The point P is simple and lies in R .

THEOREM 24. *No completely separable point set contains uncountably many edge points.*

Proof. Let H denote the collection of all edge points belonging to a completely separable point set M . If X is a point of H , there exists no sequence of simple domains closing down on X . Hence for each point X of H , there exists an integer n_x such that no region of G_{n_x} of Axiom 1 contains a simple domain containing X . If H is uncountable, there exist an integer k and an uncountable subset K of H , such that if X is a point of K , $n_x = k$. Since M is completely separable, K has a point P of condensation. Let Q denote a connected domain containing P and lying in a region of G_k of Axiom 1. By Theorem 22 there exists in Q a region R containing P such that every point X of $R - P$ is contained in a simple domain lying in Q . This is clearly a contradiction.

THEOREM 25. *If M is a countable set of simple points, $S - M$ is arc-wise connected.*

This theorem may be proved without using Axioms 4 and 5_1^* by a modification of R. L. Moore's proof[†] of Theorem 1 of Chapter II of *Foundations*.[‡] The kernel of this proof is the construction of a sequence C_1, C_2, C_3, \dots of simple chains of connected domains from a point A to a point B such that for each n , C_n^* contains \bar{C}_{n+1}^* and the common part of the point sets $C_1^*, C_2^*, C_3^*, \dots$ is an arc from A to B . Now for each n , C_n^* is a connected domain, and it is easy to see that no finite number of simple points disconnects a connected domain. Hence if (1) A and B are two distinct points of $S - M$, (2) $M = P_1 + P_2 + \dots + P_n + \dots$, and (3) for each n the chain C_n is constructed so that C_n^* contains no point of $P_1 + P_2 + \dots + P_n$, then the arc from A to B of Moore's construction will not contain any point of M .

THEOREM 26. *If M is a countable set of simple points, $S - M$ is cyclicly connected.*

Proof. If A and B are any two points of $S - M$, then by Theorem 25 there exists an arc AXB in $S - M$. With the help of Theorems 5, 7 and 17, and

[†] Cf. J. R. Kline, *Concerning the complement of a countable infinity of point sets of a certain type*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 290-292.

[‡] See footnote p. 54.

Theorem 9 of Chapter II of *Foundations*, one may show that $S - \widehat{AXB}$ is a space satisfying Axioms 0, 1, 2, and 3. Hence by Theorem 25 there exists an arc AYB lying in $S - (M + \widehat{AXB})$. The sum of the two arcs, AXB and AYB , is a simple closed curve lying in $S - M$ and containing $A + B$.

THEOREM 27. *No compact set of edge points separates two simple points from each other.*

Proof. Suppose that A and B are two simple points and that M is a compact set of edge points. Suppose that n is any fixed integer. Then there exists from A to B a simple chain of simple domains such that each element of the chain is a subset of some region of G_n of Axiom 1. For suppose the contrary. Let H_A denote the set of all points X such that there exists from A to X a chain of simple domains whose elements are each a subset of some region of G_n . Since A is a simple point, H_A is a connected domain. This domain does not contain B , and its boundary β is a subset of M . Hence β is a closed and compact point set which separates A from B . By Theorem 8 and Axiom 3, β contains a non-degenerate continuum separating A from B . Hence M is uncountable, which contradicts Theorem 24.

Now with the help of Theorem 9 of Chapter II of *Foundations* and Theorem 5 one may show that a simple domain is a space satisfying Axioms 0–4 and 5_1^* , and that the complementary domain of a point is also such a space. Let $M = P_1 + P_2 + P_3 + \dots$. Let C_1 denote a simple chain of simple domains from A to B such that (1) each element of C_1 is a subset of some region of G_1 , and (2) C_1^* does not contain P_1 . Now the boundary of the domain C_1^* is a closed and compact point set not separating A from B ; so there exists a simple closed curve separating it from $A + B$ (Theorem 3 in modification). Hence there exists in C_1^* a simple domain D_1 containing $A + B$. Let C_2 denote a simple chain of simple domains from A to B such that if c is an element of D_2 , (1) \bar{c} is a subset of some element of C_1 , (2) c is a subset of some region of G_2 , and (3) c does not contain P_2 . This process may be continued, and by Theorem 80 of Chapter I of *Foundations*, $C_1^* \cdot C_2^* \cdot C_3^* \cdot \dots$ is a continuum. Hence M does not separate A from B .

As a matter of fact the above process may be carried out along the lines of the proof of Theorem 1 of Chapter II of *Foundations*, so as to show that $S - M$ is arc-wise, and even cyclicly, connected.

Nevertheless, a countable but non-compact set of edge points may separate two simple points from each other as is shown by the following example. Let space consist of all points of the number plane whose ordinates are not zero together with those whose ordinates are zero but whose abscissas are rational. If (X, Y) is a point whose ordinate Y is not zero, then for each integer

such that $1/n < |Y|$, let $V_n(X, Y)$ denote the point set consisting of all points which lie at a distance less than $1/n$ from (X, Y) . If (X, Y) is a point whose ordinate Y is zero and whose abscissa X is rational then for each integer n , let $V_n(X, Y)$ denote the point set consisting of (X, Y) together with all points which are interior to the isosceles triangles bounded by the lines whose equations are $y = 1/n$, $y = -1/n$, $y = n(x - X)$, and $y = -n(x - X)$. These point sets V_n shall be called regions, and for each integer n , G_n of Axiom 1 shall denote the collection of all of the point sets V_{n+q} , $q = 0, 1, 2, \dots$. It can be shown that Axioms 0-4 and 5_1^* hold true in this space. Furthermore, the space is completely separable and locally compact at all but a countable nowhere-dense set of points. Let M denote the set of all points whose abscissas are rational and whose ordinates are zero. Each point of M is an edge point. If A and B are points with positive and negative ordinates respectively, M separates A from B , and the point set H_A of the above argument consists of all points whose ordinates are positive.

THEOREM 28. *No compact set of edge points separates space.*

Theorem 28 is a consequence of Theorems 23 and 27.

DEFINITIONS. If D_1 and D_2 are domains such that D_1 contains \bar{D}_2 , and \hat{T} is a segment lying in $D_1 - \bar{D}_2$ having one end point on the boundary of D_1 and the other on the boundary of D_2 , then \hat{T} is said to *cross* $D_1 - \bar{D}_2$. Furthermore, if C is the component of $D_1 - \bar{D}_2$ which contains \hat{T} , then \hat{T} is said to *cross* C . Also the arc T is said to *cross* $D_1 - \bar{D}_2$ and to *cross* C .

DEFINITION. If D_1 and D_2 are domains such that D_1 contains \bar{D}_2 , C is a component of $D_1 - \bar{D}_2$, and \hat{T}_1 , \hat{T}_2 , and \hat{T}_3 are segments crossing C , then if \hat{T}_2 separates \hat{T}_1 from \hat{T}_3 in C , \hat{T}_2 is said to be *between* \hat{T}_1 and \hat{T}_3 in C , and T_2 is said to be *between* T_1 and T_3 in C .

THEOREM 29. *Suppose that D_1 and D_2 are simply connected domains such that D_1 contains \bar{D}_2 but no simple domain lying in D_1 contains D_2 , C is a component of $D_1 - \bar{D}_2$, and \hat{T}_1 , \hat{T}_2 , and \hat{T}_3 are three mutually exclusive segments crossing C . Then one and only one of these segments is between the other two in C .*

Proof. That not more than one of these segments lies between the other two in C is an immediate consequence of a well known theorem.

Suppose that no one of them is between the other two in C . Then either (1) $C - (\hat{T}_1 + \hat{T}_2 + \hat{T}_3)$ contains a component W which has limit points in each of the three segments, or (2) $C - (\hat{T}_1 + \hat{T}_2 + \hat{T}_3)$ contains three mutually exclusive components W_{12} , W_{13} , and W_{23} having limit points in the segments indicated by their subscripts.

Case I. Suppose (1) holds. Then there exists in W three mutually exclusive segments \widehat{AA}_1 , \widehat{AA}_2 , and \widehat{AA}_3 having a common end point A in W such that A_1 , A_2 , and A_3 belong to \widehat{T}_1 , \widehat{T}_2 , and \widehat{T}_3 respectively. For each i , $i=1, 2, 3$, let O_i and P_i denote the end points of T_i which belong to the boundaries of D_1 and D_2 respectively. Since D_2 is connected, there exist in $D_1 - (\widehat{T}_1 + \widehat{T}_2 + \widehat{T}_3 + \widehat{AA}_1 + \widehat{AA}_2 + \widehat{AA}_3)$ three mutually exclusive segments \widehat{BB}_1 , \widehat{BB}_2 , and \widehat{BB}_3 having a common end point B in D_2 and having their other end points B_1 , B_2 , and B_3 in the intervals A_1P_1 of T_1 , A_2P_2 of T_2 , and A_3P_3 of T_3 respectively. If AB_nB denotes $\widehat{AA}_n + A_nB$ (of A_nP_n) + \widehat{BB}_n , $n=1, 2, 3$, then the segments AB_1B , AB_2B , and AB_3B are mutually exclusive. By Theorem 5 of Chapter III of *Foundations*, the sum of two of these arcs, say AB_1B and AB_3B , forms a simple closed curve J whose interior I with respect to O_1 contains the segment from A to B of the other one. Since D_1 is simply connected, I is a subset of D_1 . But T_2 contains B_2 and no point of J and is, therefore, a subset of I . Hence I contains O_2 . This is a contradiction.

Case II. Suppose (2) holds. There exist segments $\widehat{A_1B_3}$, $\widehat{B_2A_3}$, and $\widehat{B_1A_2}$ lying in W_{13} , W_{23} , and W_{12} respectively and having end points on the arcs T_1 , T_2 , and T_3 as indicated by the subscripts in the notation. The point set $A_1B_3 + A_1B_1$ (of T_1) + B_1A_2 (of T_2) + $B_2A_3 + A_3B_3$ (of T_3) is a simple closed curve J lying in $D_1 - D_2$. Since D_1 is simply connected, one of the complementary domains I of J is a subset of D_1 . The domain I either contains D_2 or is a subset of $D_1 - \overline{D_2}$. By hypothesis D_1 contains no simple domain containing D_2 . Hence I is a subset of $C - (\widehat{T}_1 + \widehat{T}_2 + \widehat{T}_3)$ and has limit points in each of the three segments \widehat{T}_1 , \widehat{T}_2 , and \widehat{T}_3 , which is Case I again.

The reader can see that any collection of mutually exclusive segments crossing C would, in accordance with this notion of *between*, have a *linear order* in C .

THEOREM 30. *Suppose that D_1 , D_2 , and D_3 are simply connected domains containing the point P such that (1) D_3 contains $\overline{D_2}$, (2) D_3 is surrounded minimally with respect to D_1 and P by a finite collection Δ of continua lying in $D_1 - D_3$, and (3) D_1 contains no simple domain containing P . Then if C is a component of $D_1 - \overline{D_2}$, T is an arc crossing C , and J is a simple closed curve separating T from P , J contains two arcs T_a and T_b which cross C such that T is between them in C .*

Proof. Let R denote a connected domain lying in $D_1 - D_1 \cdot J$ and containing the end point A of T which is on the boundary of D_2 . Now in $R + D_2$ there exists an arc AP from A to P , such that $J \cdot AP$ is a subset of an interval $A'P'$ of AP lying in D_2 . By Theorem 10, $J + AP$ contains a simple closed curve J_1

separating A from P such that $J_1 \cdot AP$ is connected; but $J_1 \cdot AP$ is a subset of $A'P'$ and is, therefore, a subset of D_2 . Since D_1 is simply connected but contains no simple domain containing P , J_1 contains a point O not belonging to D_1 . The point set $J_1 - (J_1 \cdot AP + O)$ is the sum of two mutually exclusive segments \widehat{OX} and \widehat{OY} , where X and Y belong to AP (and Y belongs to the interval XP of AP) and the arcs OX and OY abut on AP from different sides. Both OX and OY contain arcs which cross $D_1 - \bar{D}_2$. Let T_a denote the first interval of OX in the order from X to O that crosses $D_1 - \bar{D}_2$, and let T_b denote the first interval of OY in the order from Y to O which crosses $D_1 - \bar{D}_2$. Now C contains an element of Δ and if U denotes the component of $D_1 - P$ which contains C , U has no point in common with any other component of $D_1 - \bar{D}_2$ because Δ is a minimal collection with respect to D_1 and P . Hence T_a and T_b cross C . Let X' and Y' denote the end points of T_a and T_b respectively which are on the boundary of D_2 . The intervals XX' of OX and YY' of OY lie in D_1 .

Suppose that T is not between T_a and T_b in C . Then there exists an arc T_1 in C irreducible from T_a to T_b not intersecting T . Let W denote the last point that OX has in common with T_1 , and let Z denote the last point that OY has in common with T_1 . The intervals WX of OX and ZY of OY are subsets of $T_a + XX'$ and $T_b + YY'$ respectively, and hence, are subsets of D_1 ; furthermore, WX and ZY abut on AP from different sides. Now $J_2 = J_1 \cdot AP + WX + WZ$ (of T_1) $+ ZY$ is a simple closed curve lying in D_1 not containing P , or a point of T , or a point of the segment \widehat{PY} of AP . Since D_1 is simply connected, one of the complementary domains I of J_2 is a subset of D_1 . But since D_1 does not contain a simple domain containing P , both P and T must be subsets of the other complementary domain E of J_2 . Now AX contains an arc which abuts on J_2 from the side opposite PY of AP , and since both A and \widehat{PY} are in D , \widehat{AX} must intersect J_2 . Let A_1 and A_2 denote the first and last points respectively that \widehat{AX} has in common with J_2 . The segments $\widehat{AA_1}$ and $\widehat{A_2X}$ of AX lie in E and I respectively. The segment \widehat{AX} does not intersect J_1 ; so $\widehat{AX} \cdot J_2$ is a subset of T_1 . Hence $\widehat{AA_1}$ and $\widehat{A_2X}$ abut on T_1 from different sides. Let R' denote a connected domain lying in D_1 and containing A but no point of $T_1 + T_a + T_b$. Then in $R' + D_2$ there exists an arc FG irreducible from AA_1 to A_2X . The simple closed curve $J_3 = FG + GA_2$ (of A_2X) $+ A_1A_2$ (of T_1) $+ A_1F$ (of AA_1) lies in D_1 , contains no point of $T_a + T_b$, but crosses the arc T_1 . Hence, T_a and T_b lie in different complementary domains of J_3 . This is impossible, for since D_1 is simply connected, one complementary domain of J_3 is a subset of D_1 .

THEOREM 31. *In order that space be metric, it is necessary and sufficient that space be completely separable.*

Proof. That a space satisfying Axioms 0 and 1 is metric if it is completely separable has been shown by R. L. Moore.† Hence, this space is metric if it is completely separable.

It will now be shown that if space is metric, it is completely separable. Let P denote an edge point, and let R denote a region containing P but containing no simple domain containing P . Since R is not S , by Theorems 12, 14, and 15, there exist in R three simply connected domains D_1 , D_2 , and D_3 such that D_3 contains \bar{D}_2 , and D_3 is surrounded minimally with respect to D_1 and P by a collection Δ of continua lying in $D_1 - D_3$. The domain D_1 contains no simple domain containing P . Let C denote a component of $D_1 - \bar{D}_2$.

Suppose that there are two mutually exclusive arcs T_1 and T_{-1} crossing C . Let α denote a well-ordered sequence (whose first element is T_1) of all arcs T which cross C such that either (1) T is T_1 , or (2) T_1 lies between T_{-1} and T in C . Let α' denote a subsequence of α such that (1) the first element of α' is T_1 and (2) if an element T of α' is not T_1 , then T is the first element of α which neither intersects a preceding element of α' nor lies between two preceding elements of α' . Every element of α either intersects an element of α' or lies between T_1 and some element of α' , and between any element of α' and the next following element in α' there is no element of α' .

The sequence α' is countable. For suppose that α' is uncountable. For each arc T of α' let d_T denote the distance from T to the next element of α' . There exist a number ϵ and an uncountable subsequence α'_ϵ of α' such that if T belongs to α'_ϵ , $d_T > \epsilon$. Let T_2 denote the first element of α' such that T is preceded in α' by infinitely many elements of α'_ϵ . Let L denote an arc from T_1 to T_2 lying in C . Infinitely many elements of α'_ϵ are between T_1 and T_2 in C and therefore intersect L . Consequently, there exists in C a connected domain D of diameter less than $\epsilon/2$ which contains points of two different elements T and T' of α'_ϵ , such that T precedes T' in α' . Then, the first element of α' which follows T in α' is either T' or lies between T and T' in C and intersects D , which is a contradiction. The same may be done for the collection of all arcs T such that T_{-1} lies between T_1 and T in C . Hence there exists a countable collection G of arcs crossing C , such that any arc which crosses C either intersects an arc of G or lies between two arcs of G . This was on the assumption that there exist two mutually exclusive arcs crossing C . It is evident that G exists if this is not the case, for there exists at least one arc which crosses C .

For each pair of arcs of G let L denote an arc in C which contains points of both of them, and let M_ϵ denote the sum of all these arcs together with the sum of the elements of G . Then if T is any arc which crosses C , T intersects

† See *Foundations*, p. 464.

M_c . Since each component C of $D_1 - \bar{D}_2$ contains an element of Δ , there are only a finite number of them. Hence $M = \overline{\sum M_c}$ is a closed and completely separable subset of $\bar{D}_1 - P$ which separates P from the boundary of D_1 . Let Q denote the component of $\bar{D}_1 - M$ which contains P . Then Q contains P and lies in R , and the boundary of Q is completely separable. Thus, if P is an edge point of a region R , there exists in R a domain containing P whose boundary is separable. This is also true if P is a simple point of a region R . Therefore, space is locally peripherally separable. By a theorem of the author's such a connected, connected im kleinen metric space is completely separable.†

THEOREM 32. *If space is completely separable and P is a point, there exists a sequence of simple domains Q_1, Q_2, Q_3, \dots bounded by simple closed curves J_1, J_2, J_3, \dots respectively, such that (1) for each n , Q_n contains \bar{Q}_{n+1} , and (2) if M is a closed and compact point set not containing P , there exists an integer n such that \bar{Q}_n does not contain a point of M .*

Proof. Let G denote the collection of all simple domains D such that \bar{D} is a subset of $S - P$. By Theorem 2, G covers $S - P$, and since space is completely separable, G contains a countable subcollection D_1, D_2, D_3, \dots which covers $S - P$. For each n let C_n denote the boundary of D_n . Let J_1 denote the boundary of a simple domain containing P . For each n , $n \geq 2$, let J_n denote a simple closed curve separating P from the closed and compact point set $C_1 + C_2 + \dots + C_{n-1} + J_1 + J_2 + \dots + J_{n-1}$, and let Q_n denote the complementary domain of J_n which contains P . Then Q_1, Q_2, Q_3, \dots is the required sequence of simple domains. For if M is any closed and compact point set, there exists an integer k such that $D_1 + D_2 + \dots + D_k$ covers M . Hence, if $n > k$, \bar{Q}_n contains no point of M .

DEFINITION. Suppose that D is a connected domain. If there exist a collection Δ of open curves, a simply connected domain Q , and a point P of D , such that (1) Q contains no simple domain containing P , and (2) Δ surrounds D minimally with respect to Q and P , then D is said to be *pseudo-simple*, and in particular, D is said to be *pseudo-simple* with respect to Q and P .

THEOREM 33. *If space is completely separable and D_1 is a simply connected domain containing an edge point P but containing no simple domain containing P , then D_1 contains a domain D which is pseudo-simple with respect to D_1 and P .*

Proof. Let D_2 and D_3 denote simply connected domains containing P such that (1) D_3 contains \bar{D}_2 , and (2) D_3 is surrounded minimally with respect to D_1

† A theorem concerning locally peripherally separable spaces, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 437-439.

and P by a finite collection Δ' of continua lying in $D_1 - D_3$. By Theorem 32 there exists a sequence of simple domains Q_1, Q_2, Q_3, \dots bounded by simple closed curves J_1, J_2, J_3, \dots respectively, such that (1) for each n , Q_n contains \bar{Q}_{n+1} , and (2) if M is a closed and compact subset of $S - P$, there exists an integer n such that \bar{Q}_n contains no point of M . Let C denote a component of $D_1 - \bar{D}_2$, and let G denote the collection of all arcs T such that T crosses C and for some n is a subset of J_n . If T and T' are arcs of G , $\widehat{T} \cdot \widehat{T}' = 0$. Since D_1 is simply connected but contains no simple domain containing P , for each n , J_n contains a point of $S - D_1$. Furthermore, let AP denote an arc lying in D_2 and in the component of $D_1 - P$ which contains C ; then there exists an integer n_1 such that \bar{Q}_{n_1} does not contain A ; so if $n > n_1$, J_n contains a point of the segment \widehat{AP} and a point of $S - D_1$, and therefore contains at least two arcs of G . However for each n , J_n contains at most a finite number of arcs of G . Hence G is countably infinite. Furthermore, suppose that L is an arc lying in C from one arc of G to another arc of G . There exists an integer n_2 such that if $n > n_2$, \bar{Q}_n contains no point of L . Hence, between any two arcs of G there are only a finite number of arcs of G . Each arc of G is between some two arcs of G . For, if T is an arc of G , there exists an integer n such that J_n separates T from P . By Theorem 30, J_n contains two arcs which cross C such that T is between them in C .

Let α denote a well-ordered sequence whose elements are the elements of G . Let N_1 and N_0 denote the first two elements of α . Let N_2 denote the first element of α such that N_1 is between N_0 and N_2 in C . Let N_{-1} denote the first element of α such that N_0 is between N_{-1} and N_1 in C . This process may be continued. The sequences $N_1, N_2, N_3, \dots, N_\omega, \dots$ and $N_0, N_{-1}, N_{-2}, \dots, N_{-\omega}, \dots$ may or may not be simple sequences, but since each is countable, the first contains a simple countable subsequence T_1, T_2, T_3, \dots running through it and the second contains a simple countable subsequence $T_0, T_{-1}, T_{-2}, \dots$ running through it, such that if T is any arc crossing C , there exists an integer n such that T is between T_n and T_{-n} in C , and such that the linear order of these arcs in C is the same as the order in the sequences.

Since C contains one and only one of the continua of Δ' and by Theorem 31 space is metric, there exists in C a connected domain C' such that (1) C' contains this element of Δ' , and (2) \bar{C}' is a subset of C . For each pair of consecutive integers (positive or negative), a and b with $a < b$, let U_a denote an arc irreducible from T_a to T_b lying in C' , and let M_a denote the interval of T_a between the end points of U_a and U_{a-1} in T_a . The point set $\sum M_a$ is closed, since each M_a is for some n a closed subset of $C \cdot J_n$, for each n J_n contains only a finite number of the arcs of G , and the limiting set of J_1, J_2, J_3, \dots is

the point P . Also $\sum U_a$ is closed; for suppose that O is a limit point of the set $\sum U_a$ not belonging to it. The point O belongs to \bar{C}' and therefore, to C . There exist a connected domain V lying in C and containing O but not intersecting more than one of the simple closed curves J_1, J_2, J_3, \dots , and an infinite collection H of segments \widehat{U}_a such that if \widehat{U}_a is a segment of H , \widehat{U}_a intersects V . Hence, $V+H^*$ is a connected subset of C not intersecting more than one of the arcs $T_1, T_2, T_3, \dots, T_0, T_{-1}, T_{-2}, \dots$, but having limit points in infinitely many of them. This is a contradiction. So $L = \sum(M_a + U_a)$ is closed and is, because of its method of construction, an open curve.

Now suppose that T is any arc crossing C . There exists an integer n such that T lies between T_n and T_{-n} in C . Hence T intersects that interval of L which is irreducible from T_n to T_{-n} . Furthermore, if this process is carried out in each component of $D_1 - \bar{D}_2$ and Δ denotes the collection of open curves so obtained, Δ^* separates P from the boundary of D_1 , and since no component of $D_1 - P$ contains more than one element of Δ' , no component contains more than one element of Δ . Hence, Δ is minimal with respect to D_1 and P .

Let D denote the component of $S - \Delta^*$ which contains P . The domain D is pseudo-simple with respect to D_1 and P .

THEOREM 34. *If L is an element of a collection of open curves which surrounds a pseudo-simple domain D minimally with respect to a simply connected domain Q and a point P , and C is the component of $D - P$ whose boundary contains a point of L , then no ray of L separates C from the boundary of Q in $S - P$.*

Proof. Suppose the contrary is true. Then L contains a ray \vec{B}_1B_2 such that B_1 belongs to \bar{C} and $\vec{B}_1B_2 + P$ separates C from β , the boundary of Q . By Theorem 18, $C + P + B_1$ contains an arc PB_1 from P to B_1 . The ray $\vec{PB}_2 = PB_1 + \vec{B}_1B_2$ separates $C - C \cdot PB_1$ from β , and by Theorem 20, $S - \vec{PB}_2$ is the sum of two connected domains I and E where I contains $C - C \cdot \vec{PB}_2$ and is a subset of Q and E contains β . Let E_1F_1 and E_2F_2 denote two arcs in C which abut on PB_1 from different sides. The points E_1 and E_2 both belong to $C - C \cdot PB_1$ and, hence, to I . Therefore I contains an arc E_1E_2 from E_1 to E_2 , and $E_1E_2 + E_1F_1 + F_1F_2$ (of PB_1) $+ E_2F_2$ contains a simple closed curve lying in Q one of whose complementary domains lies in Q and contains P . This is a contradiction.

THEOREM 35. *If L is an element of a collection Δ of open curves which surrounds a pseudo-simple domain D minimally with respect to a simply connected domain Q and a point P , then no two segments lying in D abut on L from different sides.†*

† Two segments abut on a ray or open curve from different sides provided the ray or open curve contains an arc on which the segments abut from different sides.

Proof. Suppose, on the contrary, that D contains two mutually exclusive segments which abut on L from different sides. It follows, using Theorem 11(b), that there exist an arc A_1A_2 and two segments, $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$, all lying in the same component C of $D-P$, such that the segments $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$ abut on L from different sides and the point set $J = A_1B_1 + B_1B_2(\text{of } L) + A_2B_2 + A_1$ is a simple closed curve lying in Q . One of the complementary domains of J is a subset of Q . Hence, L contains two rays, $\overrightarrow{B_1B_2}$ and $\overrightarrow{B_2B_1}$, whose common part is the arc B_1B , one of which has the property that it together with P separates C from the boundary of Q . This is a contradiction of the preceding theorem.

THEOREM 36. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P , β is the boundary of D , and J is a simple closed curve lying in \overline{D} such that $J \cdot \beta$ is connected, then one of the complementary domains of J is a subset of D .*

Proof. Let Δ denote a collection of open curves which surrounds D minimally with respect to Q and P , let L denote the element of Δ which contains $J \cdot \beta$, and let C denote the component of $D-P$ which contains $J \cdot (D-P)$. By Theorems 7 and 9, $J \cdot \beta + P$ does not separate space, and there exists an arc T from a point of $L - J \cdot L$ to a point on the boundary of Q which contains no point of J . One of the complementary domains I of J is a subset of Q . Therefore, I contains no point of T and, with the help of Theorem 35, it may be shown that I contains no point of L . But I does not contain P .[†] Hence, since I contains points of C , I is a subset of C and, consequently, of D .

THEOREM 37. *If D is a pseudo-simple domain, L is an element of a collection Δ of open curves which surrounds D minimally with respect to a simply connected domain Q and a point P , and OX and $O'X'$ are arcs lying in $S - \overline{D}$ except for their end points X and X' which lie on L , then the arcs OX and $O'X'$ abut on L from the same side.*

Proof. With the help of Theorems 11(b) and 34 it can be shown that there exists an arc FF' lying in D except for the end points F and F' , which lie on L , such that the interval FF' of L contains X and X' as interior points. Then by Theorem 36, $J = FF' + FF'$ (of L) is a simple closed curve one of whose complementary domains is a subset of D . The other complementary domain of J contains the segments \widehat{OX} and $\widehat{O'X'}$. Hence, the arcs approach FF' of L and, therefore, L from the same side.

THEOREM 38. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P , and OX and $O'X'$ are arcs which have their*

[†] From the definition of a pseudo-simple domain, Q contains no simple domain containing P .

end points X and X' in an element L of a collection Δ of open curves which surrounds D , but have no other points in common with the boundary of D , and approach L from different sides, then one of these arcs lies in D and the other lies in $S - \bar{D}$.

Theorem 38 is a consequence of Theorems 35 and 37.

THEOREM 39. *If Δ is a collection of open curves which surrounds a pseudo-simple domain minimally with respect to a simply connected domain Q and a point P , then the boundary of D is Δ^* , and every point of Δ^* is accessible from D .*

Proof. Let X denote a point of an open curve L of Δ , and let AXB denote an interval of Δ containing X as an interior point. By Theorem 19 there exist two segments, \widehat{EX} and $\widehat{E'X}$, which abut on AXB at X from different sides. The segments, \widehat{EX} and $\widehat{E'X}$, contain segments, \widehat{FX} and $\widehat{F'X}$, respectively, which abut on L from different sides and do not contain P or any point of the boundary of D except X . By Theorem 38 one of the latter two segments is a subset of D .

THEOREM 40. *If X is a point of the component L of the boundary β of a domain D which is pseudo-simple with respect to a simply connected domain Q and a point P , and M is a closed and compact point set not containing X , there exists a simple closed curve lying in $S - (D + M)$ whose common part with β is an interval of L containing X as an interior point, and whose interior with respect to P contains no point of M .*

Proof. Let XP denote an arc from X to P lying in $D + X$, and let J denote a simple closed curve separating X from P such that $J \cdot XP$ is a connected subset of D . The curve J contains a point of $S - D$. Let O denote the first point that XP has in common with J , and let AOB denote the interval of J containing O and lying in D except for its end points A and B which lie on L . The interval AB of L contains X because one of the complementary domains of the simple closed curve $AOB + AB$ is a subset of D and since it can not contain P , it must contain the segment \widehat{OX} of XP . Let AZB denote the arc of J such that $J = AXB + AZB$. In the order A, Z, B , let A_0 denote the last point that AZB has in common with the ray \overrightarrow{XA} of L , and let B_1 denote the next point that AZB has in common with β if it is not in L . Let L_1 denote the component of β containing B_1 , and let A_1 denote the last point that AZB has in common with L_1 . Let B_2 denote the next point that AZB has in common with β if it is not in L , let L_2 denote the open curve of β containing B_2 , and let A_2 denote the last point that AZB has in common with L_2 . This process can be continued (for a finite number of steps) until for some integer n the next point that AZB has in common with β following A_n is a point B_0 of

L . Since A_0 precedes B_0 in AZB , B_0 belongs to the ray \overrightarrow{AB} of L . Let J_1 denote the simple closed curve A_0XB_0 (of L) + A_0B_1 (of AZB) + A_1B_1 (of L_1) + A_1B_2 (of AZB) + \cdots + $A_{n-1}B_n$ (of AZB) + A_nB_n (of L_n) + A_nB_0 (of AZB). (See Figure 3.) It is clear that if B_1 exists, J_1 lies in $S - D$, because by Theorem 11(b) any

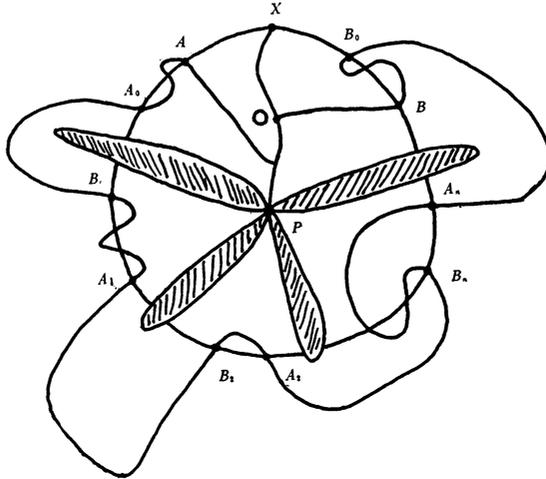


FIGURE 3

segment lying in D having its end points in different components of β contains P , and J_1 does not contain P . Now in case $J_1 = A_0XB_0$ (of L) + A_0B_0 (of AZB), suppose that $\widehat{A_0B_0}$ lies in D . Then J_1 lies in \bar{D} and $\beta \cdot J_1$ is connected; so by Theorem 36 one of the complementary domains I of J_1 is a subset of D . Let $P'X$ denote an arc lying in I except for X which is on the boundary of I . Now I does not contain P and J_1 does not intersect $\widehat{P'X} + P$; hence, I does not contain $\widehat{P'X}$, and therefore PX and $P'X$ abut on AXB from different sides. But since both $\widehat{P'X}$ and \widehat{PX} lie in D , this contradicts Theorem 35.

Thus in either case J_1 is a subset of $S - D$. By Theorem 16 one of the complementary domains I_1 of J_1 is a subset of $S - \bar{D}$ and is the interior of J_1 with respect to P . Let X' denote a point of $J_1 - J_1 \cdot L$, and let J_2 denote a simple closed curve separating X from $P + M + X'$. Let I_2 denote the interior of J_2 with respect to P . By Theorem 11 of Chapter III of *Foundations*, $J_1 + J_2$ contains a simple closed curve J_3 such that (1) I_3 , the interior of J_3 with respect to P , is a subset both of I_1 and of I_2 , and (2) $J_3 \cdot L$ is an arc containing X as an interior point. Let $A'X'B'$ denote an arc lying in I_3 except for points A' and B' which belong to $J_3 \cdot \widehat{A_0X}$ and $J_3 \cdot \widehat{B_0X}$ respectively. Then $J_4 = A'XB'$ (of L) + $A'X'B'$ is a simple closed curve lying in $S - (D + M)$ whose interior with respect to P is a subset of I_3 and hence, contains no point of M . Furthermore,

the common part of J_4 with β is the arc $A'XB'$ which is an interval of L containing X as an interior point.

THEOREM 41. *If D is a pseudo-simple domain, then $S - \bar{D}$ is connected.*

Proof. The boundary β of D is the sum of a finite number of mutually exclusive open curves L_1, L_2, \dots, L_j . By Theorem 40 there exists, for each point X of β , a simple closed curve J_X such that J_X is a subset of $S - D$ and $J_X \cdot \beta$ is for some $n \leq j$ an interval T_X of L_n containing X as an interior point. Let I_X denote the complementary domain of J_X which does not contain D . By Theorem 16, I_X is a subset of $S - \bar{D}$. Now for each $n, n \leq j$, let $D_n = \sum I_X$ for X on L_n . All the segments $J_X - T_X$ for X on L_n lie in $S - \bar{D}$ and abut on L_n from the same side. Hence the domains I_X overlap, so that D_n is a connected subset of $S - \bar{D}$. Furthermore, if T is any arc in $S - D$ which abuts on L_n at a point X , it abuts on L_n from the same side as $J_X - T_X$, and hence, contains points of I_X and D_n . So for each $n \leq j$, only one component of $S - \bar{D}$ has boundary points on L_n .

Now suppose that $S - \bar{D}$ is not connected. Then there exist two mutually exclusive domains H and K such that $S - \bar{D} = H + K$. Since space is connected, both H and K have boundary points in β , but for no $n, n \leq j$, does L_n contain boundary points of both H and K . Hence $\beta \cdot \bar{H}$ and $\beta \cdot \bar{K}$ are mutually exclusive closed point sets. Let AB denote an arc in $S - P$ irreducible from $\beta \cdot \bar{H}$ to $\beta \cdot \bar{K}$. Now since each $D_n, n \leq j$, belongs either to H or to K , $\beta = \beta \cdot \bar{H} + \beta \cdot \bar{K}$; so AB contains only two points A and B of β , and these points belong to different open curves of the collection L_1, L_2, \dots, L_j . Therefore, since AB does not contain P , the segment AB lies in $S - \bar{D}$, and hence, contains a point of I_A and a point of I_B . But, since I_A and I_B belong to H and K respectively, this is a contradiction.

THEOREM 42. *Every pseudo-simple domain is strongly simply connected.*

This theorem is a consequence of Theorems 16 and 41.

DEFINITION. A domain D is said to be *internally simple* provided that for each point X of D there exists a simple domain containing X and lying together with its boundary in D .

THEOREM 43. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P , then each component of $D - P$ is internally simple.*

Proof. Let U denote a component of $D - P$, and let X denote a point of U . Let AP denote an arc from A to P lying except for P in U . By Theorems 2 and 10 there exists a simple closed curve J crossing XP . If J intersects the boundary of D , then J contains an arc AYB crossing XP and lying except

for the points A and B of L in D . Let J_1 denote the simple closed curve $AB(\text{of } L) + AYB$. One of the complementary domains of J_1 is a subset of D and contains X . Let J_X denote a simple closed curve separating $J_1 + P$ from X . If J does not intersect the boundary of D , let J_X denote J . Then, one of the complementary domains I_X of J_X contains X , and $I_X + J_X$ is a subset of U . Hence, U is internally simple.

THEOREM 44. *If D is a pseudo-simple domain, $S - \bar{D}$ is internally simple.*

Proof. The domain D is pseudo-simple with respect to a domain Q and a point P . Let Δ denote the collection of mutually exclusive open curves whose sum is the boundary of D , and let L denote an element of Δ . Let X denote a point of $S - \bar{D}$, and let AX denote an arc irreducible from L to X and containing no point of $\Delta^* - L$. Suppose that $S - (\bar{D} + AX)$ is the sum of two mutually exclusive domains H and K . Because no arc separates space, both H and K have boundary points in Δ^* . Furthermore, because of Theorem 40 no element of $\Delta - L$ contains points of both $\Delta^* \cdot \bar{H}$ and $\Delta^* \cdot \bar{K}$, nor does either of the rays of L from A contain points of both $\Delta^* \cdot \bar{H}$ and $\Delta^* \cdot \bar{K}$. Let AP denote an arc in $D + A$ from A to P . Since $AX + AP$ does not separate space, there exists an arc T irreducible from $\Delta^* \cdot \bar{H}$ to $\Delta^* \cdot \bar{K}$ in $S - (AX + AP)$. Now the segment \hat{T} is a subset of $S - \bar{D}$, for if \hat{T} were a subset of D , it would have one end on one ray of L from A and the other end on the other ray from A without intersecting AP , which by Theorems 35 and 36 is impossible. But it is also impossible for \hat{T} to lie in $S - \bar{D}$, for in this case H and K would have boundary points in the same open curve of Δ . Hence, $S - (\bar{D} + AX)$ is connected.

Let EYF denote an arc lying except for Y in $S - (\bar{D} + AX)$ and crossing AX at the point Y . Since $S - (\bar{D} + AX)$ is connected, there exists an arc EZF lying in it from E to F . The sum of the arcs, EYF and EZF , contains a simple closed curve J crossing AX at the point Y whose interior with respect to P is a simple domain I containing X such that $I + J$ is a subset of $S - \bar{D}$.

THEOREM 45. *If D_1 and D_2 are pseudo-simple domains with respect to domains Q_1 and Q_2 and a point P respectively, Q_2 is a subset of D_1 , and C is a component of $D_1 - \bar{D}_2$, then C is an internally simple, strongly simply connected domain.*

Proof. The component C is strongly simply connected; for if a simple closed curve J is a subset of \bar{C} , then by Theorems 16 and 41 the interior I of J with respect to P is a subset of both D_1 and $S - \bar{D}_2$.

If X is a point of C , then by Theorems 43 and 44 there exist two simple domains, I_1 and I_2 , each containing X such that \bar{I}_1 and \bar{I}_2 are subsets of $D_1 - P$ and $S - \bar{D}_2$ respectively. By Theorem 12 of Chapter III of *Foundations*, $I_1 \cdot I_2$

contains a simple domain I_X which contains X and whose boundary is a subset of the sum of the boundaries of I_1 and I_2 .† Therefore, \bar{I}_X is a subset of C , and C is internally simple.

THEOREM 46. *If D is an internally simple, simply connected domain and AB is an arc lying in D except for its end points which belong to the boundary of D , then $D - D \cdot AB$ is the sum of two mutually exclusive, internally simple, simply connected domains, each having AB on its boundary.*

Proof. For each point X of \widehat{AB} let I_X denote a simple domain containing X such that \bar{I}_X is a subset of D . Let J_X denote the boundary of I_X and let C denote a simple closed curve separating X from $A + B + J_X$. By Theorem 10, $C + \widehat{AB}$ contains a simple closed curve C' separating X from $A + B + J_X$, whose common part with AB is the sum of two continua. Let $A'XB'$ denote the interval of AB containing X such that $A'XB' \cdot C' = A' + B'$. One of the complementary domains I' of C' is a subset of D and contains the segment $A'XB'$. The point set $I' - \widehat{A'XB'}$ is the sum of two simple domains I_{X_1} and I_{X_2} , one on one side of AB and the other on the other side of AB . Now if the subscripts 1 and 2 are so used so that all the domains I_{X_1} for X on AB are on the same side of AB and all the domains I_{X_2} are on the other side of AB , $D_1 = \sum I_{X_1}$ and $D_2 = \sum I_{X_2}$ are two connected domains which are subsets of D . Now each point O of $D - \widehat{AB}$ may be joined to a point of \widehat{AB} by an arc OE lying in $D - \widehat{AB}$ except for its one end point E , and \widehat{OE} must intersect either D_1 or D_2 . If for each i , $i = 1, 2$, H_i denotes the set of all points O such that the segment \widehat{OE} intersects D_i , then H_1 and H_2 are connected domains and $H_1 + H_2 = D - \widehat{AB}$. But if D is simply connected, no arc segment lying in D can abut on AB from different sides since neither A nor B belongs to D . Hence H_1 and H_2 are mutually exclusive, and AB belongs to the boundary of each of them.

Now suppose that J is a simple closed curve lying in H_1 . One of the complementary domains I of J is a subset of D . Therefore, I contains no point of AB . But I has limit points in H_1 , namely, the points of J , and hence contains points of H_1 . Consequently I is a subset of H_1 , and therefore H_1 is simply connected. Likewise, H_2 is simply connected.

Let X denote a point of H_1 . Since D is internally simple, there exists a simple domain I_X' containing X and lying together with its boundary J_X' in D . By Theorem 3 there exists a simple closed curve J_X separating X from $J_X' + AB$. The curve J_X is a subset of H_1 , and hence one of its complementary domains I_X is a subset of H_1 and contains X . Consequently, H_1 is internally simple and likewise H_2 is internally simple.

† The notation here does not correspond to that of Theorem 12 of Chapter III of *Foundations*.

THEOREM 47. *If D is an internally simple, simply connected domain, β is the boundary of D , J is a simple closed curve lying in D , and AB is an arc irreducible from J to the boundary of D , then $D - (J + \widehat{AB})$ is the sum of two mutually exclusive, internally simple, simply connected domains, one having J for its boundary, and the other having for its boundary $J + AB + \beta$.*

Theorem 47 may be established by methods similar to those used in proving Theorem 46.

THEOREM 48. *If D is a pseudo-simple domain and AB is an arc lying in $S - \overline{D}$ except for the points A and B which lie on the boundary of D , then $S - (\overline{D} + AB)$ is the sum of two connected domains whose boundaries have as their common part the arc AB .*

Proof. From Theorems 16, 41, 44, and 46, $S - (\overline{D} + AB)$ is the sum of two connected domains whose boundaries contain AB in their common part. And if β denotes the boundary of D , it is clear from Theorems 38 and 40 that this common part cannot contain a point of $\beta - (A + B)$.

THEOREM 49. *If under the hypothesis of Theorem 45, AXB is an arc lying in C except for the points A and B which lie on the boundary of C , then $C - \overline{AXB}$ is the sum of two connected domains whose boundaries have as their common part the arc AXB .*

Theorem 49 may be proved by an argument similar to that of the preceding theorem if we use Theorems 45 and 46 and a theorem similar to Theorem 40 and established in a similar manner.

DEFINITION. Suppose that X_1, X_2, X_3 , and X_4 are points of β , the boundary of a pseudo-simple domain. Then $X_1 + X_3$ is said to *ordinally separate* X_2 from X_4 on β if there exists a simple closed curve J whose common part with β is $X_1 + X_3$ and which separates X_2 from X_4 .

THEOREM 50. *Suppose that D is a pseudo-simple domain, β is the boundary of D , and X_1, X_2, X_3 , and X_4 are four distinct points of β . Then one pair of them ordinally separates the other two on β ; and if $X_1 + X_3$ ordinally separates X_2 from X_4 on β , then (1) $X_2 + X_4$ ordinally separates X_1 from X_3 on β , (2) every simple closed curve J containing a point of D and $S - D$ such that $J \cdot \beta = X_1 + X_3$ separates X_2 from X_4 , but (3) $X_1 + X_2$ does not ordinally separate X_3 from X_4 on β .*

Proof. Let X_1AX_2 and X_1BX_2 denote arcs lying except for X_1 and X_2 in D and $S - \overline{D}$ respectively. Suppose that $J' = X_1AX_2 + X_1BX_2$ does not separate X_3 from X_4 . Now let AX_3 and BX_3 denote arcs lying except for X_3 in D and $S - \overline{D}$ respectively, such that $AX_3 \cdot J' = A$ and $BX_3 \cdot J' = B$. With respect to X_4 , the interior of the simple closed curves formed by one pair of the three arcs AX_1B , AX_2B , and AX_3B contains the other arc segment. Hence, one

pair of the points $X_1, X_2, X_3,$ and X_4 ordinally separates the other two on β .

Suppose that X_1+X_3 ordinally separates X_2 from X_4 on β . Then there exists by a construction similar to that of the above paragraph two arcs X_2AX_4 and X_2BX_4 lying except for X_2 and X_4 in D and $S-\bar{D}$ respectively whose sum is a simple closed curve separating X_1 from X_3 . If J is a simple closed curve containing a point of D and a point of $S-\bar{D}$ whose common part with β is X_1+X_3 , then J is the sum of two arcs $X_1A'X_3$ and $X_1B'X_3$ lying except for X_1 and X_3 in D and $S-\bar{D}$ respectively. Let C denote the simple closed curve $X_2AX_4+X_2BX_4$. By a double application of Theorem 10, $C+\overline{X_1A'X_3}+\overline{X_1B'X_3}$ contains a simple closed curve C' whose common part with $\overline{X_1A'X_3}$ is connected and whose common part with $\overline{X_1B'X_3}$ is also connected and which separates X_2 from X_4 . It is clear now that J separates X_2 from X_4 .

Now suppose that X_1+X_2 ordinally separates X_3 from X_4 on β . Then there would exist a simple closed curve, having the properties of J in the preceding paragraph, which does not separate X_2 from X_4 . This is impossible if X_1+X_3 ordinally separates X_2 from X_4 .

Arguments of a similar nature show that if order on β is interpreted by the notion of ordinal separation, then we have

THEOREM 51. *If β is the boundary of a pseudo-simple domain, the points of β have a cyclic order which preserves the ordinary order on any open curve component of β .*

THEOREM 52. *Suppose that D_1 and D_2 are pseudo-simple domains with respect to domains Q_1 and Q_2 and a point P respectively, Q_2 is a subset of D_1 , $X_1, X_2, X_3,$ and $X_4,$ and $Y_1, Y_2, Y_3,$ and Y_4 are the end points of four mutually exclusive arcs $T_1, T_2, T_3,$ and T_4 which cross $D_1-\bar{D}_2$. Then if X_1+X_3 ordinally separates X_2 from X_4 on β_1 , the boundary of D_1 , then Y_1+Y_2 ordinally separates Y_2 from Y_4 on β_2 , the boundary of D_2 , and conversely.*

Proof. Let X_1AX_2 denote an arc lying except for the points X_1 and X_3 in $S-\bar{D}_1$, and let Y_1BY_3 denote an arc lying except for the points Y_1 and Y_3 in D_2 . Let J denote the simple closed curve $T_1+X_1AX_3+T_3+Y_1BY_3$. Since X_1+X_3 ordinally separates X_2 from X_4 on β_1 , J separates X_2 from X_4 . But J does not intersect T_2 or T_4 and hence, J separates Y_2 from Y_4 . Therefore, Y_1+Y_2 ordinally separates Y_2 from Y_4 on β_2 . Likewise, the converse is true.

2. CONSEQUENCES OF AXIOMS 0-4, 5_1^* AND 7

AXIOM 7. *The set of all points is completely separable.†*

† This axiom is numbered 7 because it is so numbered in *Foundations*. See *Foundations*, p. 412.

The purpose of this section is to outline an argument showing that a space satisfying Axioms 0-4, 5_1^* , and 7 is a subset of a space satisfying Axioms 0-4, 5_1^* , and 7 which contains no edge points. Considerable detail will be left to the reader, and notation introduced in a definition, when used subsequently without explanation, will be understood to have the significance given it in the definition.

DEFINITION. Suppose that P is an edge point. Suppose, furthermore, that there exist for each n , a pseudo-simple domain D_n containing \bar{D}_{n+1} whose boundary β_n is the sum of a finite number of open curves, $L_{n1}, L_{n2}, \dots, L_{nq_n}$, and for each n and q , $q \leq q_n$, two sequences of mutually exclusive arcs, $T_{1,nq}, T_{2,nq}, \dots$ and $T_{-1,nq}, T_{-2,nq}, \dots$, crossing $D_n - \bar{D}_{n+1}$ such that, denoting $T_{m,nq} \cdot \beta_n$ and $T_{m,nq} \cdot \beta_{n+1}$ by $X_{m,nq}$ and $Y_{m,nq}$ respectively, †

(0) for each n , D_n is pseudo-simple with respect to P and a domain Q_n which is a subset of a region of G_n of Axiom 1, which is in turn a subset of D_{n-1} , $D_0 = S$;

(1) for each m , L_{nq} contains $X_{m,nq}$;

(2) for each n , the points $X_{1,n1}, X_{2,n1}, \dots, \dots, X_{-2,n2}, X_{-1,n2}, X_{1,n2}, X_{2,n2}, \dots, \dots, \dots, X_{-2,nq_n}, X_{-1,nq_n}, X_{1,nq_n}, X_{2,nq_n}, \dots, \dots, X_{-2,n1}, X_{-1,n1}$ have on β_n the cyclic order indicated; ‡

(3) for each $Y_{m,nq}$, there exists an integer r , $r \geq q$, such that $X_{m,(n+1)r} = Y_{m,nq}$;

(4) for each n and q , $q \leq q_n$, neither of the sequences, $T_{1,nq}, T_{2,nq}, \dots$ and $T_{-1,nq}, T_{-2,nq}, \dots$ has a sequential limit set. Let Ws denote the point set consisting of P together with all the sets L_{nq} and $T_{m,nq}$. Such a set, Ws , is said to be a *radial web skeleton* about P .

THEOREM 53. *If R is a region containing an edge point P , R contains a radial web skeleton about P .*

Theorem 53 may be proved with the help of Theorems 15, 18, 32, 33, 45, and 52 and a method similar to that of Theorem 33.

DEFINITION AND NOTATION. Suppose that Ws is a radial web skeleton about an edge point P . For each n and q , $q \leq q_n$, L_{nq} of Ws is the sum of two rays L_{nq}^+ and L_{nq}^- having only one point in common and containing $X_{1,nq}, X_{2,nq}, \dots$ and $X_{-1,nq}, X_{-2,nq}, \dots$ respectively. Such rays are said to be *rays on β_n* and to have the cyclic order of the points $X_{m,nq}$. The point set Ws is said to connect L_{nq}^+ to $L_{(n+1)r}^+$ if $T_{1,nq}$ contains points of both of them. Likewise, Ws is said to connect L_{nq}^- to $L_{(n+1)r}^-$ if $T_{-1,nq}$ contains points of both of them.

† When an integral subscript letter is followed by a comma, its range is both the positive and negative integers.

‡ This order is selected arbitrarily on β_1 ; then it is transferred from β_1 to β_2 , from β_2 to β_3 , \dots by the arcs $T_{m,nq}$ using Theorem 52.

THEOREM 54. *Suppose that Ws is a radial web skeleton about an edge point P , and Ws connects L_{nq}^+ to $L_{(n+1)r}^+$ and L_{nq}^- to $L_{(n+1)r'}^-$. Then if $q' = q + 1, r' = r + 1$, and conversely.*

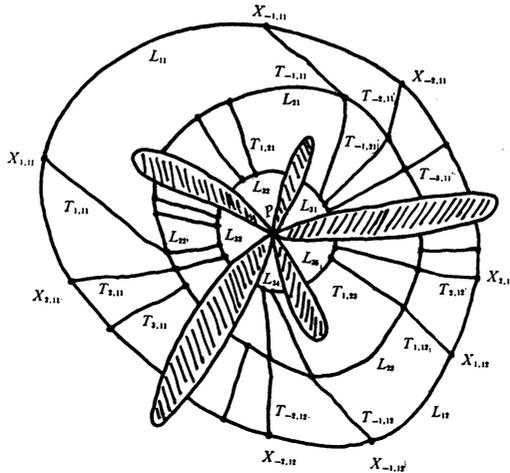


FIGURE 4

Proof. Suppose on the contrary, for instance, that $T_{1,11}$ contains a point of L_{11}^+ and of L_{23}^+ while $T_{-1,12}$ contains a point of L_{12}^- and of L_{25}^- . Since $T_{1,11} + T_{-1,12} + P$ does not separate space, there exists an arc T from L_{24} to β_1 of Ws lying except for its end points in $D_1 - \bar{D}_2$. Let $T \cdot \beta_1 = X$ and $T \cdot \beta_2 = Y$. Since Y is between $Y_{1,11}$ and $Y_{-1,12}$ on β_2 , X is between $X_{1,11}$ and $X_{-1,12}$ on β_1 by Theorem 52. Hence X is either on the subray of L_{11}^+ from $X_{1,11}$ or on the subray of L_{12}^- from $X_{-1,12}$. Suppose the former, and a similar argument will apply to the other case. Since the sequence of arcs $T_{1,11}, T_{2,11}, T_{3,11}, \dots$ has no sequential limit set, there is some one of them, $T_{m,11}$, which has no point in common with T such that $X_{m,11}$ is between X and $X_{-1,12}$ on β_1 . But $Y_{m,11}$ belongs to L_{23}^+ and is not between Y and $Y_{-1,12}$ on β_2 , which contradicts Theorem 52.

A similar argument will show the converse to be true; that is, if $r' = r + 1$, then $q' = q + 1$.

DEFINITION. Suppose that Ws is a radial web skeleton. Then L_{nq}^+ and $L_{n(q+1)}^-$ of Ws are said to be *adjacent rays* on β_n .†

THEOREM 55. *Suppose that Ws is a radial web skeleton. Then Ws connects each pair of adjacent rays on β_n to a pair of adjacent rays on β_{n+1} , and if Ws*

† When $q = q_n, q + 1$ is to be interpreted as 1.

connects either of a pair of adjacent rays on β_n to a ray on β_{n-1} , then Ws connects this pair to a pair of adjacent rays on β_{n-1} .

Notation. If Ws is a radial web skeleton, let C_{nq} denote the component of $D_n - \bar{D}_{n+1}$ whose boundary β_{nq} contains L_{nq} of Ws .

THEOREM 56. *If Ws is a radial web skeleton, then $\beta_{nq} \cdot \beta_{n+1}$ is the sum of a finite collection H_{nq} of consecutive rays on β_{n+1} consisting of two extreme rays, $L_{(n+1)q}^-$ and $L_{(n+1)q'}^+$, $q \leq q'$, and all other rays on β_{n+1} between $L_{(n+1)q}^-$ and $L_{(n+1)q'}^+$; and $L_{(n+1)q}^-$ and $L_{(n+1)q'}^+$ are the only rays of H_{nq} which are connected by Ws to rays of β_n .*

THEOREM 57. *If Ws is a radial web skeleton and n and q are integers such that neither L_{nq}^+ nor $L_{n(q+1)}^-$ of Ws is connected to a ray on β_{n-1} by Ws , then there exists a sequence of mutually exclusive arcs $U_{1[nq]}, U_{2[nq]}, U_{3[nq]}, \dots \dagger$ such that (1) for each $m, m > 0$, $U_{m[nq]}$ is an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these two points in $D_{n-1} - \bar{D}_n, D_0 = S$, and (2) the sequence has no sequential limit set.*

Proof. Let C denote the component of $D_{n-1} - \bar{D}_n$ whose boundary contains $L_{nq}^+ + L_{n(q+1)}^-$, and let L denote $\bar{C} \cdot \beta_{n-1}$.[‡] Let G denote the collection of all simple domains which lie together with their boundaries in C . By Theorems 44 and 45, G covers C . Since space is completely separable, G contains a countable subcollection I_1, I_2, I_3, \dots covering C . For each integer $m, m > 0$, let M_m denote an arc irreducible from J_m , the boundary of I_m , to L lying except for one end point in C . By Theorem 47, $C - C \cdot (J_m + M_m)$ is the sum of two connected domains, I_m and E_m , where the boundary of E_m consists of $J_m + M_m$ together with the boundary of C . For each m , let $U'_{m[nq]}$ denote an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these end points in E_m . By Theorem 49, $U'_{m[nq]}$ separates C into two internally simple, simply connected domains. One of these domains V'_m has for its boundary $U'_{m[nq]}$ plus the subray of L_{nq}^+ from $X_{m,nq}$ and the subray of $L_{n(q+1)}^-$ from $X_{-m,n(q+1)}$, and contains no point of I_m . If the arcs $U'_{m[nq]}$ are mutually exclusive, then they form the sequence $U_{1[nq]}, U_{2[nq]}, U_{3[nq]}, \dots$. If this is not the case, let $U_{1[nq]} = U'_{1[nq]}$ and $V_1 = V'_1$, and suppose that $U_{1[nq]} \cdot U'_{2[nq]}$ is not vacuous. Let A_1B_1 and A_2B_2 denote intervals of $U_{1[nq]}$ and $U'_{2[nq]}$ respectively which lie in C and contain $U_{1[nq]} \cdot U'_{2[nq]}$. By Theorems 10 and 45 and Theorem 13 of Chapter III of *Foundations*, there exists a simple closed curve J lying in C and separating $A_1B_1 + A_2B_2$ from $X_{2,nq} + X_{-2,n(q+1)}$ such that the common part of J with each of the intervals $A_2X_{2,nq}$ and $B_2X_{-2,n(q+1)}$ of $U'_{2[nq]}$ is connected. Now since

[†] This notation, $[nq]$, is used to restrict n and q to values for which the hypothesis of Theorem 57 is satisfied.

[‡] When $n = 1$, $\bar{C} \cdot \beta_{n-1}$ is to be interpreted as a point of L_{nq}^- .

$A_2\widehat{X}_{2,nq}$ and $B_2\widehat{X}_{-2,n(q+1)}$ are subsets of V_1 and contain no points of V_2' , J contains an arc AB lying in $V_1 \cdot V_2'$ except for A and B irreducible from $A_2X_{2,nq}$ to $B_2X_{-2,n(q+1)}$. Let $U_{2[nq]} = AB + AX_{2,nq}$ of $A_2X_{2,nq} + BX_{-2,n(q+1)}$ of $B_2X_{-2,n(q+1)}$, and let V_2 denote the component of $V_1 - \widehat{U}_{2[nq]}$ which has the subray of L_{nq}^+ from $X_{2,nq}$ and the subray of $L_{n(q+1)}^-$ from $X_{-2,n(q+1)}$ on its boundary. The domain V_2 is a subset of $V_1 \cdot V_2'$. This process may be continued and the required sequence of arcs constructed, for it is clear that since $V_1 \supset V_2 \supset V_3 \supset \dots$ and $\overline{V_1} \cdot \overline{V_2} \cdot \overline{V_3} \cdot \dots = 0$, the sequence of arcs has no sequential limit set.

Notation and construction. Now on a radial web skeleton it is possible to construct what will be called a *radial web*. Let Ws denote a radial web skeleton about an edge point P . For each integer triplet, m (positive or negative), $n > 0$, and $0 < q \leq q_n$, there exists a continuous reversible transformation of the point set $T_{m,nq} - Y_{m,nq}$ of Ws into the number set $1/n \geq t > 1/(n+1)$.

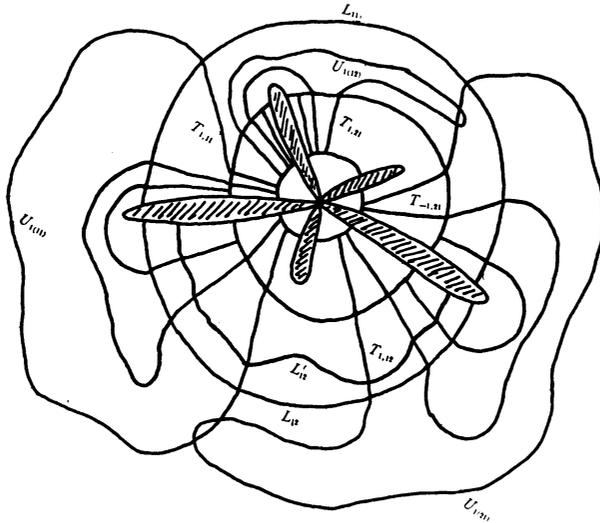


FIGURE 5

Let $X_{m,nq}^t$ denote the point of $T_{m,nq}$ whose transform is the number t . Then, of course, $X_{m,nq} = X_{m,nq}^{1/n}$. Let ρ denote a countable set of numbers everywhere dense on $0 < t < 1$ and containing all the reciprocals of positive integers except 1. For each pair of positive integers, n and q , $q \leq q_n$, and for each t of ρ such that $1/n \geq t > 1/(n+1)$, let L_{nq}^t denote an open curve lying in C_{nq} such that (1) $L_{nq}^t = L_{nq}$ if $t = 1/n$, (2) L_{nq}^t intersects $T_{m,nq}$ in the point $X_{m,nq}^t$, and (3) if t and t' are different numbers of ρ , $L_{nq}^t \cdot L_{nq}^{t'} = 0$. Since ρ is countable, these open curves may be constructed one at a time by the use of Theorem 18 and repeated applications of Theorem 46.

For each pair of integers, n and q , $n > 0$ and $q \leq q_n$, such that L_{nq}^+ is not

connected by W s to a ray on β_{n-1} , there exists by Theorem 57 a sequence $U_{1[nq]}, U_{2[nq]}, U_{3[nq]}, \dots$ of mutually exclusive arcs such that (1) for each $m, m > 0, U_{m[nq]}$ is an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these points in $D_{n-1} - \bar{D}_n, D_0 = S$, and (2) the sequence has no sequential limit set. For each t of ρ , let $\beta^t = \sum_{q=1}^{q_n} L_{nq}^t$. The point set W consisting of W s together with all the point sets β^t and $U_{m[nq]}$ is called a *radial web* about P , and W s is said to be the skeleton of W .

Notation and definition. If $X_{m,nq}$ is a point of a ray on β_n of the skeleton W s of the radial web W which is not connected to a ray on β_{n-1} by W s, let $N_{m,[nq]}$ denote $P + \sum_{r=n}^{\infty} T_{m,rp_r}$, where $p_n = q$ and for $r > n, p_r$ is the integer such that $X_{m,rp_r} = Y_{m,(r-1)p_{r-1}}$. For each $[nq]$ and $m, m > 0$, let $J_{m[nq]}$ denote the simple closed curve $N_{m,[nq]} + U_{m[nq]} + N_{-m,[n(q+1)]}$, and let $I_{m[nq]}$ denote the complementary domain of $J_{m[nq]}$ which contains $J_{m+1[nq]} - P$.[†] For each t of ρ , let $U_{m[nq]}^t$ denote the interval of $J_{m[nq]} - P$ which is irreducible from one component of β^t to another component of β^t with $t \leq 1/n$.

For each pair of values, t' and t'' of $\rho, t' > t''$, let $\beta_{m,[nq]}^{t''}$ denote the point set consisting of the interval of $N_{m,[nq]}$ between $\beta^{t'}$ and $\beta^{t''}$, when both t' and t'' are less than $1/n$, together with the rays of $\beta^{t'}$ and $\beta^{t''}$ from the end points of this interval which intersect $N_{m+1,[nq]}$. It is clear that $\beta_{m,[nq]}^{t''}$ separates space into two mutually exclusive connected domains, one of which, $I_{m,[nq]}^{t''}$, is a subset of $I_{m[nq]}$, is the sum of an infinite number of simple domains, and has $\beta_{m,[nq]}^{t''}$ for its complete boundary. For each $m, m > 0$, let $R_{m,[nq]}^{t''}$ denote $I_{m,[nq]}^{t''} + I_{-m,[n(q+1)]}^{t''}$. The boundary, $\beta_{m,[nq]}^{t''}$, of $R_{m,[nq]}^{t''}$ is $\beta_{m,[nq]}^{t''} + \beta_{-m,[n(q+1)]}^{t''}$. Let $P_{[nq]}^t$ denote the collection of all domains $R_{m,[nq]}^{t''}$, where $t'' < t < t'$. The collection $P_{[nq]}^t$ will be hereinafter referred to as an *ideal point*.[‡] The set of ideal points, $\sum_t P_{[nq]}^t$, will be called the *spur* $N_{[nq]}$ and $\sum N_{[nq]}$ will be called the *cluster* at P or about P .

THEOREM 58. For each $[nq]$, the sequence $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \dots$ has P for its sequential limit set and $I_{1[nq]} \cdot I_{2[nq]} \cdot I_{3[nq]}, \dots = 0$.

THEOREM 59. For each $m, W - W \cdot \sum_{[nq]} I_{m[nq]}$ is a closed and compact point set.

THEOREM 60. If $P_{[nq]}^t$ is an ideal point and M is a compact point set, there exists a number δ and an integer \bar{m} , such that if $0 < t' - t'' < \delta, m > \bar{m}$, and $R_{m,[nq]}^{t''}$ belongs to $P_{[nq]}^t$, then $\bar{R}_{m,[nq]}^{t''}$ contains no point of M .

THEOREM 61. If t is a number of ρ and M is a compact point set, there exists a number \bar{m} such that if $m > \bar{m}$, then for no $[nq]$ does $U_{m[nq]}^t$ contain a point of M .

[†] The point sets $U_{m[nq]}$ may intersect the sets β^t in more than one point.

[‡] This is analogous to a method used by R. L. Moore to define point in terms of *piece*. See R. L. Moore, *A set of axioms for plane analysis situs*, *Fundamenta Mathematicae*, vol. 25 (1935), pp. 13-28.

Notation and construction. By Theorem 24 the set of edge points may be put into a simple countable sequence P_1, P_2, P_3, \dots . Let W_1, W_2, W_3, \dots and, for each integer j , $R_{1j}, R_{2j}, R_{3j}, \dots$ and $D_{1j}, D_{2j}, D_{3j}, \dots$ denote sequences of point sets such that (1) for each n and j , R_{nj} is a region of G_{n+j} of Axiom 1, (2) for each n and j , D_{nj} is a pseudo-simple domain with respect to P_j and a simply connected domain Q_{nj} lying with its boundary in R_{nj} , and D_{nj} contains $R_{(n+1)j}$, (3) for each j , W_j is a radial web about P_j whose skeleton is constructed on the domains, $D_{1j}, D_{2j}, D_{3j}, \dots$, and lies in R_{1j} , (4) if $j' > j$ and $R_{1j'}$ contains a point of D_{1j} , then $\bar{R}_{1j'}$ is a subset of R_{1j} , and (5) for each integer j' , there exists an integer m such that if $j < j'$, then $\bar{R}_{1j'}$ contains no point of $\sum_{[nq]} I_{m[nq]}$ of W_j . Let G denote a countable collection of simple domains such that if X is a simple point of S and R is a region containing X , some domain of G contains X and lies together with its boundary in R . For each integer i , let H_i denote the collection of all the domains D such that either (1') D is a simple domain of G which is a subset of a region of G_i of Axiom 1, or (2') for some pair of integers n and j , $n+j \geq i$, D is a pseudo-simple domain D_{nj} , or (3') for some triplet of integers, m, n , and q , and some pair of numbers, t' and t'' , of ρ , with $m \geq i$ and $0 < t' - t'' \leq 1/i$, $D = R_{m[nq]}^{t't''}$ of W_j for some j .

DEFINITION. If for some i and j , D is a domain of H_i and $P_{[nq]}^i$ is an ideal point defined from W_j , then D is said to *inclose* $P_{[nq]}^i$ provided that some domain of $P_{[nq]}^i$ together with its boundary is a subset of D .

DEFINITION. Let S' be a space in which *point* is to be interpreted as meaning either a point of S or one of the ideal points defined by W_j for some j , and in which *region* is to be interpreted as meaning a point set g' consisting of the points of some domain D of H_i , for some i , together with the ideal points which D incloses. A point of S' which is also a point of S will be called an "ordinary" point and a point of S' which is not a point of S will be called an "ideal" point. Regions in S' will be referred to as G' -regions.

From here on the space to be considered is S' .

THEOREM 63. *In S , point and limit point are unchanged by the above definition of point and region in S' .*

THEOREM 64. *Every point of S' is a limit point of S .*

THEOREM 65. *If g' is a G' -region, then $\bar{g}' = \overline{g' \cdot S}$.*

THEOREM 66. *If J is a simple closed curve of "ordinary" points, then $S' - J$ is the sum of two mutually exclusive connected domains each having J for its complete boundary.*

Proof. The point set $S - J$ is the sum of two mutually separate connected sets H and K . Let $H' = \bar{H} - \bar{H} \cdot J$ and $K' = \bar{K} - \bar{K} \cdot J$. Suppose that H' has a point A in common with K' . The point A is an "ideal" point $P'_{[nq]}$. By Theorem 60 there exists a number \bar{m}_1 such that if $m > \bar{m}_1$ and $R''''_{m[nq]}$ belongs to $P'_{[nq]}$, then $\bar{R}''''_{m[nq]} \cdot S$ contains no point of J . But $R''''_{m[nq]}$ contains points of both H and K . Hence, $I''''_{m,[nq]} + \beta''''_{m,[nq]}$ is a subset of H , say, and $I''''_{-m,[n(q+1)]} + \beta''''_{-m,[n(q+1)]}$ is a subset of K . By Theorem 61 there exists a number \bar{m}_2 such that if $m > \bar{m}_2$, then $U^t_{m[nq]}$ does not intersect J . But for each m , $U^t_{m[nq]}$ intersects both $\beta''''_{m,[nq]}$ and $\beta''''_{-m,[n(q+1)]}$, $t'' < t < t'$, which leads to a contradiction when m is greater than both \bar{m}_1 and \bar{m}_2 . Hence, H' and K' have no point in common. Since H and K are connected and have J on their respective boundaries, H' and K' are connected and have J for their respective boundaries. By Theorem 64, $S' = J + H' + K'$.

THEOREM 67. *If $P^t_{[nq]}$ belongs to the cluster of "ideal" points about the point P_j , and g' is a G' -region containing the point $P^t_{[nq]}$ such that $g' \cdot S$ is not a domain of $P^t_{[nq]}$, then $\bar{g}' \cdot S$ contains P_j .*

Proof. Suppose that $g' \cdot S$ is not a domain of $P^t_{[nq]}$ but contains a domain R of $P^t_{[nq]}$ and $\bar{g}' \cdot S$ does not contain P_j .

Case I. If $g' \cdot S$ is of type (1'), † its boundary J separates R from P_j . Therefore, infinitely many of the simple closed curves $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \dots$ of W_j would intersect J . But since J is a compact subset of S , the sequential limit set of $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \dots$ of W_j would contain points of S other than P_j , contrary to Theorem 58.

Case II. If $g' \cdot S$ is of type (2') and its boundary was used in the construction of the web $W_{j'}$, then j' is not j and one may see, from Theorem 58 and a method similar to that of Theorem 40, that the boundary of $g' \cdot S$ and a finite number of the arcs of the type $U^t_{m[nq]}$ of $W_{j'}$, contains a simple closed curve J lying in S and separating $g' \cdot S$, and hence R , from P_j . This leads to a contradiction as in Case I.

Case III. If $g' \cdot S$ is of type (3'), there exists an integer j' , such that j' is not j and $g' \cdot S$ was defined using the web $W_{j'}$. Hence, $P_{j'}$ is not P_j and the boundary of $g' \cdot S$ together with one of the simple closed curves of the type $J_{m[nq]}$ of $W_{j'}$, contains a simple closed curve J lying in S and separating $g' \cdot S$, and hence R , from P_j . This leads to a contradiction as in Case I.

DEFINITION. For each i , let G'_i denote the collection of all G' -regions g' of S' such that $D = g' \cdot S$ is an element of H_i . The region g' is said to be *defined* by or *obtained from* D .

† See Notation and construction, p. 87.

THEOREM 68. *If $P'_{[nq]}$ is an "ideal" point, there exists an integer k such that no region of G'_k contains $P'_{[nq]}$ except those regions of G'_k defined from domains belonging to $P'_{[nq]}$.*

Proof. Let P_j denote the edge point P such that the cluster about P contains $P'_{[nq]}$. Let \bar{n} denote an integer such that $t > 1/\bar{n}$. There exists an integer \bar{m} such that if i is an integer greater than \bar{m} and R is a region in S of G_i of Axiom 1 which contains P_j , then R is a subset of $D_{\bar{m}j}$ in S . Suppose that g' is a region of G'_i , $i > \bar{m}$, of the type (1') or (2') which contains $P'_{[nq]}$.[†] Then by Theorem 67, $\bar{g}' \cdot S$ contains P_j . But $\bar{g}' \cdot S$ is a subset of a region R of G_1 of Axiom 1. Hence, R contains P_j and points not belonging to $D_{\bar{m}j}$, which is a contradiction. Likewise, if $j \geq \bar{m}$ and $j \neq \bar{j}$, no region of type (3') obtained from a domain of an "ideal" point of the cluster about P_j contains $P'_{[nq]}$ because each such domain together with its boundary in S is a subset of a region of G_i of Axiom 1 with $i > \bar{m}$. Now by Theorem 58, for each integer j , $j < \bar{m}$ and $j \neq \bar{j}$, there exists an integer \bar{m}_j such that for $m > \bar{m}_j$, $\sum_{[nq]} I_{m[nq]}$ defined from W_j does not contain P_j . Then if g' is a region of type (3') of G'_i , $i > \bar{m}_j$, defined by W_j , $\bar{g}' \cdot S$ does not contain P_j . Hence, by Theorem 67, R does not contain $P'_{[nq]}$. Let k denote an integer greater than \bar{m} and for each j , $j < \bar{m}$ and $j \neq \bar{j}$, greater than \bar{m}_j . Thus, if $i > k$ and g' is a region of G'_i which contains $P'_{[nq]}$, then g' was obtained from a domain of $P'_{[nq]}$, for no two regions of type (3') defined by W_j containing points of different spurs of the cluster about P_j have a point in common.

THEOREM 69. *The space S' satisfies Axiom 1.*

Proof. It can be shown with the help of Theorems 67 and 68 that S' satisfies parts (1), (2), and (3) of Axiom 1.[‡] Considerable argument is required but is left to the reader.

It will now be shown that part (4) of Axiom 1 is satisfied. Suppose that M_1, M_2, M_3, \dots is a sequence of closed subsets of S' such that for each i , M_i contains M_{i+1} and is a subset of a region g'_i of G'_i .

Case I. Suppose that for infinitely many values of i , g'_i is of type (1') or (2'). Suppose for simplicity that this is true for all integers i . Then for each i , $K_i = (\bar{g}'_1 \cdot S) \cdot (\bar{g}'_2 \cdot S) \cdot \dots \cdot (\bar{g}'_i \cdot S)$ is a non-vacuous point set closed with respect to S such that K_i contains K_{i+1} . But for each i , $\bar{g}'_i \cdot S$, and hence K_i , is a subset of a region R_i of G_i of Axiom 1. By Axiom 1 for S , there exists a point P of S which for each i belongs to K_i . Hence in S' , P belongs to the

[†] See Notation and construction, p. 87.

[‡] See footnote p. 54. The role of the collections G_1, G_2, G_3, \dots is now to be taken by the collections G'_1, G'_2, G'_3, \dots .

sequential limit set of M_1, M_2, M_3, \dots and therefore for each i , P belongs to M_i .

Case II. If Case I does not occur, then there exists an integer \bar{j} such that for infinitely many values of i , g'_i is a G' -region of type (3') defined by the web $W_{\bar{j}}$; for otherwise infinitely many of these regions of type (3') could be replaced by regions of type (2') satisfying the above conditions. Suppose for simplicity that for each i , g'_i is of type (3') and defined by $W_{\bar{j}}$. Since no two regions of type (3') defined by $W_{\bar{j}}$ which have a point in common contain points of different spurs of the cluster about $P_{\bar{j}}$, there exist for each i , integers m, n , and q , and numbers t' and t'' , $0 < t'' < t' < 1$, such that (1) $R_{m[nq]}^{t''}$ defined from $W_{\bar{j}}$ is $g'_i \cdot S$, (2) $[nq]$ does not vary with i , and (3) $m \rightarrow \infty$ and $(t' - t'') \rightarrow 0$, as $i \rightarrow \infty$. Since for each i , g'_i contains points of g'_i , there exists a number t , $0 < t < 1$, such that as $i \rightarrow \infty$, $t' \rightarrow t$ and $t'' \rightarrow t$. Hence $P_{[nq]}^t$ is common to M_1, M_2, M_3, \dots .

THEOREM 70. *Every G' -region is a simple domain.*

THEOREM 71. *The space S' satisfies Axioms 2, 3, 5_1^* , and 7 and contains no edge points.*

Theorem 71 is a consequence of Theorems 64, 69, and 70 and the fact that the collection of all G' -regions is countable.

THEOREM 72. *If J is a simple closed curve of "ordinary" points, I is one of its complementary domains, the points A and B separate the points C and D on J , and AXB is an arc of "ordinary" points such that \widehat{AXB} is a subset of I , then $I - \widehat{AXB}$ is the sum of two simple domains I_1 and I_2 such that the boundary of I_1 is the simple closed curve ACB (of J) + AXB and the boundary of I_2 is the simple closed curve ADB (of J) + AXB .*

Proof. Since the theorem holds for S , it can be shown to hold for S' with the help of Theorem 66.

THEOREM 73. *Theorem 72 remains true when the segment AXB contains just one "ideal" point.*

Proof. Suppose for simplicity of notation that X is the "ideal" point of the segment AXB . Let W_j denote the web used in the definition of the domains of X , and let $N_{[nq]}$ denote the spur which contains X . A number of cases occur depending upon where the edge point P_j is. Only one of these cases will be treated. The others may be dealt with in a somewhat similar but more complicated manner.

Suppose that P_j does not belong to $J + AXB$. Let $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \dots$ denote the sequence of simple closed curves in W_j which has as its sequential limit set $P_j + N_{[nq]}$. It can be shown by using Theorem 10 that there exists a

sequence C_1, C_2, C_3, \dots of simple closed curves whose sequential limit set is $P_j + N_{[nq]}$ such that for each k , (1) there is an integer $m, m \geq k$, so that C_k is a subset of $J_{m[nq]} + AXB$, (2) $C_k \cdot \overrightarrow{AX}$ is an arc $A'_k A''_k$ and $C_k \cdot \overrightarrow{BX}$ is an arc $B'_k B''_k$, (3) the components of $C_k - C_k \cdot AXB$ are two segments, $A'_k X_k B'_k$ and $A''_k Y_k B''_k$, which approach the rays \overrightarrow{AX} and \overrightarrow{BX} in X from different sides but $A'_k X_k B'_k$ and $A''_{k+1} X_{k+1} B'_{k+1}$ approach these rays from the same side, and (4) the common part of C_k and C_{k+1} is at most P_j , C_k separating $C_{k+1} - C_{k+1} \cdot C_k$ from J . For each k , let I_{1k} and I_{2k} denote respectively the complementary domains of AA'_k (of AX) + $A'_k X_k B'_k$ + BB'_k (of BX) + ACB or AA''_k (of AX) + $A''_k Y_k B''_k$ + BB''_k (of BX) + ACB and AA'_k (of AX) + $A'_k X_k B'_k$ + BB'_k (of BX) + ADB or AA''_k + $A''_k Y_k B''_k$ + BB''_k (of BX) + ADB which are subsets of I . Let I_1 denote $I_{11} + I_{12} + I_{13} + \dots$ together with the limit points of this set which do not belong to $J + AXB$, and let I_2 denote $I_{21} + I_{22} + I_{23} + \dots$ together with the limit points of this set which do not belong to $J + AXB$. These point sets, I_1 and I_2 , can be shown to be domains satisfying the conclusions of Theorem 72.

THEOREM 74. *Theorem 72 remains true if the arc AXB contains only a finite number of "ideal" points.*

Proof. The method of proof will be that of mathematical induction. Theorem 73 shows that Theorem 74 holds true when the number of "ideal" points is one. We shall suppose that the theorem holds for $j-1$ or less points and show that it holds for j points. Suppose that $AXB \cdot (S' - S) = Z_1 + Z_2 + \dots + Z_j$. Let $N_{[nq]}$ denote a spur containing one of these "ideal" points, and let Z_r denote that one of them on $N_{[nq]}$ which is farthest in $N_{[nq]}$ from its end point. Now the argument of Theorem 73 may be applied, Z_r playing the role of X .

THEOREM 75. *If J is a simple closed curve containing only a finite number of "ideal" points, then $S' - J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

Proof. Let O denote an ordinary simple point of J , and let R denote a simple domain whose boundary contains no "ideal" points such that $R \cdot J$ is a subset of an arc in J of "ordinary" points. Let ACB denote an arc of "ordinary" points in R which has only the two points A and B in common with J and such that an arc ADB of J is a subset of R . It is now easy to see that, with the help of Theorem 74, Theorem 75 may be established.

THEOREM 76. *Theorem 72 remains true regardless of the composition of the arc AXB .*

Proof. By slight modifications of the arguments of R. L. Moore a number

of the theorems of Chapter III of *Foundations* can be shown to hold true in S' if the simple closed curves involved contain at most a finite number of "ideal" points.† In particular Theorem 13 of Chapter III of *Foundations* holds true in S' for simple closed curves which contain at most a finite number of "ideal" points. Theorem 76 may then be established in the following manner.

Since the boundaries of G' -regions are simple closed curves containing at most a finite number of "ideal" points, it is easy to see that there exist two sequences of mutually exclusive arcs $A'_1 X_1 B'_1, A'_2 X_2 B'_2, A'_3 X_3 B'_3, \dots$ and $A''_1 Y_1 B''_1, A''_2 Y_2 B''_2, A''_3 Y_3 B''_3, \dots$ such that (1) for each k the segments $A'_k X_k B'_k$ and $A''_k Y_k B''_k$ are subsets of I , contain at most a finite number of "ideal" points and the points A'_k, B'_k, A''_k , and B''_k belong to segments AC, BC, AD , and BD of J respectively, (2) for each k , the complementary domain of $A'_k X_k B'_k + A'_k A A'_k$ (of J) $+ B'_k B B'_k$ (of J) $+ A''_k Y_k B''_k$ which is a subset of I contains the segment of AXB and the corresponding complementary domain of $A'_{k+1} X_{k+1} B'_{k+1} + A'_{k+1} A A'_{k+1}$ (of J) $+ B'_{k+1} B B'_{k+1}$ (of J) $+ A''_{k+1} Y_{k+1} B''_{k+1}$, and (3) the sequential limit set of either of these two sequences of arcs is AXB . If for each k , I_{1k} and I_{2k} denote respectively the complementary domains of $A'_k C B'_k$ (of J) $+ A'_k X B'_k$ and $A''_k D B''_k$ (of J) $+ A''_k Y B''_k$ which are subsets of I , then $I_1 = \sum I_{1k}$ and $I_2 = \sum I_{2k}$ satisfy the conclusions of Theorem 72.

Now as in Theorem 75 it is possible to establish

THEOREM 77. *If J is a simple closed curve containing an arc of "ordinary" points, then $S' - J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

With the help of Theorem 77 and methods similar to those in *Foundations* it can be shown that Theorem 74 remains true when the simple closed curve J is of any nature, provided it contains an arc of "ordinary" points. With this in mind, it is easy to see that Theorem 76 remains true when J is of any nature, provided it contains an arc of "ordinary" points. Then as in Theorem 74 it is possible to establish

THEOREM 78. *If J is a simple closed curve, $S' - J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

THEOREM 79. *The space S' satisfies Axiom 4.*

THEOREM 80. *The space S is homeomorphic with a subset of a space S' which satisfies Axioms 0-4, 5_1^* and 7 and contains no edge points.*

† It may be helpful for the reader to note that an arc in S' may be approximated by an arc in S which contains at most a finite number of "ideal" points since a G' -region has at most two "ideal" points in its boundary.

THEOREM 81. *The space S is homeomorphic with a subset of a plane or a sphere.*

Proof. J. H. Roberts† has shown that S' of Theorem 80 is homeomorphic with a subset of a plane or a sphere.

Some of the preceding results will now be summarized for metric spaces

THEOREM 82. *A locally connected, complete metric space in which the Jordan Curve Theorem‡ is satisfied non-vacuously and in which Axiom 5_1^* holds true, is a cyclicly connected subset of a plane (if not compact) or a sphere (if compact).*

Proof. A locally connected complete metric space satisfies Axioms 0, 1, and 2. And such a space in which Axiom 5_1^* holds true satisfies Axioms 4 and 3 if the Jordan Curve Theorem holds true non-vacuously (Theorem B). Axiom 7 holds true in such a space by Theorem 31. Thus Theorem 82 follows from Theorem 81.

THEOREM 83. *A locally connected, locally peripherally connected, complete metric space in which the Jordan Curve Theorem is non-vacuously satisfied is a cyclicly connected subset of a plane or a sphere.§*

THEOREM 84. *A locally connected, locally peripherally compact, complete metric space in which the Jordan Curve Theorem is non-vacuously satisfied is a cyclicly connected subset of a plane or a sphere.*

† J. H. Roberts, *Concerning compact continua in certain spaces of R. L. Moore*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 615–621.

‡ Axiom 4.

§ See p. 53.