ON EFFECTIVE SETS OF POINTS IN RELATION TO INTEGRAL FUNCTIONS*

BY

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1. **Introduction.** Let $f(z)$ be an integral function and let $M(r, f) = \max_{|z| \leqq r} |f(z)|$. The order ρ and the type $\kappa(f)$ of $f(z)$ are defined by the relations

$$\rho = \limsup_{r\to\infty} \frac{\log \log M(r, f)}{\log r}; \qquad \kappa(f) = \limsup_{r\to\infty} \frac{\log M(r, f)}{r^\rho}.$$

Let $[z_n]$, $z_n = r_n e^{i\theta_n}$ be a distinct sequence of complex numbers such that

$$0 < r_1 \leqq r_2 \leqq \cdots \leqq r_n \to \infty$$

as $n \to \infty$. Let $\rho_1 > 0$ be any number. The type $\kappa(f, \rho_1, [z_n])$ of $f(z)$ over the set $[z_n]$ is defined by the relation

$$\kappa(f, \rho_1, [z_n]) = \limsup_{n\to\infty} \frac{\log |f(z_n)|}{|z_n|^{\rho_1}}.$$

If $f(z)$ is of order ρ, it is evident that $\kappa(f, \rho_1, [z_n]) \leqq 0$ when $\rho_1 > \rho$. If $\rho_1 \leqq \rho$, the value of $\kappa(f, \rho_1, [z_n])$ can vary from $-\infty$ to ∞.

1.1. Definition. Let $f(z)$ be a function of order ρ; we shall say that $[z_n]$ is an effective set, or briefly an E-set, for $f(z)$ when $\kappa(f, \rho, [z_n]) = \kappa(f)$.

1.2. It is easy to see that any given function $f(z)$ always possesses an E-set; for, on $|z| = r$, there is at least one point $z(r)$ such that $M(r, f) = |f(z(r))|$; also, a sequence $[r_n]$, $r_1 < r_2 < \cdots < r_n \to \infty$ as $n \to \infty$, exists for which

$$\kappa(f) = \lim_{n\to\infty} \frac{\log M(r_n, f)}{r_n^{\rho}};$$

hence $[z(r_n)]$ is an effective set for $f(z)$. A more interesting question is to ascertain whether all functions of a given class specified by some simple property possess an *E-set in common.* In this paper an attempt is made to answer this question.

1.3. We denote $C(\rho, d)$ the class of all functions of order ρ and type less than d where ρ and d are any two given positive numbers. We regard all functions of order less than ρ as of order ρ and minimal type, that is $\kappa(f) = 0$, un-

* Presented to the Society, September 10, 1937; received by the editors January 21, 1937.

less $f(z) \equiv 0$ in which case $\kappa(f) = \kappa(f, \rho, [z_n]) = -\infty$ for all ρ and $[z_n]$. These are included in $C(\rho, d)$ for the purposes of this paper.

2. We shall, first, discuss a few general properties of an E-set for a given class $C(\rho, d)$.

THEOREM 1. *In order that a set* $[z_n]$ *may be an E-set for a class* $C(\rho, d)$ *it is necessary that*

(i) *the exponent of convergence (which we shall speak of as the order) of* $[z_n]$ *cannot be less than* ρ;

(ii) *if the order of* $[z_n]$ *be* ρ, *any function with zeros at* $z = z_n$ *must be of order* ρ *and type not less than d unless such a function is identically zero*;

(iii) *the set* $[\theta_n]$ *of amplitudes of* $[z_n]$ *must be everywhere dense in* $0 \leqq \theta \leqq 2\pi$.

Proof. If $[z_n]$ were of order $\rho' < \rho$, the canonical product $\sigma(z)$ with simple zeros at $[z_n]$ is of order ρ' and therefore is of order ρ and minimal type so that, by the definition of $C(\rho, d)$

$$\kappa(\sigma) = \kappa(\sigma, \rho, [z_n]) = 0.$$

But $\sigma(z_n) = 0$ so that

$$\kappa(\sigma, \rho, [z_n]) = -\infty.$$

This contradiction shows that the order of $[z_n]$ cannot be less than ρ. A similar argument proves (ii). To prove (iii), suppose that $[\theta_n]$ is not everywhere dense in $(0, 2\pi)$. Then there is a θ_0 such that $\theta_0 - \delta \leqq \theta \leqq \theta_0 + \delta$ contains no θ_n, $\delta > 0$ being sufficiently small. We can suppose without loss of generality that $\theta_0 = 0$ so that the angle $|\theta| \leqq \delta$ does not contain any point of $[z_n]$. Now, let $H_\rho(z)$ be defined by*

$$H_\rho(z) = \begin{cases} \displaystyle\prod_{n=1}^{\infty} \left(1 + \frac{z}{n^{1/\rho}}\right), & 0 < \rho \leqq \frac{1}{2}, \\ \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\dfrac{n}{\rho} + 1\right)}, & \rho > \frac{1}{2}. \end{cases}$$

It is known that, if z is outside $|\theta| \leqq \delta$,

$$\limsup_{|z| \to \infty} \frac{\log |H_\rho(z)|}{|z|^\rho} < \kappa(H_\rho).$$

By considering a function of the form $H_\rho(\eta z)$ where η is such that $\eta^\rho \kappa(H_\rho) < d$, we conclude that $[z_n]$ cannot be an E-set for $H_\rho(\eta z)$ which obviously belongs to $C(\rho, d)$. Hence $[\theta_n]$ is everywhere dense in $(0, 2\pi)$. It may be noted that

* For $0 < \rho \leqq \frac{1}{2}$, see Paley and Wiener, *Fourier Transforms in the Complex Domain*, p. 79; for $\rho > \frac{1}{2}$, $H_\rho(z)$ are Mittag-Leffler's functions, Acta Mathematatica, vol. 29 (1905), pp. 101–181.

since an E-set remains an E-set when any other set is added to it, we cannot expect to improve upon the result (i) of Theorem 1.

2.1. We shall now give a general criterion for a set $[z_n]$ to form an E-set for a class $C(\rho, d)$. Let $A_n(h)$ denote the circle with center z_n and radius $|z_n|^{-h}$; and let $A(h)$ denote the system of circles $A_n(h)$, $n=1, 2, \cdots$. We prove

THEOREM 2. *The set* $[z_n]$ *of order* ρ *will form an E-set for* $C(\rho, d)$ *provided there exists a function* $g(z)$ *with simple zeros at* $z=z_n$ *and* $h>\rho$ *such that the following relations hold*:

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{\log |g'(z_n)|}{|z_n|^\rho} = d; \qquad \text{(ii)} \qquad \lim \frac{\log |g(z)|}{|z|^\rho} = d$$

as $|z| \to \infty$ *outside the circles* $A(h)$.

2.2. We shall establish two lemmas in the first place.

LEMMA 1. *There exists a sequence* $[R_n]$, $R_1 < R_2 < \cdots < R_n \to \infty$ *as* $n \to \infty$, $R_{n+1} \leqq aR_n$, $a>1$ *being given, such that no circle* $|z| = R_n$, $n=1, 2, \cdots$, *cuts any circle of* $A(h)$.

Proof. Let $b>1$. Consider the ring $r \leqq |z| \leqq br$. The sum of the diameters of those circles of $A(h)$ whose centers lie in the ring cannot exceed $\sum_{r\leqq |z_n| \leqq br} |z_n|^{-h}$ which is less than a fixed positive constant since $\sum |z_n|^{-h}$ converges when $h>\rho$. Therefore if r is sufficiently large, there is at least one circle $|z| = R$ in the ring $r \leqq |z| \leqq br$ which does not cut any circle of $A(h)$. Hence, there is an n_0 such that for all $n \geqq n_0$, the ring $b^n \leqq |z| \leqq b^{n+1}$ contains a circle of the type required. Taking $b=a^{1/2}$ we get the required result.

LEMMA 2. *Any function* $g(z)$ *satisfying the condition* (ii) *of Theorem* 2 *is of order* ρ *and type* d.

Proof. Let $a>1$ be given and let $[R_n]$ be the sequence of Lemma 1. On $|z| = R_n$, we have, by (ii)

$$M(R_n, g) \leqq \exp \left[(d + \epsilon)R_n^\rho\right]$$

for $n \geqq n_0 = n_0(\epsilon)$. Since $M(r, g)$ is an increasing function of r and $R_{n+1} \leqq aR_n$, we get for all $r \geqq r_0 = r_0(\epsilon)$

$$M(r, g) \leqq \exp \left[a^\rho(d + \epsilon)r^\rho\right],$$

so that $\kappa(g) \leqq a^\rho d$ and since a is any number greater than one, we get $\kappa(g) \leqq d$. But obviously $d \leqq \kappa(g)$. Hence $\kappa(g) = d$.

2.3. **Proof of Theorem** 2. Let $f(z)$ be any function of $C(\rho, d)$ and $[R_\nu]$ the sequence of Lemma 1 for some $a>1$. Let

$$I_\nu = \frac{1}{2\pi i}\int_{|x|=R_\nu} \frac{x^m f(x)}{g(x)} \frac{dx}{x - z},$$

where $m \geqq 0$ is an integer. By (ii) we find that

$$I_\nu \to 0 \tag{1}$$

as $\nu \to \infty$ uniformly in any fixed circle $|z| \leqq R$. But

$$I_\nu = \frac{z^m f(z)}{g(z)} - \sum_{|z_n|<R_\nu} \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z - z_n}, \tag{2}$$

while by (i) the series

$$\sum_{n=1}^{\infty} \left| \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z - z_n} \right|$$

converges uniformly except at the points $z = z_n$. Therefore (1) and (2) give

$$z^m \frac{f(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z - z_n}. \tag{3}$$

Let $\kappa(f, \rho, [z_n]) = \beta \leqq \kappa(f) < d$. Choose η so that $0 < \eta < d - \beta$ and λ so that $d\lambda^\rho = d - \beta - \eta$. Let $\chi(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ be any integral function of order ρ and type not exceeding $d\lambda^\rho$. Then by (i) the double series

$$\sum_{(m,n)} \left| \frac{c_m z_n^m f(z_n)}{g'(z_n)} \frac{1}{z - z_n} \right|$$

converges uniformly except at $z = z_n$, so that (3) gives

$$\frac{f(z)\chi(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{f(z_n)\chi(z_n)}{g'(z_n)} \frac{1}{z - z_n}. \tag{4}$$

In (4) we can take $\chi(z) = g(\lambda z)$ since in this case $\kappa(\chi) = d\lambda^\rho$ by Lemma 2. So (4) gives

$$f(z) = \frac{g(z)}{g(\lambda z)} \sum_{n=1}^{\infty} \frac{f(z_n) g(\lambda z_n)}{g'(z_n)} \frac{1}{z - z_n}. \tag{5}$$

Let $A_\lambda(h)$ denote the circles around the zeros of $g(\lambda z)$ similar to $A(h)$. Then, given $a > 1$, we can, just as in Lemma 1, choose a sequence $[R_n]$, $R_{n+1} \leqq a R_n$, such that the circles $|z| = R_n$ do not cut any circle of either $A(h)$ or $A_\lambda(h)$. Using (i), (ii) and the choice of λ, we get from (5),

$$M(R_n, f) \leqq \exp\left[(d - d\lambda^\rho + \epsilon) R_n^\rho\right] \tag{6}$$

for $n \geqq n_0 = n_0(\epsilon)$. Starting from (6), an argument of the type used in Lemma 2 shows that $k(f) \leqq d - d\lambda^\rho = \beta + \eta$ and since η is subject to the sole restriction $0 < \eta < d - \beta$ we get $\kappa(f) \leqq \beta$. Since $\beta \leqq \kappa(f)$, we get $\kappa(f) = \beta$ which is the result required.

2.4. In some cases it is possible to conclude that the relation (i) of Theorem 2 follows from (ii). The circles of $A(h)$ determine a sequence of non-overlapping domains $D_1, D_2, \cdots, D_n, \cdots$. Let p_n denote the number of points of $[z_n]$ lying in D_n. We shall prove

LEMMA 3. *If* $p_n \leqq P$, *a fixed positive number, then* (ii) *of Theorem* 2 *involves* (i).

Proof. Let z_n be contained in D_{q_n}. Let

$$P_n(z) = \prod_{z_\nu \subset D_{q_n}} \left(1 - \frac{z}{z_\nu}\right),$$

and

$$g(z) = P_n(z) Q_n(z).$$

The greatest and the least distances of the boundary of D_{q_n} from the origin lie in the interval $(|z_n| - H, |z_n| + H)$ where $H = \sum_{n=1}^{\infty} |z_n|^{-h}$. Since the degree of $P_n(z)$ does not exceed P, we have

$$\lim \frac{\log |P_n(z)|}{|z|^\rho} = 0,$$

as $|z| \to \infty$ outside the domains D_ν, uniformly in n. Therefore on the boundary of D_{q_n}, we have by (ii)

$$\exp\left[(d - \epsilon)(|z_n| - H)^\rho\right] \leqq |Q_n(z)| \leqq \exp\left[(d + \epsilon)(|z_n| + H)^\rho\right]. \tag{7}$$

Since $Q_n(z)$ does not vanish in D_{q_n}, (7) holds in the interior of D_{q_n}, in particular, at $z = z_n$. Hence

$$\lim_{n \to \infty} \frac{\log |Q_n(z_n)|}{|z_n|^\rho} = d.$$

Moreover

$$g'(z_n) = P_n'(z_n) Q_n(z_n),$$

and arguing as before we get

$$\lim_{n \to \infty} \frac{\log |g'(z_n)|}{|z_n|^\rho} = \lim_{n \to \infty} \frac{\log |Q_n(z_n)|}{|z_n|^\rho} = d.$$

So the lemma is proved.

3. Using Theorem 2, we shall set up an E-set for a given class $C(\rho, d)$. We first establish the following

LEMMA 4. *Let* $\rho = 2/\alpha$ *and*

$$\sigma_\rho(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^n}{n^{\alpha n}}\right). \tag{8}$$

Then $\sigma_\rho(z)$ *is an integral function of order* ρ *satisfying* (i) *and* (ii) *of Theorem* 2, *with* $d = \alpha/4 = 1/(2\rho)$.

Proof. If we prove (ii) with some $h > \rho$, (i) will follow from Lemma 3 since the circles of $A(h)$ are obviously non-overlapping after a certain stage. Let $p > 0$ be any integer and let z lie in the ring $p^\alpha \leqq |z| \leqq (p+1)^\alpha$.

$$\sigma_\rho(z) = \left(1 - \frac{z^p}{p^{\alpha p}}\right)\left(1 - \frac{z^p}{(p+1)^{\alpha(p+1)}}\right)F_p(z), \tag{9}$$

where

$$F_p(z) = \prod_{n=1}^{p-1}\left(1 - \frac{z^n}{n^{\alpha n}}\right)\prod_{n=p+2}^{\infty}\left(1 - \frac{z^n}{n^{\alpha n}}\right) = S_1 \times S_2, \tag{10}$$

say. We have

$$\begin{aligned}\log|S_1| &= \tfrac{1}{2}p(p-1)\log|z| - \alpha\sum_{n=1}^{p-1} n\log n + \log\prod_{n=1}^{p-1}\left|1 - \frac{n^{\alpha n}}{z^n}\right| \\ &= \frac{\alpha}{4}p^2 + O(p\log p) + \log\prod_{n=1}^{p-1}\left|1 - \frac{n^{\alpha n}}{z^n}\right| \qquad (11) \\ &= \frac{\alpha}{4}|z|^{2/\alpha} + O(|z|^{1/\alpha}\log|z|) + \log\prod_{n=1}^{p-1}\left|1 - \frac{n^{\alpha n}}{z^n}\right|.\end{aligned}$$

Now, since $p^\alpha \leqq |z| \leqq (p+1)^\alpha$, we have

$$\prod_{n=1}^{p-1}\left(1 - \frac{n^{\alpha n}}{p^{\alpha n}}\right) \leqq \prod_{n=1}^{p-1}\left|1 - \frac{n^{\alpha n}}{z^n}\right| \leqq \prod_{n=1}^{p-1}\left(1 + \frac{n^{\alpha n}}{p^{\alpha n}}\right). \tag{12}$$

Using the fact that $(1+1/x)^x$ steadily increases and $(1-1/x)^{-x}$ steadily decreases to e as x varies in $0 < x < \infty$, we get

$$\begin{aligned}\left(\frac{n}{p}\right)^{\alpha n} &= \left\{\left(1 - \frac{p-n}{p}\right)^{p/(p-n)}\right\}^{\alpha n(p-n)/p} \leqq \exp\left[-\frac{\alpha n(p-n)}{p}\right] \\ &\leqq \exp\left[-\frac{\alpha}{2}\cdot\min(n, p-n)\right].\end{aligned}$$

Therefore (12) gives

$$0 < a = \left\{\prod_{n=1}^{\infty}(1 - e^{-\alpha n/2})\right\}^2 \leqq \prod_{n=1}^{p-1}\left|1 - \frac{n^{\alpha n}}{z^{\alpha n}}\right| \leqq \left\{\prod_{n=1}^{\infty}(1 + e^{-\alpha n/2})\right\}^2 = b,$$

so that by (11),

$$\log |S_1| = \frac{\alpha}{4}|z|^{2/\alpha} + O(|z|^{1/\alpha}\log|z|). \tag{13}$$

A similar argument shows that

$$\log |S_2| = O(1),$$

so that (9), (10), and (13) give

$$\begin{aligned}\log|\sigma_\rho(z)| &= \frac{\alpha}{4}|z|^{2/\alpha} + O(|z|^{1/\alpha}\log|z|)\\ &\quad + \log\left|1 - \frac{z^p}{p^{\alpha p}}\right|\left|1 - \frac{z^{p+1}}{(p+1)^{\alpha(p+1)}}\right|,\end{aligned} \tag{14}$$

where $p^\alpha \leqq |z| \leqq (p+1)^\alpha$. Taking $h > \rho = 2/\alpha$, we find from (14) that when z is outside the circles of $A(h)$ but inside the ring $p^\alpha \leqq |z| \leqq (p+1)^\alpha$,

$$\log|\sigma_\rho(z)| = \frac{\alpha}{4}|z|^{2/\alpha} + O(|z|^{1/\alpha}\log|z|), \tag{15}$$

and since p is any integer, we get

$$\lim \frac{\log|\sigma_\rho(z)|}{|z|^\rho} = \frac{\alpha}{4} = \frac{1}{2\rho}$$

as $|z| \to \infty$ outside the circles of $A(h)$. So the lemma is proved.

3.1. It is easy to see that if $[z_n]$ is an E-set for $C(\rho, d)$ then $[\eta z_n]$ is an E-set for $C(\rho, d/\eta^\rho)$. Hence Lemma 4 enables us, in conjunction with Theorem 2, to state

THEOREM 3. *Let $\rho > 0$, $d > 0$ be given. The set of points*

$$[(2\rho d)^{-1/\rho} n^{2/\rho} e^{2\nu\pi i/n}], \qquad n = 1, 2, 3, \cdots; \nu = 0, 1, 2, \cdots, n-1,$$

forms an E-set for the class $C(\rho, d)$. In other words, if $f(z)$ is any function of order ρ and type less than d, then

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{r^\rho} = \kappa(f) = \limsup_{n\to\infty} \frac{\log|f[(2\rho d)^{-1/\rho} n^{2/\rho} e^{2\nu\pi i/n}]|}{(2\rho d)^{-1} n^2}.$$

3.2. As a corollary we get, taking $d = 1/(2\rho)$, in Theorem 3,

THEOREM 4. *If $f(z)$ is any integral function of order less than ρ or of order ρ and minimal type ($\kappa(f) = 0$), then*

$$\limsup_{n\to\infty} \frac{\log|f(n^{2/\rho} e^{2\nu\pi i/n})|}{n^2} = 0.$$

3.3. If $\sigma(z)$ is the canonical product with simple zeros at the lattice points $z=m+in$, $m, n=0, \pm 1, \pm 2, \cdots$, it is known from the pseudo-periodic properties of $\sigma(z)$ that (i) and (ii) of Theorem 2 hold for $\sigma(z)$ with $\rho=2$ and $d=\pi/2$. Hence the class $C(2, \pi/2)$ has the peculiarly simple E-set $z=m+in$, $m, n=0, \pm 1, \pm 2, \cdots$.

3.4. I have shown elsewhere* that a function of $C(2, \pi/2)$ bounded at the lattice-points must be a constant. The question may be asked whether the same is true of an E-set for $C(\rho, d)$ for which the conditions of Theorem 2 hold. That this is in fact the case can be shown by using exactly the same method followed in the case of the lattice points.† So we can state

THEOREM 5. *Let $[z_n]$ be a set of points satisfying the conditions* (i) *and* (ii) *of Theorem* 2. *Then any function of order ρ and type less than d bounded at the points $[z_n]$ must reduce to a constant.*

3.5. As a particular case of Theorem 5 we get

THEOREM 6. *An integral function of order ρ and type less than* $1/(2\rho)$ *bounded at the points*

$$n^{2/\rho}e^{2\nu\pi i/n}, \qquad n = 1, 2, 3, \cdots; \nu = 0, 1, 2, \cdots, n-1$$

reduces to a constant.

3.6. It may be noted that an E-set for a class $C(\rho, d)$ is a fixture to that class and is independent of the individual functions of the class. Theorems 2–4 throw a good deal of light on the peculiar behaviour of the functions of $C(2, \pi/2)$ at the lattice points. These latter were, in fact, the starting point of the investigations of this paper. It is very probable that conditions closely allied to those of Theorem 2 are also necessary for an E-set although I have not succeeded in discovering exactly what these conditions are. The question whether (ii) of Theorem 2 always involves (i) is also unsolved.

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* Journal of the London Mathematical Society, vol. 11 (1936), pp. 247–250.

† Since $f(z_n)=0$, $\kappa(f)\leqq 0$; if $\kappa(f)<0, f(z)\equiv 0$; if $\kappa(f)=0$, then also $\kappa(f^p)=0$, so that formula (3) holds with $m=0$ and $f^p/p!$, $p=0, 1, 2, \cdots$, in place of $f(z)$. An addition and an argument, as in Lemma 2, will show that $e^{f(z)}$ is of finite order, that is, $f(z)$ is a polynomial which must be a constant since $f(z_n)=O(1)$.

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