THE CHARACTERIZATION OF THE CLOSED n-CELL*

BY

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1. **Introduction.** Various characterizations of the closed 1-cell and of the closed 2-cell have been given. But, with the exception of a paper by Alexandroff,† there is no record of previous attempts to give a characterization of the closed n-cell which is uniformly valid for all values of $n>0$.

Characteristic of the present work are (1) the use of the notion of *strong homeomorphism* (§2) by means of which is defined a very useful concept, that of *the descendant of a set*, and (2) the emphasis placed upon an essential property of the closed n-cell as given in Corollary, Theorem P_1 (§3).

In Theorem I (§4) there is presented a characterization of the closed n-cell without reference to the euclidean spaces. The definition of the closed n-cell implied by this theorem, although given by means of recursive statements, is essentially set-theoretic in character. The words and symbols constituting this theorem may be regarded as defining a function of n, $F(n)$, such that if k is a positive integer, $F(k)$ is a closed k-cell. The space $F(n)$ is defined in terms of certain of its subsets as given by $F(n-1)$. By definition, $F(0)$ consists of a single point.

The proof of Theorem I is based upon Theorem I′ (§4). This latter theorem gives a characterization of the closed n-cell in terms of the closed $(n-1)$-cell.

2. **Definitions.** The set $\overline{M}-M$, where $\overline{M}$ designates the closure of the set M, is called the λ-set of M and is denoted by $\lambda[M]$.

A set M_1 is said to be *strongly homeomorphic* with a set M_2 provided there exists a homeomorphism, $H(\overline{M}_1)=\overline{M}_2$, of such nature that $H(M_1)=M_2$.‡

Let P be a non-vacuous subset of a set M such that $M-P=M_1+M_2$, $\overline{M}_1\cdot\overline{M}_2=\overline{P}$ and M_i, $i=1, 2$, is strongly homeomorphic with M. Each of the sets M_i is called a *proper descendant* of M. This relation is expressed symbolically: $M_i=D_i^P(M)$. The set P is said to generate the descendants. The

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† P. Alexandroff, *Zur Begründung der n-dimensionalen mengentheoretischen Topologie*, Mathematische Annalen, vol. 94, p. 296.

‡ The use of the expression "strongly homeomorphic" in this connection was suggested to the author by Professor J. R. Kline.

complex whose elements are the descendants of M generated by P is represented by $\Delta[P]M$.

$$\Delta[P]M = (D_1^P(M), D_2^P(M)).$$

A set P is said to generate *improper descendants* for a set M as follows:

(1) $P\neq 0$, $M\neq 0$, but P does not generate proper descendants for M: $D_i^P(M)=0$, $i=1, 2$; $\Delta[P]M=(0, 0)$.

(2) $P=0$: $D_i^P(M)=M$, $i=1, 2$; $\Delta[P]M=(M, M)$.

(3) $M=0$: $D_i^P(M)=0$, $i=1, 2$; $\Delta[P]M=(0, 0)$.

We define two additional complexes:

$$\Delta[P](M_1, M_2, \cdots, M_n) = (D_1^P(M_1), D_2^P(M_1), \cdots, D_1^P(M_n), D_2^P(M_n)),$$

$$\mathop{\Delta}_{i=1}^{n}[P_i]M = \Delta[P_1]\left(\mathop{\Delta}_{i=2}^{n}[P_i]M\right).$$

That the complex $\Delta_{i=1}^n[P_i]M$ contains at least one non-vacuous element is indicated as follows: $\Delta_{i=1}^n[P_i]M\neq 0$. It is to be noted that, if $P_i=0$ for every value of i, every element of $\Delta_{i=1}^n[P_i]M$ is identical with M.

It will be of advantage to extend the notion of a descendant of a set and speak of every descendant, proper or improper, of a descendant of a set M as a descendant of M. When there is no doubt as to the identity of the set which generates a given descendant, the symbol for the generating set will be omitted. Frequently multiple subscripts will be used in denoting descendants, as in $D_{12}(M)$. In such cases the meaning will be clear from the context.

An n-cell, $n>0$, is a subset of a space S which is strongly homeomorphic with the set in euclidean n-space which is the interior of the $(n-1)$-sphere whose equation is $\sum_{i=1}^n x_i^2=1$. A 0-cell is a set consisting of a single point.

3. **Preliminary Theorems.** We prove first

THEOREM P_1. *Let C^n and K^n be two n-cells, $n>0$. If there is a homeomorphism, $H_1(\lambda[C^n])=\lambda[K^n]$, there exists a homeomorphism, $H_2(\overline{C^n})=\overline{K^n}$, such that $H_2(\lambda[C^n])=H_1(\lambda[C^n])$.*

In view of the definition of the n-cell (§2), K^n may be taken to be the set in euclidean n-space which is the interior of the $(n-1)$-sphere whose equation is $\sum_{i=1}^n x_i^2=1$. Since C^n is strongly homeomorphic with K^n, there is a homeomorphism, $H_\alpha(\overline{C^n})=\overline{K^n}$, such that $H_\alpha(\lambda[C^n])=\lambda[K^n]$. Denote by O the point $(0, 0, \cdots, 0)$ in euclidean n-space. Let p be any point of $\lambda[C^n]$, $H_1(p)=q$ and $H_\alpha(p)=q'$. Let Oq and Oq' be the straight line intervals in $\overline{K^n}$ joining O to q and q' respectively. Make the points of Oq and Oq' to correspond in such a manner that a point q_1 of Oq corresponds to a point q_1' of Oq' if, and only if, $d(O, q_1)=d(O, q_1')$.*

* If p and q are two points, the symbol $d(p, q)$ is used to designate the distance from p to q.

Since p is any point of $\lambda[C^n]$, this procedure results in a transformation of $\overline{K^n}$ into itself which is a homeomorphism. If $H_\beta(\overline{K^n})=\overline{K^n}$ is this transformation, $H_\beta(q_1)=q_1'$. Let H_2 denote the transformation $H_\beta^{-1}H_\alpha$. Then H_2 is a homeomorphism. $H_2(\overline{C^n})=\overline{K^n}$. Since $H_2(p)=H_\beta^{-1}H_\alpha(p)=H_\beta^{-1}(q')=q$, $H_2(\lambda[C^n])=H_1(\lambda[C^n])$.

COROLLARY. *If C^n is an n-cell and if there is a homeomorphism, $H_1(\lambda[C^n])=\lambda[C^n]$, there exists a homeomorphism, $H_2(\overline{C^n})=\overline{C^n}$, such that $H_2(\lambda[C^n])=H_1(\lambda[C^n])$.*

THEOREM P_2. *Let S_1^n and S_2 be two topological n-spheres such that $S_i^n=\overline{C_{i1}^n}+\overline{C_{i2}^n}$, $\overline{C_{i1}^n}\cdot\overline{C_{i2}^n}=\lambda[C_{i1}^n]=\lambda[C_{i2}^n]$, $i=1, 2$, and C_{ij}^n is an n-cell. Then there exists a homeomorphism, $H(S_1^n)=S_2^n$, such that $H(\overline{C_{1i}^n})=\overline{C_{2i}^n}$, $i=1, 2$.*

$\overline{C_{11}^n}$ is strongly homeomorphic with $\overline{C_{21}^n}$. There is a homeomorphism $H_1(\overline{C_{11}^n})=\overline{C_{21}^n}$, such that $H_1(\lambda[C_{11}^n])=\lambda[C_{21}^n]$. But $\lambda[C_{i1}^n]=\lambda[C_{i2}^n]$, $i=1, 2$. Then $H_1(\lambda[C_{11}^n])=H_1(\lambda[C_{12}^n])=\lambda[C_{22}^n]$. By Theorem P_1 there is a homeomorphism, $H_2(\overline{C_{12}^n})=\overline{C_{22}^n}$ such that $H_2(\lambda[C_{12}^n])=H_1(\lambda[C_{12}^n])$. The existence of the required transformation is evident.

COROLLARY. *If S^n is an n-sphere, $S^n=\overline{C_{11}^n}+\overline{C_{12}^n}=\overline{C_{21}^n}+\overline{C_{22}^n}$ and $\overline{C_{i1}^n}\cdot\overline{C_{i2}^n}=\lambda[C_{i1}^n]=\lambda[C_{i2}^n]$, $i=1, 2$, there exists a homeomorphism, $H(S^n)=S^n$, such that $H(\overline{C_{1i}^n})=\overline{C_{2i}^n}$, $i=1, 2$.*

THEOREM P_3. *If C^n is an n-cell, $\lambda[C^n]=\overline{C_1^{n-1}}+\overline{C_2^{n-1}}$ and $\overline{C_1^{n-1}}\cdot\overline{C_2^{n-1}}=\lambda[C_i^{n-1}]$, $i=1, 2$, there exists an $(n-1)$-cell C_3^{n-1} such that $\lambda[C_3^{n-1}]=\lambda[C_i^{n-1}]$, $i=1, 2$, and $\Delta[C_3^{n-1}]\, C^n\neq 0$.*

Let K^n be the same subset of euclidean n-space as in the proof of Theorem P_1. The $(n-1)$-dimensional plane $x_1=0$ has in common with $\overline{K^n}$ the $(n-1)$-cell K_3^{n-1}. The set $\lambda[K^n]$ is the sum of two closed $(n-1)$-cells, $\overline{K_1^{n-1}}$ and $\overline{K_2^{n-1}}$, such that $\overline{K_1^{n-1}}\cdot\overline{K_2^{n-1}}=\lambda[K_i^{n-1}]$, $i=1, 2, 3$. By Theorem P_2, there is a homeomorphism, $H_1(\lambda[C^n])=\lambda[K^n]$, such that $H_1(\overline{C_i^{n-1}})=\overline{K_i^{n-1}}$, $i=1, 2$. By Theorem P_1 there is a homeomorphism, $H_2(\overline{C^n})=\overline{K^n}$, such that $H_2(\lambda[C^n])=H_1(\lambda[C^n])$. The set $H_2^{-1}[K_3^{n-1}]$ is an $(n-1)$-cell C_3^{n-1}. Obviously $\Delta[C_3^{n-1}]C^n\neq 0$. The set C_3^{n-1} separates C^n into two n-cells.

Remark. In the sequel the symbol C^k (or $C_\delta{}^k$) is always to be understood as designating a k-cell.

4. **Principal theorems.** We state now our main theorems.

THEOREM I. *In order that a space Z^n be a closed n-cell, the following conditions are necessary and sufficient:*

(4.1) *Z^n is a connected and locally compact Hausdorff space* which is de-*

* A Hausdorff space is a space defined by a system of neighborhoods which satisfy the Hausdorff neighborhood axioms. See F. Hausdorff, *Grundzüge der Mengenlehre*, 1914, p. 213.

fined by a countable set of neighborhoods: $\{N_i\}$, $i=1, 2, 3, \cdots$.

(4.2) *Z^n contains a proper subset I^n such that* (i) $\overline{I^n}=Z^n$ *and* (ii) *if there is a homeomorphism, $H_1(\lambda[I^n])=\lambda[I^n]$, there exists a homeomorphism, $H_2(Z^n)=Z^n$, for which $H_2(\lambda[I^n])=H_1(\lambda[I^n])$.*

(4.3) *Let W_1 and W_2 be two sets such that $0\neq W_1\cdot W_2 \not\supset W_j$, $j=1, 2$, W_1 is a neighborhood N_ξ and W_2 is an element of $\Delta_{i=1}^a[P_i]V$, where V is either I^n or a neighborhood belonging to I^n and $0\subseteq P_i\subset\lambda[N_{\alpha_i}]$, $N_{\alpha_i}\not\subset V$, $\xi\neq\alpha_i$. Then $\overline{W}_2\cdot\lambda[W_1]=\sum_{j=1}^d Z_j^{n-1}$, $\Delta_{j=1}^d[I_j^{n-1}]W_2\neq 0$, and, if K_{β_j} is a component of $\lambda[W_2]-\lambda[W_2]\cdot Z_j^{n-1}$, there exists $Z_{\beta_j}^{n-1}$ such that $K_{\beta_j}=I_{\beta_j}^{n-1}$.*

(4.4) *$Z^0(\equiv I^0)$ is a set consisting of a single point.*

THEOREM I$'$. *This theorem differs from Theorem* I *only in the following respects*: (1) *in Condition* (4.3), *Z_j^{n-1}, I_j^{n-1}, $Z_{\beta_j}^{n-1}$, and $I_{\beta_j}^{n-1}$ are replaced by $\overline{C_j^{n-1}}$, C_j^{n-1}, $\overline{C_{\beta_j}^{n-1}}$, and $C_{\beta_j}^{n-1}$ respectively, and* (2) *Condition* (4.4) *is omitted.*

5. **Proof that the conditions in Theorem I$'$ are sufficient.** We state first

5.1. LEMMA 1. *There exists a set G of compact neighborhoods which is a subset of the set of all neighborhoods in Z^n having the following properties*:

(a) *The set G is equivalent to the set of all neighborhoods.*

(b) *Corresponding to each point p, there exists a subset of G: $G(p)=\{N_{a_i}\}$, $i=1, 2, 3, \cdots$, such that $N_{a_i}\supset p$ and $N_{a_{i+1}}\subset N_{a_i}$.*

(c) *If $\{N_{b_i}\}$, $i=1, 2, 3, \cdots$, is a set of neighborhoods belonging to G for which $N_{b_{i+1}}\subset N_{b_i}$, then the set $\prod_{i=1}^\infty \overline{N}_{b_i}$ consists of a single point.**

Hereafter all neighborhoods mentioned will be members of the set G.

5.2. LEMMA 2. *The set $\lambda[I^n]$ is an $(n-1)$-sphere.*

There exists a neighborhood N_a such that $N_a\cdot\lambda[I^n]\neq 0$ and $N_a\not\supset I^n$. By Theorem I$'$ (4.3), $Z^n\cdot\lambda[N_a]=\sum_{j=1}^d\overline{C_j^{n-1}}$ and $\Delta[C_d^{n-1}]I^n\neq 0$.

$$I^n = D_1(I^n) + D_2(I^n) + C_d^{n-1};\ \overline{D_1(I^n)}\cdot\overline{D_2(I^n)} = \overline{C_d^{n-1}}.$$

The set $N_a\cdot\lambda[I^n]$ contains a point p which belongs to one and only one of the sets $\lambda[I^n]\cdot\lambda[D_i(I^n)]$, $i=1, 2$. Suppose that $\lambda[D_1(I^n)]\supset p$. There is a neighborhood N_b such that $N_b\supset p$ and $\overline{N}_b\cdot\overline{D^2(I^n)}=0$.† Then $\lambda[N_b]$ contains a closed $(n-1)$-cell $\overline{C_\alpha^{n-1}}$ such that $\Delta[C_\alpha^{n-1}]\ D_1(I^n)\neq 0$ (Theorem I$'$ (4.3)).

* For the proof of this lemma, see I. Gawehn, *Über unberandete 2-dimensionale Mannigfaltigkeiten*, Mathematische Annalen, vol. 98, p. 339. An understanding of the method by which the sets $G(p)$ are obtained is assumed in 5.8 and 5.9.

† The fact that Z^n is a locally compact Hausdorff space assures us of the existence of a neighborhood N_b having the desired properties. In fact, the following proposition, of which we shall make frequent use, holds; *If F is a closed set and p, a point not belonging to F, there exists a neighborhood $N_b\supset p$ such that $\overline{N}_b\cdot F=0$.*

$$D_1(I^n) = D_{11}(I^n) + D_{12}(I^n) + C_\alpha^{n-1}; \ \overline{D_{11}(I^n)} \cdot \overline{D_{12}(I^n)} = \overline{C_\alpha^{n-1}}.$$

Every point of $\lambda[D_1(I^n)]$ not belonging to $\overline{C_\alpha^{n-1}}$ belongs to one and only one of the sets $\lambda[D_{1i}(I^n)]$, $i=1, 2$. Since $\overline{C_d^{n-1}}$ is connected and $\overline{C_d^{n-1}} \cdot \overline{C_\alpha^{n-1}}=0$, $\overline{C_d^{n-1}}$ belongs to one of the sets $\lambda[D_{1i}(I^n)]$. Suppose that $\overline{C_d^{n-1}} \subset \lambda[D_{12}(I^n)]$. Then $\overline{C_d^{n-1}} \cdot \lambda[D_{11}(I^n)]=0$.

Case 1. $n<3$. Assume that, in this case, $\lambda[I^n]-\lambda[I^n]\cdot\overline{C_d^{n-1}}$ contains an infinite number of components. $\overline{C_d^{n-1}} \subset I^n$. Every component of $\lambda[I^n]-\lambda[I^n] \cdot\overline{C_d^{n-1}}$ is an $(n-1)$-cell (Theorem I′ (4.3)).

If $n=1$, C_d^0 consists of a single point and $\lambda[I^1]\cdot\overline{C_d^0}=0$. Then $\lambda[I^1]$ is not a connected set and consists of infinitely many points.

Let $n=2$. The set C_d^1 is a 1-cell and $\lambda[C_d^1]$ consists of two points, g_1 and g_2. $\overline{C_d^1}$ belongs to a component C_m^1 of $\lambda[D_1(I^2)]-\lambda[D_1(I^2)]\cdot\overline{C_\alpha^1}$. The points g_1 and g_2 separate C_m^1 into three 1-cells. Of these three 1-cells, one is C_d^1 and each of the other two 1-cells is a subset of a component of $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ which belongs to $\lambda[D_1(I^2)]$. By a similar process it can be proved that each of the points g_i belongs to the λ-set of one and only one component of $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ which is contained in $\lambda[D_2(I^2)]$. Every component of $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ belongs to one and only one of the sets $\lambda[D_i(I^2)]$. At least one of the sets $\lambda[D_i(I^2)]$ contains an infinite number of these components. Let $\lambda[D_1(I^2)]$ be such a set.

The set g_1+g_2 cannot belong to the λ-set of a single component of $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ which belongs to $\lambda[D_1(I^2)]$. For, suppose that C_e^1 is a component of $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ which belongs to $\lambda[D_1(I^2)]$ and whose λ-set is g_1+g_2. There exists N_f such that $N_f\cdot\lambda[D_1(I^2)]\neq 0$ and $\overline{N}_f\cdot(\overline{C_d^1}+C_e^1)=0$. Then $\lambda[N_f]$ contains a 1-cell C_x^1 such that $\Delta[C_x^1]D_1(I^2)\neq 0$. $\overline{C_d^1}+C_e^1$ is a 1-sphere which belongs to a component of $\lambda[D_1(I^2)]-\lambda[D_1(I^2)]\cdot\overline{C_x^1}$. But such a component would not be a 1-cell. We can now conclude that $\lambda[D_1(I^2)]$ consists of infinitely many components and that each of these components is a 1-cell. This last statement holds true for $\lambda[I^2]$ which is homeomorphic with $\lambda[D_1(I^2)]$.

Since $\lambda[D_2(I^n)]-\overline{C_d^{n-1}}\neq 0$, there exists a closed 1-cell $\overline{C_\beta^{n-1}}$ belonging to the λ-set of a neighborhood and $\Delta[C_\beta^{n-1}]D_2(I^n)\neq 0$, $n=1, 2$.

$$D_2(I^n) = D_{21}(I^n) + D_{22}(I^n) + C_\beta^{n-1}; \ \overline{D_{21}(I^n)}\cdot\overline{D_{22}(I^n)} = \overline{C_\beta^{n-1}}.$$

Suppose that $\overline{C_d^{n-1}} \subset \lambda[D_{22}(I^n)]$. Then $\overline{C_d^{n-1}}\cdot\lambda[D_{21}(I^n)]=0$. $\lambda[D_{i1}(I^n)]$, $i=1, 2$, being homeomorphic with $\lambda[I^n]$, consists of infinitely many components, each of which is an $(n-1)$-cell.

There exists a homeomorphism, $H(\lambda[D_{11}(I^n)]=\lambda[D_{21}(I^n)])$. It can be shown that $\lambda[D_{11}(I^n)]=G_1+G_2$, where $G_i\neq 0$, $\overline{G}_1\cdot G_2+G_1\cdot\overline{G}_2=0$,

$G_1 \subset \lambda[D_{11}(I^n)] \cdot \lambda[I^n]$, $H(G_1) \subset \lambda[D_{21}(I^n)] \cdot \lambda[I^n]$ and both of the sets, $\lambda[D_{11}(I^n)] \cdot \lambda[I^n] - G_1$ and $\lambda[D_{21}(I^n)] \cdot \lambda[I^n] - H(G_1)$, are non-vacuous. Let $H(G_1) = G_1'$.

There exists a homeomorphism, $H_1(\lambda[I^n]) = \lambda[I^n]$, such that $H_1(G_1) = H(G_1) = G_1'$, $H_1(G_1') = H^{-1}(G_1') = G_1$ and, if x is a point of $\lambda[I^n] - G_1 - G_1'$, $H_1(x) = x$. By Theorem I (4.2), there is a homeomorphism, $H_2(Z^n) = Z^n$, for which $H_2(\lambda[I^n]) = H_1(\lambda[I^n])$.

Under the transformation H_2, a subset of $\overline{D_{i1}(I^n)}$, $i = 1, 2$, is transformed into itself and a subset of $\overline{D_{j1}(I^n)}$ is transformed into a subset of $\overline{D_{k1}(I^n)}$, $j \neq k$; $j, k = 1, 2$. The set $\overline{D_{i1}(I^n)}$, being homeomorphic with Z^n, is a connected set. Since H_2 is a homeomorphism, $H_2(\overline{D_{i1}(I^n)})$ is a connected set. $H_2(\overline{D_{i1}(I^n)})$ contains non-vacuous subsets in each of the two sets, $\overline{D_1(I^n)} - \overline{C_d^{n-1}}$ and $\overline{D_2(I^n)} - \overline{C_d^{n-1}}$. Neither of the two last-named sets contains a point or limit point of the other. Therefore, $H_2(\overline{D_{i1}(I^n)}) \cdot \overline{C_d^{n-1}} \neq 0$.

Let $n = 1$. In this case the point C_d^0 is the transform of two distinct points belonging respectively to $\overline{D_{11}(I^n)}$ and $\overline{D_{21}(I^n)}$. This is impossible since H_2 is a homeomorphism. Therefore, our assumption that $\lambda[I^1] - \lambda[I^1] \cdot \overline{C_d^0}$ has an infinite number of components has led to a contradiction. Hence $\lambda[I^1]$ consists of a finite number of points. Let n_1 be the number of points in $\lambda[I^1]$.

$$
\begin{aligned}
\lambda[I^1] &= \lambda[D_1(I^1)] \cdot \lambda[I^1] + \lambda[D_2(I^1)] \cdot \lambda[I^1], \\
n_1 &= 2(n_1 - 1), \\
n_1 &= 2.
\end{aligned}
$$

Therefore, $\lambda[I^1]$ is a 0-sphere.

Let $n = 2$. Since each point of $\lambda[C_d^1]$ is a limit point of $\lambda[I^2] - G_1 - G_1'$, $H_2(\lambda[C_d^1]) = \lambda[C_d^1]$ and $H_2(\overline{D_{i1}(I^2)}) \cdot \overline{C_d^1} \subset C_d^1$, $i = 1, 2$. Then $\overline{C_d^1} - \sum_{i=1}^{2} H_2(\overline{D_{i1}(I^2)}) \cdot \overline{C_d^1}$ is not a connected set. $H_2(\overline{C_d^1}) \not\subset \overline{C_d^1}$. For, if $H_2(\overline{C_d^1}) \subset \overline{C_d^1}$, $H_2(\overline{C_d^1}) \subset \overline{C_d^1} - \sum_{i=1}^{2} H_2(\overline{D_{i1}(I^2)}) \cdot \overline{C^1}$. But no subset of $\overline{C_d^1} - \sum_{i=1}^{2} H_2(\overline{D_{i1}(I^2)}) \cdot \overline{C_d^1}$ containing $\lambda[C_d^1]$ is a connected set. Therefore, $H_2(C_d^1)$ contains a point q not belonging to C_d^1. This point q belongs either to $D_1(I^2)$ or to $D_2(I^2)$. The discussion is of the same character in either case. Suppose that $D_2(I^2) \supset q$ and $H_2^{-1}(q) = q'$. Since C_d^1 contains no point which is a limit point of G_1, q' is not a limit point of G_1, $H_2(\lambda[D_1(I^2)] \cdot \lambda[I^2] - G_1) = \lambda[D_1(I^2)] \cdot \lambda[I^2] - G_1$. The point q, which belongs to $D_2(I^2)$, is not a limit point of $H_2(\lambda[D_1(I^2)] \cdot \lambda[I^2] - G_1)$. Therefore, q' is not a limit point of $\lambda[D_1(I^2)] \cdot \lambda[I^2] - G_1$. Then there exists N_r such that $N_r \supset q'$ and $\overline{N}_r \cdot (\lambda[D_1(I^2)] - C_d^1) = 0$. The set $\lambda[N_r]$ contains a closed 1-cell $\overline{C_\kappa^1}$ such that $\Delta[C_\kappa^1] D_1(I^2) \neq 0$. Since the λ-set of each descendant of $D_1(I^2)$ generated by C_κ^1 is homeomorphic with $\lambda[D_1(I^2)]$, each component of the λ-set of such a descendant, under our assumption, is a 1-cell. $\overline{C_\kappa^1}$ belongs to a component of the λ-set of each descendant of $D_1(I^2)$ generated by C_κ^1 and $\overline{C_\kappa^1}\cdot(\lambda[D_1(I^2)]-C_d^1)=0$. Then $\lambda[C_\kappa^1]\subset C_d^1$. The set C_d^1 contains a 1-cell C_μ^1 such that $\lambda[C_\mu^1]=\lambda[C_\kappa^1]$. The λ-set of one of the descendants of $D_1(I^2)$ generated by C_κ^1 contains the 1-sphere $\overline{C_\kappa^1}+C_\mu^1$. This contradicts the fact that every component of the λ-set of a descendant of $D_1(I^2)$ generated by C_κ^1 is a 1-cell. Hence our assumption that $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ contains infinitely many components has led to a contradiction.

The set $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ is non-vacuous and consists of a finite number of components $C_{\alpha_1}^1, C_{\alpha_2}^1, \cdots, C_{\alpha_f}^1$. There exists N_s such that $N_s\cdot C_{\alpha_1}^1\neq 0$ and $\overline{N}_s\cdot\sum_{i=2}^{f}\overline{C_{\alpha_i}^1}=0$. The set $\lambda[N_s]$ contains a closed 1-cell $\overline{C_t^1}$ such that $\Delta[C_t^1]I^2\neq 0$. As in the above, it can be shown that the λ-set of one of the descendants of I^2 generated by C_t^1 contains a 1-sphere. Then $\lambda[I^2]$ contains a 1-sphere S^1. Suppose that $\lambda[I^2]$ contains a point g not belonging to S^1. There exists N_m such that $N_m\supset g$ and $\overline{N}_m\cdot S^1=0$. Then $\lambda[N_m]$ contains a closed 1-cell $\overline{C_\rho^1}$ such that $\Delta[C_\rho^1]I^2\neq 0$. S^1, being a connected set, belongs to a component of $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_\rho^1}$. But such a component, containing a 1-sphere, would not be a 1-cell. Hence $\lambda[I^2]=S^1$.

The set $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ consists of two components belonging to $\lambda[D_1(I^2)]$ and $\lambda[D_2(I^2)]$ respectively and $\lambda[C_d^1]$ is the λ-set of each component.

Case 2. $n\geqq 3$. The set $\overline{C_d^{n-1}}$, being a connected set, belongs to a component C_ξ^{n-1} of $\lambda[D_1(I^n)]-\lambda[D_1(I^n)]\cdot\overline{C_\alpha^{n-1}}$ (see the first part of this proof). By the Jordan-Brouwer Theorem,* $C_\xi^{n-1}-\lambda[C_d^{n-1}]=M_1+M_2$, $\overline{M}_1\cdot\overline{M}_2=\lambda[C_d^{n-1}]$, and M_i, $i=1, 2$, is a connected set. Since C_d^{n-1} is connected, one of the sets M_i contains no point of C_d^{n-1}. Let M_1 be this set. Then M_1 is a subset of a component C_h^{n-1} of $\lambda[I^n]-\lambda[I^n]\cdot\overline{C_d^{n-1}}$ which belongs to $\lambda[D_1(I^n)]$ and which has $\lambda[C_d^{n-1}]$ for its λ-set. Then $\lambda[D_1(I^n)]$ contains the $(n-1)$-sphere $\overline{C_d^{n-1}}+C_h^{n-1}$. Therefore, $\lambda[I^n]$ contains an $(n-1)$-sphere. By an argument used in connection with the proof for the case when $n=2$, it can be shown that $\lambda[I^n]$ is an $(n-1)$-sphere. The set $\lambda[I^n]-\lambda[I^n]\cdot\overline{C_d^{n-1}}$ consists of two components belonging to $\lambda[D_1(I^n)]$ and $\lambda[D_2(I^n)]$ respectively, and $\lambda[C_d^{n-1}]$ is the λ-set of each component.

5.3. Lemma 3. *If N_α is a neighborhood such that $\overline{N}_\alpha\subset I^n$, the set $\lambda[N_\alpha]$ is an $(n-1)$-sphere.*

Since N_α is an open proper subset of the connected space Z^n, $\lambda[N_\alpha]\neq 0$. $\overline{N}_\alpha$ is compact (Lemma 1).

* L. E. J. Brouwer, *Beweis des Jordanschen Satzes für den n-dimensionalen Raum*, Mathematische Annalen, vol. 71, p. 314; J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, these Transactions, vol. 23, p. 333.

Case 1. $n=1$. There exists a set of neighborhoods $\{N_{\phi_i}\}$, $i=1, 2, \cdots, m$, such that $I^n \supset N_{\phi_i} \not\supset N_\alpha$, $N_{\phi_i} \cdot \lambda[N_\alpha] \neq 0$ and $\lambda[N_\alpha] \subset \sum_{i=1}^{m} N_{\phi_i}$. Then $\overline{N}_{\phi_1} \cdot \lambda[N_\alpha] = \sum_{j=1}^{d_i} \overline{C_{ij}^0}$, $i=1, 2, \cdots, m$. Hence $\lambda[N_\alpha]$ consists of a finite number of points. By a method used in Lemma 2, it can be proved that $\lambda[N_\alpha]$ is a 0-sphere.

Case 2. $n>1$. There exists N_ψ such that $N_\psi \cdot \lambda[N_\alpha] \neq 0$, $N_\psi \subset I^n$ and $N_\psi \not\supset N_\alpha$. Then $\overline{N}_\psi \cdot \lambda[N_\alpha] = \sum_{j=1}^{d} \overline{C_j^{n-1}}$ and $\Delta_{j=1}^{d}[C_j^{n-1}]N_\psi \neq 0$. Let p be a point of C_1^{n-1}. The point p is not a limit point of the set $\sum_{j=2}^{d} \overline{C_j^{n-1}}$. There is a neighborhood N_δ such that $N_\delta \supset p$, $\overline{N}_\delta \subset N_\psi$ and $\overline{N}_\delta \cdot \sum_{j=2}^{d} C_j^{n-1} = 0$. Then $\lambda[N_\delta]$ contains a closed $(n-1)$-cell $\overline{C_t^{n-1}}$ such that $\Delta[C_t^{n-1}]N_\alpha \neq 0$. By arguments given in the proof of Lemma 2, it can be proved that C_1^{n-1} contains an $(n-1)$-cell C_a^{n-1} whose λ-set is $\lambda[C_t^{n-1}]$. It is evident that the λ-set of one of the descendants of N_α generated by C_t^{n-1} contains the $(n-1)$-sphere $\overline{C_t^{n-1}} + C_a^{n-1}$ and, furthermore, that the λ-set of this descendant is identical with this $(n-1)$-sphere. Hence $\lambda[N_\alpha]$ is an $(n-1)$-sphere.

5.4. Lemma 4. *Let W_1, W_2, and V be the sets given in Theorem* I′ (4.3) *with the following restrictions*: $\overline{W}_1 \subset I^n$ *and, if* $V \neq I^n$, $\bar{V} \subset I^n$. *Then* $W_2 = \sum_{i=1}^{d+1} D_i + \sum_{j=1}^{d} C_j^{n-1}$, *$D_i$ is a proper descendant of W_2 and* $D_r \cdot D_s = 0$, $r \neq s$. *If μ is a fixed value of j, there are two sets, D_α and D_β, such that* $\lambda[D_\alpha] \cdot \lambda[D_\beta] = \overline{C_\mu^{n-1}}$ *and* $C_\mu^{n-1} \cdot \lambda[D_\delta] = 0$, $\delta \neq \alpha, \beta$. *The set* $\lambda[D_\alpha] - \overline{C_\mu^{n-1}}$ *is an $(n-1)$-cell whose λ-set is* $\lambda[C_\mu^{n-1}]$.

Since $\lambda[W_i]$, $i=1, 2$, is an $(n-1)$-sphere, it can be shown that $\lambda[C_j^{n-1}] \subset \lambda[W_2]$. The other desired results can be obtained by referring to the definitions in §2, Theorem P_1 and results previously established.

The set of descendants given in the lemma is called the set of final descendants of W_2 generated by $\lambda[W_1]$.

Corollary. *If V is I^n or is strongly homeomorphic with I^n,* $\Delta[C_j^{n-1}]W_2 \neq 0$ *for all values of j.*

This proposition can be proved by means of the results given in the lemma, Theorem I′ (4.2) and Theorems P_1 and P_2.

5.5. Lemma 5. *Let C_1^{n-1} and C_2^{n-1} be two $(n-1)$-cells such that* $\lambda[I^n] = \overline{C_1^{n-1}} + \overline{C_2^{n-1}}$ *and* $\overline{C_1^{n-1}} \cdot \overline{C_2^{n-1}} = \lambda[C_i^{n-1}]$, $i=1, 2$. *Then there exists an $(n-1)$-cell C_3^{n-1} having the following properties*: $\lambda[C_3^{n-1}] = \lambda[C_i^{n-1}]$, $i=1, 2$, *and* $\Delta[C_3^{n-1}]I^n \neq 0$.

There exists N_f such that $\lambda[N_f]$ contains a closed $(n-1)$-cell $\overline{C_a^{n-1}}$ and $\Delta[C_a^{n-1}]I^n \neq 0$. From the proof of Lemma 2 it is known that $\lambda[I^n] = \overline{C_{\beta_1}^{n-1}} + \overline{C_{\beta_2}^{n-1}}$, where $\overline{C_{\beta_1}^{n-1}} \cdot \overline{C_{\beta_2}^{n-1}} = \lambda[C_a^{n-1}] = \lambda[C_{\beta_i}^{n-1}]$, $i=1, 2$. By Corollary, Theorem P_2, there exists a homeomorphism, $H_1(\lambda[I^n]) = \lambda[I^n]$, such that $H_1(\overline{C_{\beta_i}^{n-1}}) = \overline{C_i^{n-1}}$, $i=1, 2$. There is a homeomorphism, $H_2(Z^n) = Z^n$, of such nature that $H_2(\lambda[I^n])=H_1(\lambda[I^n])$. The set $H_2(C_\alpha^{n-1})$ is an $(n-1)$-cell C_3^{n-1}. Since H_2 is a homeomorphism, $\Delta[C_3^{n-1}]I^n\neq 0$.

5.6. LEMMA 6. *I^n is a connected set.*

Case 1. $n=1$. $\lambda[I^1]$ consists of two points, p_1 and p_2. Suppose that I^1 is not connected. Then

$$I^1 = M_1 + M_2;\ \overline{M}_1\cdot M_2 + M_1\cdot\overline{M}_2 = 0;\ M_i \neq 0.$$

The two sets, $M_1+p_1+p_2$ and $M_2+p_1+p_2$, cannot both be connected sets. For, assume that each of these sets is connected. There exists N_h such that $N_h \supset p_1$ and $\overline{N}_h \not\supset p_2$. Since $M_1+p_1+p_2$ is a connected set, $M_1+p_1+p_2$ contains a point q belonging to $\lambda[N_h]$ and $\Delta[q]I^1\neq 0$. $M_1\supset q$. Then $M_2\not\supset q$.

$$I^1 = D_1(I^1) + D_2(I^1) + q;\ \overline{D_1(I^1)}\cdot\overline{D_2(I^1)} = q.$$

Each of the points p_i, $i=1$, 2, belongs to one and only one of the sets $\lambda[D_j(I^1)]$, $j=1$, 2. The connected set $M_2+p_1+p_2$ is the sum of two non-vacuous sets belonging respectively to $\overline{D_1(I^1)}-q$ and $\overline{D_2(I^1)}-q$. This is impossible. Therefore, at least one of the sets $M_i+p_1+p_2$ is not connected. Suppose that $M_1+p_1+p_2$ is not connected.

$$M_1 + p_1 + p_2 = P_1 + P_2;\ \overline{P}_1\cdot P_2 + P_1\cdot\overline{P}_2 = 0;\ P_i \neq 0.$$

The set p_1+p_2 cannot belong to one of the sets P_i. For, suppose that $P_1\supset p_1+p_2$. $Z^1=(P_1+M_2)+P_2$. But Z^1 is a connected set and $\overline{P_1+M_2}\cdot P_2+(P_1+M_2)\cdot\overline{P}_2=0$. Then let $P_i\supset p_i$, $i=1$, 2. The set P_1 is connected. For, suppose that

$$P_1 = P_{11} + P_{12};\ \overline{P}_{11}\cdot P_{12} + P_{11}\cdot\overline{P}_{12} = 0;\ P_{1i} \neq 0.$$

The point p_1 belongs to one of the sets P_{1i}. Assume that $P_{11}\supset p_1$. $Z^1=(P_{11}+P_2+M_2)+P_{12}$. Again we have an impossible situation, since $\overline{P_{11}+P_2+M_2}\cdot P_{12}+(P_{11}+P_2+M_2)\cdot\overline{P}_{12}=0$. Then P_1 is a connected set. Similarly, P_2 is a connected set. Since $M_1\neq 0$, at least one of the sets P_i contains more than one point. Let P_1 be such a set. Now assume that $M_2+p_1+p_2$ is a connected set. There exists N_g such that $N_g\supset p_1$ and $N_g\not\supset P_1$. Then P_1, being a connected set, contains a point x of $\lambda[N_g]$ and $\Delta[x]I^1\neq 0$.

$$I^1 = D_{\delta_1}(I^1) + D_{\delta_2}(I^1) + x;\ \overline{D_{\delta_1}(I^1)}\cdot\overline{D_{\delta_2}(I^1)} = x.$$

Since x belongs to M_1, the connected set $M_2+p_1+p_2$ is the sum of two non-vacuous sets belonging respectively to $\overline{D_{\delta_1}(I^1)}-x$ and $\overline{D_{\delta_2}(I^1)}-x$. Then $M_2+p_1+p_2$ is not a connected set.

$$M_2 + p_1 + p_2 = R_1 + R_2;\ \overline{R}_1\cdot R_2 + R_1\cdot\overline{R}_2 = 0;\ R_i \neq 0.$$

As in the above, it can be shown that, by a proper choice of subsets, $R_i \supset p_i$, $i=1, 2$. $Z^1=(P_1+R_1)+(P_2+R_2)$ and $\overline{P_1+R_1}\cdot(P_2+R_2)+(P_1+R_1)\cdot\overline{P_2+R_2}=0$. But Z^1 is connected. This final contradiction shows that I^1 is a connected set.

Case 2. $n>1$. There exists N_r such that $\overline{N}_r\subset I^n$. $\lambda[N_r]$ is an $(n-1)$-sphere (Lemma 4). Suppose that $\overline{N}_r$ is not connected. Then

$$\overline{N}_r = B_1 + B_2;\ \overline{B}_1\cdot B_2 + B_1\cdot\overline{B}_2 = 0;\ B_i \neq 0.$$

$\lambda[N_r]$, being a connected set, belongs to one of the sets B_i. Suppose that $\lambda[N_r]\subset B_1$. $Z^n=[B_1+(Z^n-\overline{N}_r)]+B_2$ and $\overline{B_1+(Z^n-\overline{N}_r)}\cdot B_2+[B_1+(Z^n-\overline{N}_r)]\cdot\overline{B}_2=0$. But Z^n is connected. Then $\overline{N}_r$ is connected.

Let T be a component of I^n. $Z^n=T+(\lambda[I^n]+(I^n-T))$. If a point q of T were a limit point of I^n-T, there would be a neighborhood N_b such that $\overline{N}_b\subset I^n$, $N_b\supset q$ and $N_b\cdot(I^n-T)\neq 0$. Then the set $T+\overline{N}_b$ would be a connected set. This contradicts the fact that T is a maximal connected subset of I^n. The set $\lambda[I^n]$ contains a limit point of T.

Let p be a point of $\lambda[T]\cdot\lambda[I^n]$. There exists N_s such that $N_s\supset p$ and $\overline{N}_s\not\supset T$. Then $T\cdot\lambda[N_s]\neq 0$. $Z^n\cdot\lambda[N_s]=\sum_{j=1}^d\overline{C_j^{n-1}}$, $\overline{C_j^{n-1}}\cdot I^n=C_j^{n-1}$ and $T\cdot\lambda[N_s]\subset\sum_{j=1}^d C_j^{n-1}$. Let $T\cdot C_1^{n-1}\neq 0$. Then $C_1^{n-1}\subset T$. $\Delta[C_1^{n-1}]I^n\neq 0$ (Corollary, Lemma 4).

$$I^n = D_1(I^n) + D_2(I^n) + C_1^{n-1};\ \overline{D_1(I^n)}\cdot\overline{D_2(I^n)} = \overline{C_1^{n-1}}.$$

There exist two sets, $C_{\alpha_1}^{n-1}$ and $C_{\alpha_2}^{n-1}$ such that $\lambda[I^n]=\overline{C_{\alpha_1}^{n-1}}+\overline{C_{\alpha_2}^{n-1}}$ and $\overline{C_{\alpha_1}^{n-1}}\cdot\overline{C_{\alpha_2}^{n-1}}=\lambda[C_1^{n-1}]=\lambda[C_{\alpha_i}^{n-1}]$, $i=1, 2$. Suppose that $\lambda[D_i(I^n)]\supset C_{\alpha_i}^{n-1}$, $i=1, 2$. Let S^{n-1} be the $(n-1)$-sphere in euclidean n-space E^n whose points have coordinates which satisfy the equation: $\sum_{i=1}^n x_i^2=1$. The $(n-1)$-dimensional plane $x_1=0$ separates S^{n-1} into two $(n-1)$-cells, K_1^{n-1} and K_2^{n-1}, such that $S^{n-1}=\overline{K_1^{n-1}}+\overline{K_2^{n-1}}$ and $\overline{K_1^{n-1}}\cdot\overline{K_2^{n-1}}=\lambda[K_i^{n-1}]$, $i=1, 2$. By Theorem P_2 there is a homeomorphism, $H(\lambda[I^n])=S^{n-1}$, such that $H(\overline{C_{\alpha_i}^{n-1}})=\overline{K_i^{n-1}}$, $i=1, 2$.

Let p_1 be a point of $C_{\alpha_1}^{n-1}$ and $H(p_1)=p_1'$. $K_1^{n-1}\supset p_1'$. In E^n there exists an $(n-1)$-dimensional plane E_1^{n-1} such that E_1^{n-1} contains the points p_1' and $(0, 0, \cdots, 0)$. E_1^{n-1} separates S^{n-1} into two $(n-1)$-cells, $K_{\beta_1}^{n-1}$ and $K_{\beta_2}^{n-1}$. $S^{n-1}=\overline{K_{\beta_1}^{n-1}}+\overline{K_{\beta_2}^{n-1}}$, $\overline{K_{\beta_1}^{n-1}}\cdot\overline{K_{\beta_2}^{n-1}}=\lambda[K_{\beta_1}^{n-1}]$, $\lambda[K_{\beta_i}^{n-1}]\supset p_1'$ and $\lambda[K_{\beta_i}^{n-1}]\cdot K_2^{n-1}\neq 0$, $i=1, 2$. Let $H^{-1}(K_{\beta_i}^{n-1})=C_{\beta_i}^{n-1}$. Then $\lambda[C_{\beta_i}^{n-1}]\supset p_1$, $\lambda[C_{\beta_i}^{n-1}]\cdot C_{\alpha_2}^{n-1}\neq 0$ and $\lambda[I^n]=\overline{C_{\beta_1}^{n-1}}+\overline{C_{\beta_2}^{n-1}}$, $\overline{C_{\beta_1}^{n-1}}\cdot\overline{C_{\beta_2}^{n-1}}=\lambda[C_{\beta_i}^{n-1}]$, $i=1, 2$. By Lemma 5, there exists $C_{\beta_3}^{n-1}$ such that $C_{\beta_3}^{n-1}\subset I^n$ and $\lambda[C_{\beta_3}^{n-1}]=\lambda[C_{\beta_i}^{n-1}]$, $i=1, 2$. The connected set $C_{\beta_3}^{n-1}$ contains non-vacuous subsets in each of the two sets, $D_1(I^n)$ and $D_2(I^n)$. Therefore $C_{\beta_3}^{n-1}\cdot C_1^{n-1}\neq 0$. Since $C_1^{n-1}\subset T$, $C_{\beta_3}^{n-1}\subset T$. Then p_1, any point of $C_{\alpha_1}^{n-1}$, is a limit point of T. In the same manner it can be shown that $C_{\alpha_2}^{n-1} \subset \lambda[T]$. Hence $\lambda[T] = \lambda[I^n]$. If I^n contains a component T_1 distinct from T, $\lambda[T_1] = \lambda[I^n]$. From the preceding discussion, it is seen that $T_1 \cdot C_1^{n-1} \neq 0$. Then T_1 cannot be distinct from T.

5.7. Lemma 7. *There exists a neighborhood N_α such that $\overline{N}_\alpha \subset I^n$ and N_α is strongly homeomorphic with I^n.*

There exists N_t such that $N_t \cdot \lambda[I^n] \neq 0$ and $N_t \not\supset I^n$. Then $Z^n \cdot \lambda[N_t] = \sum_{j=1}^{d} \overline{C_j^{n-1}}$. Let $D_1(I^n)$ and $D_2(I^n)$ be the two members of the set of final descendants of I^n generated by $\lambda[N_t]$ which have C_1^{n-1} on their λ-sets (Lemma 4). Let p be a point of C_1^{n-1}. There is a neighborhood N_α such that $N_\alpha \supset p$ and $\overline{N}_\alpha$ belongs to the set $D_1(I^n) + D_2(I^n) + C_1^{n-1}$. $\lambda[N_t]$ generates a set of final descendants for N_α. $\overline{N}_\alpha \cdot \lambda[N_t] = \sum_{j=1}^{m} \overline{C_{\delta_j}^{n-1}}$. Since $\overline{N}_\alpha \cdot \lambda[N_t] \subset C_1^{n-1}$, $\sum_{j=1}^{m} \overline{C_{\delta_j}^{n-1}} \subset C_1^{n-1}$. Let $D_1(N_\alpha)$ and $D_2(N_\alpha)$ be the two members of the set of final descendants of N_α generated by $\lambda[N_t]$ which have $C_{\delta_1}^{n-1}$ on their λ-sets. $\lambda[N_\alpha]$ generates a set of final descendants for each of the sets $D_i(I^n)$, $i = 1, 2$. $C_{\delta_1}^{n-1} \cdot \lambda[N_\alpha] = 0$. Then, since $C_{\delta_1}^{n-1}$ is a connected set, $C_{\delta_1}^{n-1}$ belongs to the λ-set of one and only one final descendant of each of the sets $D_i(I^n)$, $i = 1, 2$, generated by $\lambda[N_\alpha]$. Let these two final descendants be $D_{11}(I^n)$ and $D_{21}(I^n)$ belonging to $D_1(I^n)$ and $D_2(I^n)$ respectively. There exists N_e such that $N_e \cdot C_{\delta_1}^{n-1} \neq 0$ and N_e belongs to the set $D_{11}(I^n) + D_{21}(I^n) + C_{\delta_1}^{n-1}$. N_e contains a point p_1 of $D_1(N_\alpha)$. The point p_1 belongs to one of the sets $D_{i1}(I^n)$. Suppose that $D_{11}(I^n) \supset p_1$. Since I^n is a connected set, $D_{11}(I^n)$ is connected. But $D_{11}(I^n) \cdot (\lambda[N_\alpha] + \lambda[N_t]) = 0$. Then $D_{11}(I^n) \subset D_1(N_\alpha)$. Since each of the two sets, $\lambda[I^n]$ and $\lambda[N_\alpha]$, is an $(n-1)$-sphere, each of the sets, $\lambda[D_{11}(I^n)]$ and $\lambda[D_1(N_\alpha)]$, is an $(n-1)$-sphere. $\lambda[D_{11}(I^n)] \subset \lambda[D_1(N_\alpha)]$. Therefore $\lambda[D_{11}(I^n)] \equiv \lambda[D_1(N_\alpha)]$, since no proper subset of an $(n-1)$-sphere is an $(n-1)$-sphere. No point of $D_1(N_\alpha)$ can belong to any final descendant of one of the sets $D_i(I^n)$, $i = 1, 2$, generated by $\lambda[N_\alpha]$ other than $D_{11}(I^n)$. For, otherwise, such a descendant would belong to $D_1(N_\alpha)$ and its λ-set would be identical with $\lambda[D_1(N_\alpha)]$ and, therefore, with $\lambda[D_{11}(I^n)]$—an impossible situation. Then $D_1(N_\alpha) = D_{11}(I^n)$. Hence $D_1(N_\alpha)$ is strongly homeomorphic with I^n. N_α is strongly homeomorphic with I^n.

Corollary. *Let N_ρ and N_ξ be two neighborhoods such that $\overline{N}_\rho + \overline{N}_\xi \subset I^n$, N_ρ is connected, N_ξ contains a point p of $\lambda[N_\rho]$ and $N_\xi \not\supset N_\rho$. Then N_ξ is strongly homeomorphic with N_ρ and p is a limit point of $Z^n - \overline{N}_\rho$.*

This proposition can be proved by means of the procedure used in the proof of the lemma.

5.8. Lemma 8. *Let N_α be a neighborhood such that $\overline{N}_\alpha \subset I^n$, N_α is strongly homeomorphic with I^n and $\overline{N}_\alpha = \sum_{k=1}^{c} P_k$, where each set P_k is an open set with* *respect to* $\overline{N}_\alpha$. *Then* $\overline{N}_\alpha=\sum_{j=1}^{r}\bar{D}_j$ *such that* $D_\xi\cdot D_\eta=0$, $\xi\neq\eta$, *and* D_j *belongs to at least one set* P_k *and is strongly homeomorphic with* N_α. *If* μ *is a fixed value of* j, $\lambda[D_\mu]=\sum_{i=1}^{b_\mu}\overline{C_{\mu_i}^{n-1}}$, $C_{\mu_s}^{n-1}\cdot C_{\mu_t}^{n-1}=0$, $s\neq t$, *and, if* h *is a fixed value of* i, $\lambda[D_\mu]-\overline{C_{\mu_h}^{n-1}}$ *is an* $(n-1)$*-cell and either* $C_{\mu_h}^{n-1}\subset N_\alpha$ *or* $C_{\mu_h}^{n-1}\subset\lambda[N_\alpha]$. *If* $C_{\mu_h}^{n-1}\subset N_\alpha$, *there is a set* D_κ, $\kappa\neq\mu$, *such that* $\lambda[D_\mu]\cdot\lambda[D_\kappa]=\overline{C_{\mu_h}^{n-1}}$ *and* $C_{\mu_h}^{n-1}\cdot\lambda[D_\delta]=0$, $\delta\neq\mu,\kappa$. *If* $C_{\mu_h}^{n-1}\subset\lambda[N_\alpha]$, $C_{\mu_h}^{n-1}\cdot\lambda[D_\phi]=0$, $\phi\neq\mu$.

The following proof applies when $n>1$. The modifications necessary when $n=1$ are obvious.

The set $\overline{N}_\alpha$ is compact. Assign to each point p of $\overline{N}_\alpha$ the neighborhood N_{a_p} such that N_{a_p} is the neighborhood of $G(p)$ (§5.1) of least subscript having the following properties: $\overline{N}_{a_p}\subset I^n$, $N_{a_p}\not\supset N_\alpha$ and $N_{a_p}\cdot\overline{N}_\alpha$ belongs to at least one of the sets P_k. There exists a finite subset of the set of all such neighborhoods: $T_1=\{N_{\psi_j}\}$, j$=1, 2, \cdots, \rho$, such that $\sum_{j=1}^{\rho}N_{\psi_j}\supset\overline{N}_\alpha$ and $N_{\psi_c}\not\supset N_{\psi_d}$, $c\neq d$. By successive applications of Corollary, Lemma 7, it can be shown that every neighborhood belonging to T_1 is strongly homeomorphic with N_α and is, therefore, a connected set.

Let N_{ψ_m} be any neighborhood of T_1. Every point of the set $\lambda[N_{\psi_m}]\cdot\overline{N}_\alpha$ belongs to at least one of the neighborhoods of T_1. Suppose that the subscripts in the symbols for the neighborhoods constituting T_1 are so chosen that $\{N_{\psi_j}\}$, $j=1, 2, \cdots, m-1$, is the set of neighborhoods such that $\lambda[N_{\psi_m}]\cdot N_{\psi_j}\neq 0$, $j=1, 2, \cdots, m-1$.

Let A_{ψ_1} represent N_{ψ_1}. If $N_{\psi_1}\cdot\lambda[N_{\psi_2}]=0$, let $(A_{\psi_1})_{\psi_2}$ be N_{ψ_1} and, if $N_{\psi_1}\cdot\lambda[N_{\psi_2}]\neq 0$, let $(A_{\psi_1})_{\psi_2}$ be the set of final descendants of A_{ψ_1} generated by $\lambda[N_{\psi_2}]$. If $N_{\psi_2}\cdot\lambda[N_{\psi_1}]=0$, let $B_{\psi_2\psi_1}$ be N_{ψ_2} and, if $N_{\psi_2}\cdot\lambda[N_{\psi_1}]\neq 0$, let $B_{\psi_2\psi_1}$ be the set consisting of those final descendants of N_{ψ_2} generated by $\lambda[N_{\psi_1}]$ which do not lie in N_{ψ_1}.

Every descendant of N_{ψ_2} is a connected set. No final descendant of N_{ψ_2} generated by $\lambda[N_{\psi_1}]$ contains a point of the set $\lambda[N_{\psi_1}]+\lambda[N_{\psi_2}]$. Then every such final descendant lies wholly in N_{ψ_1} or wholly in $Z^n-\overline{N}_{\psi_1}$.

$$A_{\psi_1\psi_2} = (A_{\psi_1})_{\psi_2} + B_{\psi_2\psi_1},$$
$$\overline{A}_{\psi_1\psi_2} = \overline{N}_{\psi_1} + \overline{N}_{\psi_2}.$$

If $A_{\psi_1\psi_2}\cdot\lambda[N_{\psi_3}]=0$, let $(A_{\psi_1\psi_2})_{\psi_3}$ be $A_{\psi_1\psi_2}$ and, if $A_{\psi_1\psi_2}\cdot\lambda[N_{\psi_3}]\neq 0$, let $(A_{\psi_1\psi_2})_{\psi_3}$ be the set consisting of the final descendants of members of $A_{\psi_1\psi_2}$ generated by $\lambda[N_{\psi_3}]$ and those members of $A_{\psi_1\psi_2}$ which contain no point of $\lambda[N_{\psi_3}]$. If $N_{\psi_3}\cdot\lambda[N_{\psi_2}]=0$, $B_{\psi_3\psi_2}=N_{\psi_3}$ and, if $N_{\psi_3}\cdot\lambda[N_{\psi_2}]\neq 0$, $B_{\psi_3\psi_2}$ is the set of those final descendants of N_{ψ_3} generated by $\lambda[N_{\psi_2}]$ which do not lie in N_{ψ_2}. If $B_{\psi_3\psi_2}\cdot\lambda[N_{\psi_1}]=0$, $B_{\psi_3\psi_2\psi_1}=B_{\psi_3\psi_2}$ and, if $B_{\psi_3\psi_2}\cdot\lambda[N_{\psi_1}]\neq 0$, $B_{\psi_3\psi_2\psi_1}$ is the set consisting of the final descendants of members of $B_{\psi_3\psi_2}$ generated by

$\lambda[N_{\psi_1}]$ which do not lie in N_{ψ_1} and those members of $B_{\psi_3\psi_2}$ which contain no point of $\lambda[N_{\psi_1}]$.

$$A_{\psi_1\psi_2\psi_3} = (A_{\psi_1\psi_2})_{\psi_3} + B_{\psi_3\psi_2\psi_1},$$

$$\overline{A}_{\psi_1\psi_2\psi_3} = \sum_{j=1}^{3} \overline{N}_{\psi_j}.$$

Proceeding in this manner, we obtain $A_{\psi_1\psi_2\cdots\psi_{m-1}}$.

$$A_{\psi_1\psi_2\cdots\psi_{m-1}} = (A_{\psi_1\psi_2\cdots\psi_{m-2}})_{\psi_{m-1}} + B_{\psi_{m-1}\cdots\psi_2\psi_1},$$

$$\overline{A}_{\psi_1\psi_2\cdots\psi_{m-1}} = \sum_{j=1}^{m-1} \overline{N}_{\psi_j}.$$

Considering in the next step the neighborhood N_{ψ_m}, we obtain $A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_m}$. If T_1 contains additional neighborhoods $N_{\psi_{m+j}}$, $j=1, 2, \cdots, s$, $N_{\psi_{m+j}}\cdot(\lambda[N_{\psi_m}]\cdot\overline{N}_\alpha)=0$. Continuing as in the above, we obtain finally

$$\overline{A}_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho} = \sum_{j=1}^{\rho} \overline{N}_{\psi_j}.$$

Let A be the set consisting of those final descendants of members of $A_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho}$ generated by $\lambda[N_\alpha]$ which lie in N_α and those members of $A_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho}$ which lie in N_α. $\overline{A}=\overline{N}_\alpha\cdot\sum_{j=1}^{\rho}\overline{N}_{\psi_j}=\overline{N}_\alpha$. Since the members of A are connected sets and $A\cdot(\lambda[N_\alpha]+\sum_{j=1}^{\rho}\lambda[N_{\psi_j}])=0$, the subdivision of $\overline{N}_\alpha$ obtained by the above process is unique, that is, the subdivision is independent of the order in which the neighborhoods N_{ψ_j} enter into the discussion.

Since $\lambda[N_{\psi_m}]$ generates a set of final descendants for each of the neighborhoods N_{ψ_j}, $j=1, 2, \cdots, m-1$, the set $\lambda[N_{\psi_m}]\cdot\sum_{j=1}^{m-1}\overline{N}_{\psi_j}$ is, topologically, an $(n-1)$-dimensional euclidean set and $\lambda[N_{\psi_m}]\cdot\sum_{j=1}^{m-1}\lambda[N_{\psi_j}]$ is an $(n-2)$-dimensional set. Then $\lambda[N_{\psi_m}]\cdot A_{\psi_1\psi_2\cdots\psi_{m-1}}\neq 0$. Therefore $\lambda[N_{\psi_m}]$ generates sets of final descendants for certain members of $A_{\psi_1\psi_2\cdots\psi_{m-1}}$. Let E be the set of such members of $A_{\psi_1\psi_2\cdots\psi_{m-1}}$. $\overline{E}\cdot\lambda[N_{\psi_m}]=\sum_{k=1}^{t}\overline{C_k^{n-1}}$, $\overline{C_k^{n-1}}\cdot E=C_k^{n-1}$, $C_\xi^{n-1}\cdot C_\eta^{n-1}=0$, $\xi\neq\eta$, and $\lambda[C_k^{n-1}]\subset\lambda[E]$. Suppose that p is a point of $\lambda[N_{\psi_m}]\cdot\overline{A}_{\psi_1\psi_2\cdots\psi_{m-1}}$ and that $\overline{E}\not\supset p$. Then $\lambda[A_{\psi_1\psi_2\cdots\psi_{m-1}}]\supset p$. But $\lambda[A_{\psi_1\psi_2\cdots\psi_{m-1}}]=\sum_{j=1}^{m-1}\lambda[N_{\psi_j}]$. Hence p belongs to the λ-set of at least one of the neighborhoods N_{ψ_j}, $j=1, 2, \cdots, m-1$. Let $\lambda[N_{\psi_1}]\supset p$. $\overline{N}_{\psi_1}\cdot\lambda[N_{\psi_m}]=\sum_{j=1}^{b}\overline{C_{\delta_j}^{n-1}}$, $N_{\psi_1}\cdot\overline{C_{\delta_j}^{n-1}}=C_{\delta_j}^{n-1}$ and $\sum_{j=1}^{b}\overline{C_{\delta_j}^{n-1}}\supset p$. Suppose that $\overline{C_{\delta_1}^{n-1}}\supset p$. There exists N_g such that $N_g\supset p$ and $\overline{N}_g\cdot\overline{E}=0$. Then N_g contains a point q of $C_{\delta_1}^{n-1}$. There is a neighborhood N_e such that $N_e\supset q$, $\overline{N}_e\subset N_{\psi_1}$ and $\overline{N}_e\cdot\overline{E}=0$. $\overline{N}_e\cdot\lambda[N_{\psi_m}]=\sum_{j=1}^{a}\overline{C_{\theta_j}^{n-1}}$. The set $\sum_{j=1}^{a}\overline{C_{\theta_j}^{n-1}}$ is an $(n-1)$-dimensional set. $\sum_{j=1}^{a}\overline{C_{\theta_j}^{n-1}}\subset\overline{A}_{\psi_1\psi_2\cdots\psi_{m-1}}-\overline{E}$. Then $\sum_{j=1}^{a}\overline{C_{\theta_j}^{n-1}}\subset\lambda[N_{\psi_m}]\cdot\lambda[A_{\psi_1\psi_2\cdots\psi_{m-1}}]$. But $\lambda[N_{\psi_m}]\cdot\lambda[A_{\psi_1\psi_2\cdots\psi_{m-1}}]$ is an $(n-2)$-dimensional set. The contradiction here encountered shows that $\lambda[N_{\psi_m}]\cdot\overline{A}_{\psi_1\psi_2\cdots\psi_{m-1}}=\lambda[N_{\psi_m}]\cdot\overline{E}=\sum_{k=1}^{t}\overline{C_k^{n-1}}$.

Let $C_{k_1}^{n-1}$ be one of the sets C_k^{n-1} and M_1 and M_2 be the two final descendants of a member of $A_{\psi_1\psi_2\cdots\psi_{m-1}}$ generated by $\lambda[N_{\psi_m}]$ such that $\overline{M}_1\cdot\overline{M}_2=\lambda[M_1]\cdot\lambda[M_2]=\overline{C_{k_1}^{n-1}}$. The set $M_1+M_2+C_{k_1}^{n-1}$ is a descendant of one of the neighborhoods N_{ψ_j}, $j=1, 2, \cdots, m-1$. Let M denote the set $M_1+M_2+C_{k_1}^{n-1}$. Considering now the set $\lambda[N_{\psi_{m+1}}]$, two possibilities present themselves:

(1) $\lambda[N_{\psi_{m+1}}]\cdot(M_1+M_2)=0$. In this case M_1 and M_2 are members of $A_{\psi_1\psi_2\cdots\psi_m\psi_{m+1}}$.

(2) $\lambda[N_{\psi_{m+1}}]\cdot(M_1+M_2)\neq 0$. In this situation $\lambda[N_{\psi_{m+1}}]$ generates final descendants for one or both of the sets M_i. If $\lambda[N_{\psi_{m+1}}]\cdot M_i\neq 0$, $\lambda[N_{\psi_{m+1}}]\cdot\lambda[M_i]$ is an $(n-2)$-dimensional euclidean set. Then $\lambda[N_{\psi_{m+1}}]\cdot C_{k_1}^{n-1}$ is at most an $(n-2)$-dimensional set. The set M belongs to one of the members of $A_{\psi_1\psi_2\cdots\psi_{m-1}}$. Let M' be this member. M may be identical with M'. Under our present assumption, $\lambda[N_{\psi_{m+1}}]$ generates a set of final descendants for M'. Let R be the set of these final descendants. $R\subset A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_{m+1}}$, $\overline{C_{k_1}^{n-1}}\subset\overline{R}$ and $C_{k_1}^{n-1}\cdot\lambda[R]=C_{k_1}^{n-1}\cdot\lambda[N_{\psi_{m+1}}]$. Then $C_{k_1}^{n-1}\cdot R\neq 0$. Therefore $\lambda[N_{\psi_m}]$ generates sets of final descendants for certain members of R. Let R_1 be the set of such members of R. $\overline{R}_1\cdot\lambda[N_{\psi_m}]=\sum_{j=1}^{a}\overline{C_{\omega_j}^{n-1}}$ and $\overline{C_{\omega_j}^{n-1}}\cdot R_1=C_{\omega_j}^{n-1}$. Since $C_{k_1}^{n-1}\cdot\lambda[R]$ is at most an $(n-2)$-dimensional set, certain of the sets $\overline{C_{\omega_j}^{n-1}}$ belong to $\overline{C_{k_1}^{n-1}}$. $C_{k_1}^{n-1}$ may be one of the sets $C_{\omega_j}^{n-1}$. Suppose that, by a proper choice of subscripts, $\{\overline{C_{\omega_j}^{n-1}}\}$, $j=1, 2, \cdots, a_1$, is the set of the sets $\overline{C_{\omega_j}^{n-1}}$ which belong to $\overline{C_{k_1}^{n-1}}$. If $\overline{C_{k_1}^{n-1}}$ contains a point q that does not belong to $\sum_{j=1}^{a_1}\overline{C_{\omega_j}^{n-1}}$, $\lambda[R]\supset q$. It can be shown by a method analogous to one used above that $\overline{C_{k_1}^{n-1}}$ contains no such point q. Therefore, $\overline{C_{k_1}^{n-1}}=\sum_{j=1}^{a_1}\overline{C_{\omega_j}^{n-1}}$ $A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_{m+1}\psi_m}\equiv A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_m\psi_{m+1}}$. Corresponding to each set $C_{\omega_j}^{n-1}$, $j=1, 2, \cdots, a_1$, $A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_m\psi_{m+1}}$ contains two members, E_{1j} and E_{2j}, such that $\overline{E}_{1j}\cdot\overline{E}_{2j}=\lambda[E_{1j}]\cdot\lambda[E_{2j}]=\overline{C_{\omega_j}^{n-1}}$.

Let Y be the set consisting of the final descendants of members of $A_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho}$ generated by $\lambda[N_\alpha]$ and those members of $A_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho}$ which contain no point of $\lambda[N_\alpha]$. It is evident that $\overline{Y}\cdot\sum_{k=1}^{t}\overline{C_k^{n-1}}=\sum_{j=1}^{\beta}\overline{C_{\mu_j}^{n-1}}$ and, corresponding to each set $C_{\mu_j}^{n-1}$, Y contains two members, B_{1j} and B_{2j}, such that $\overline{B}_{1j}\cdot\overline{B}_{2j}=\lambda[B_{1j}]\cdot\lambda[B_{2j}]=\overline{C_{\mu_j}^{n-1}}$. $\lambda[N_{\psi_m}]\cdot\overline{A}=\lambda[N_{\psi_m}]\cdot\overline{N}_\alpha\subset\sum_{k=1}^{t}\overline{C_k^{n-1}}$. If $N_{\psi_m}\subset N_\alpha$, $\lambda[N_{\psi_m}]\cdot\overline{A}=\sum_{j=1}^{\beta}\overline{C_{\mu_j}^{n-1}}$ and, if $N_{\psi_m}\not\subset N_\alpha$, $\lambda[N_{\psi_m}]\cdot\overline{A}$ is the sum of a certain number of the sets $\overline{C_{\mu_j}^{n-1}}$.

It can be shown that $\lambda[N_\alpha]=\sum_{j=1}^{\nu}\overline{C_{\kappa_j}^{n-1}}$ and, corresponding to each set $C_{\kappa_j}^{n-1}$, Y contains two members, P_{1j} and P_{2j}, such that $\overline{P}_{1j}\cdot\overline{P}_{2j}=\lambda[P_{1j}]\cdot\lambda[P_{2j}]=\overline{C_{\kappa_j}^{n-1}}$. Therefore $\lambda[A]=\lambda[N_\alpha]+\overline{N}_\alpha\cdot\sum_{j=1}^{\rho}\lambda[N_{\psi_j}]=\sum_{i=1}^{w}\overline{C_i^{n-1}}$. Each set C_i^{n-1} belongs wholly to N_α or wholly to $\lambda[N_\alpha]$. $C_{i_1}^{n-1}\cdot C_{i_2}^{n-1}=0$, $i_1\neq i_2$. If $C_{i_1}^{n-1}\subset N_\alpha$, $C_{i_1}^{n-1}$ belongs to the λ-sets of two and only two members of A and no other member of A contains a point of $C_{i_1}^{n-1}$ on its λ-set. Since every point of $\lambda[N_\alpha]$ is a limit point of $Z^n-\overline{N}_\alpha$, if $C_{i_1}^{n-1}\subset\lambda[N_\alpha]$, $C_{i_1}^{n-1}$ belongs wholly to the λ-set of one and only one member of A and no other member of A contains a point of $C_{i_1}^{n-1}$ on its λ-set. It can be shown without difficulty that, if D is a member of A and $\lambda[D]\supset C_{i_1}^{n-1}$, then $\lambda[D]-\overline{C_{i_1}^{n-1}}$ is an $(n-1)$-cell whose λ-set is $\lambda[C_{i_1}^{n-1}]$.

Let B be any member of A. Evidently $\lambda[B]=\sum_{y=1}^{b}\overline{C_y^{n-1}}$, $C_{y_1}^{n-1}\cdot C_{y_2}^{n-1}=0$, $y_1\neq y_2$, and $\lambda[B]-\overline{C_{y_1}^{n-1}}$ is an $(n-1)$-cell whose λ-set is $\lambda[C_{y_1}^{n-1}]$. Since every neighborhood of T_1 is strongly homeomorphic with I^n, it follows that B is strongly homeomorphic with I^n and Lemma 5 holds if B is substituted for I^n. Then there exists C_{11}^{n-1} such that $\overline{C_{11}^{n-1}}\cdot B=C_{11}^{n-1}$, $\lambda[C_{11}^{n-1}]=\lambda[C_1{}^{-1}]$ and $\Delta[C_{11}^{n-1}]B\neq 0$.

$$B = D_1(B) + D_2(B) + C_{11}^{n-1};\ \overline{D_1(B)}\cdot\overline{D_2(B)} = \overline{C_{11}^{n-1}}.$$

Suppose that $C_1^{n-1}\subset\lambda[D_1(B)]$. Then $\sum_{y=2}^{b}\overline{C_y^{n-1}}\subset\lambda[D_2(B)]$. It can be shown that the set $\lambda[D_2(B)]-\overline{C_2^{n-1}}$ is homeomorphic with the set $\lambda[B]-\overline{C_2^{n-1}}$. Therefore $\lambda[D_2(B)]-\overline{C_2^{n-1}}$ is an $(n-1)$-cell whose λ-set is $\lambda[C_2^{n-1}]$. There exists C_{21}^{n-1} such that $\lambda[C_{21}^{n-1}]=\lambda[C_2^{n-1}]$ and $\Delta[C_{21}^{n-1}]D_2(B)\neq 0$.

$$D_2(B) = D_{21}(B) + D_{22}(B) + C_{21}^{n-1};\ \overline{D_{21}(B)}\cdot\overline{D_{22}(B)} = \overline{C_{21}^{n-1}}.$$

Suppose that $C_2{}^{-1}\subset\lambda[D_{21}(B)]$. Then $\sum_{y=3}^{b}\overline{C_y^{n-1}}\subset\lambda[D_{22}(B)]$. Proceeding in this manner, we finally arrive at the following result: $B=\sum_{j=1}^{b+1}B_j+\sum_{j=1}^{b}C_{j1}^{n-1}$, where B_j is a descendant of B, $\lambda[B_j]=\overline{C_j^{n-1}+C_{j1}^{n-1}}$, $j=1, 2, \cdots, b$, and $\lambda[B_{b+1}]=\sum_{j=1}^{b}\overline{C_{j1}^{n-1}}$.

Subject each member of A to the same sort of subdivision. The result is a set of descendants of neighborhoods satisfying the requirements of the lemma.

5.9. Let N_α be the same as in Lemma 8 and the sets P_k, which occur in this lemma, be the sets $\overline{N}_\alpha\cdot N_{\rho_k}$, where N_{ρ_k} is a neighborhood belonging to a finite set of neighborhoods which cover $\overline{N}_\alpha$. Then $\overline{N}_\alpha=\sum_{j=1}^{r}\overline{D}_j$ (Lemma 8). Each set D_j is the descendant of a neighborhood. As in Lemma 8, denote by T_1 the set of all such neighborhoods. Let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ be a sequence of positive numbers such that $\epsilon_{n+1}<\epsilon_n$ and $\lim_{n\to\infty}\epsilon_n=0$.

Designate by K the set in euclidean n-space E^n whose points have coordinates which satisfy the inequality $\sum_{i=1}^{n}x_i^2<1$. Then K is an n-cell and $\lambda[K]$ is an $(n-1)$-sphere. It will now be proved that $\overline{K}=\sum_{j=1}^{r}\overline{K}_j$, where the sets K_j are n-cells and that there exists a homeomorphism, $H(\sum_{j=1}^{r}\lambda[D_j])=\sum_{j=1}^{r}\lambda[K_j]$, such that $H(\lambda[D_j])=\lambda[K_j]$, $j=1, 2, \cdots, r$. We shall show that, if the above statement is true for $r=m$, the statement is true for $r=m+1$.

Suppose that we have a subdivision of $\overline{N}_\alpha$ of the sort described in Lemma 8 in which $r=m+1$. Then $\overline{N}_\alpha=\sum_{j=1}^{m+1}\bar{D}_j$. Consider the set D_1. The discussion in Lemma 8 shows that there exists a set D_κ such that $\bar{D}_1\cdot\bar{D}_\kappa=\overline{C_{1\kappa}^{n-1}}$, none of the sets $\lambda[D_j]$, other than $\lambda[D_1]$ and $\lambda[D_\kappa]$, contains a point of $C_{1\kappa}^{n-1}$, $\lambda[D]-\overline{C_{1\kappa}^{n-1}}$ is an $(n-1)$-cell $C_{\mu 1}^{n-1}$, $\lambda[D_\kappa]-\overline{C_{1\kappa}^{n-1}}$ is an $(n-1)$-cell $C_{\mu\kappa}^{n-1}$, and $\overline{C_{\mu 1}^{n-1}}\cdot\overline{C_{\mu\kappa}^{n-1}}$ $=\lambda[C_{1\kappa}^{n-1}]=\lambda[C_{\mu 1}^{n-1}]=\lambda[C_{\mu\kappa}^{n-1}]$. Suppose that $\kappa=2$. Let D represent the set $D_1+D_2+C_{12}^{n-1}$. Since we are assuming that our present sets D_j result from a process similar to that employed in Lemma 8, it is clear that D is strongly homeomorphic with N_α. $\lambda[D]$ is the $(n-1)$-sphere $\overline{C_{\mu 1}^{n-1}}+\overline{C_{\mu 2}^{n-1}}$. $\overline{N}_\alpha=\bar{D}+\sum_{j=3}^{m+1}\bar{D}_j$. The sets D, D_j, $\lambda[D]$, $\lambda[D_j]$, $j=3, 4, \cdots, m+1$, have the properties and the relations among themselves ascribed to the corresponding sets obtained in Lemma 8. Since it is assumed that the statement under consideration is true for $r=m$, $\overline{K}=\sum_{j=1}^{m}\overline{K}_j$ and there exists a homeomorphism, $H_1(\lambda[D]+\sum_{j=3}^{m+1}\lambda[D_j])=\sum_{j=1}^{m}\lambda[K_j]$, such that $H_1(\lambda[D])=\lambda[K_1]$ and $H_1(\lambda[D_j])=\lambda[K_k]$, $j=k+1$, $k=2, 3, \cdots, m$. $H_1(\overline{C_{\mu j}^{n-1}})$ is a closed $(n-1)$-cell $\overline{C_j^{n-1}}$, $j=1, 2$. $\overline{C_1^{n-1}}\cdot\overline{C_2^{n-1}}=\lambda[C_j^{n-1}]$, $j=1, 2$. Then, by Theorem P_3 there exists an $(n-1)$-cell C_3^{n-1} such that $\lambda[C_3^{n-1}]=\lambda[C_j^{n-1}]$, $j=1, 2$, and $\Delta[C_3^{n-1}]K_1\neq 0$. $H_1(\lambda[C_{12}^{n-1}])=\lambda[C_3^{n-1}]$. By Theorem P_1 there is a homeomorphism, $H_2(\overline{C_{12}^{n-1}})=\overline{C_3^{n-1}}$, such that $H_2(\lambda[C_{12}^{n-1}])=H_1(\lambda[C_{12}^{n-1}])$. Since $\Delta[C_3^{n-1}]K_1\neq 0$, $K_1=K_{11}+K_{12}+C_3^{n-1}$. K_{1j}, $j=1, 2$, is an n-cell. There exists a homeomorphism, $H_3(\sum_{j=1}^{m+1}\lambda[D_j])=\lambda[K_{11}]+\lambda[K_{12}]+\sum_{j=2}^{m}\lambda[K_j]$, in which $H_3(\lambda[D_j])=H_3(\overline{C_{\mu j}^{n-1}}+\overline{C_{12}^{n-1}})=H_1(\overline{C_{\mu j}^{n-1}})+H_2(\overline{C_{12}^{n-1}})=\lambda[K_{1j}]$, $j=1, 2$, and $H_3(\lambda[D_j])=\lambda[K_k]$, $j=k+1$, $k=2, 3, \cdots, m$. The statement under consideration holds for $r=1$. The induction is complete.

We have the desired result $\overline{K}=\sum_{j=1}^{r}\overline{K}_j$, and there is a homeomorphism, $H(\sum_{j=1}^{r}\lambda[D_j])=\sum_{j=1}^{r}\lambda[K_j]$, such that $H(\lambda[D_j])=\lambda[K_j]$, $j=1, 2, \cdots, r$. Suppose that K_ξ is one of the sets K_j and that $d(K_\xi)\geqq\epsilon_1$.* Let R be the set of all points of E^n which have coordinates x_i such that $0<x_i<1$, $i=1, 2, \cdots, n$. There exists a homeomorphism $H_\theta(\overline{R})=\overline{K}_\xi$. The correspondence established by H_θ is uniformly continuous both ways. Therefore, there is a number $\delta_1>0$ such that, if the distance between any two points of $\overline{R}$ is less than δ_1, the distance between the corresponding two points of $\overline{K}_\xi$ as given by H_θ is less than ϵ_1. Let m be the smallest integer greater than $n^{1/2}/\delta_1$ and let $t=1/m$. Then $t<\delta_1/n^{1/2}$.

Consider the $(n-1)$-dimensional planes in E^n whose equations are

$$x_i = yt, \quad i = 1, 2, \cdots, n; \; y = 1, 2, \cdots, m-1.$$

These $(n-1)$-dimensional planes are $n(m-1)$ in number and separate R into m^n n-cells R_j such that $\overline{R}=\sum_{j=1}^{m^n}\overline{R}_j$. Let $H_\theta(\overline{R}_j)=\overline{K}_{\xi j}$, $j=1, 2, \cdots, m^n$. If

* The symbol $d(M)$ is used to denote the diameter of the set M.

two points belong to a set $\overline{R}_j$, the difference between the x_i-coordinates ($i=1, 2, \cdots, n$) of these two points is at most t. Since $t<\delta_1/n^{1/2}$, the distance between these two points is less than δ_1. Then $d(\overline{K}_{\xi j})<\epsilon_1, j=1, 2, \cdots, m^n$. Corresponding to K_ξ is the set D_ξ which belongs to N_α. By successive applications of an argument used above, it can be shown that $\bar{D}_\xi=\sum_{j=1}^{m^n}\bar{D}_{\xi j}$, where $D_{\xi j}$ is a descendant of D_ξ, and there exists a homeomorphism, $H_\xi(\sum_{j=1}^{m^n}\lambda[D_{\xi j}])=\sum_{j=1}^{m^n}\lambda[K_{\xi j}]$, such that $H_\xi(\lambda[D_{\xi j}])=\lambda[K_{\xi j}]$ and $H_\xi(\lambda[D_\xi])=H(\lambda[D_\xi])=\lambda[K_\xi]$.

Suppose that this procedure is followed in the case of every set K_j whose diameter is not less than ϵ_1. The final result is that $\overline{N}_\alpha$ and $\overline{K}$ can be expressed as follows:

$$\overline{N}_\alpha = \sum_{j=1}^{r_1}\overline{D_j^{(1)}},$$

$$\overline{K} = \sum_{j=1}^{r_1}\overline{K_j^{(1)}}.$$

$d(K_j^{(1)})<\epsilon_1$, $D_j^{(1)}$ is strongly homeomorphic with N_α, $K_j^{(1)}$ is an n-cell and $\sum_{j=1}^{r}\lambda[D_j]\subset\sum_{j=1}^{r_1}\lambda[D_j^{(1)}]$. There exists a homeomorphism, $H_{\psi_1}(\sum_{j=1}^{r_1}\lambda[D_j^{(1)}])=\sum_{j=1}^{r_1}\lambda[K_j^{(1)}]$, such that $H_{\psi_1}(\lambda[D_j^{(1)}])=\lambda[K_j^{(1)}]$, $j=1, 2, \cdots, r_1$, and $H_{\psi_1}(\lambda[D_j])=H(\lambda[D_j]), j=1, 2, \cdots, r$.

Assign to each point p of $\overline{N}_\alpha$ the neighborhood N_{b_p} such that N_{b_p} is the neighborhood of $G(p)$ (§5.1) of least subscript greater than 2 having the following properties: $\overline{N}_{b_p}$ belongs to every neighborhood of T_1 which contains p; if N_ξ belongs to T_1 and $\overline{N}_\xi \not\ni p$, then $\overline{N}_{b_p}\cdot\overline{N}_\xi=0$; $N_{b_p}\not\ni D_j^{(1)}$ for all values of j. There exists a finite subset of the set of all such neighborhoods: $T_2=\{N_{\phi_j}\}, j=1, 2, \cdots, h$, such that $\sum_{j=1}^{h}N_{\phi_j}\supset\overline{N}_\alpha$ and $N_{\phi_s}\not\ni N_{\phi_t}$, $s\neq t$.

Let $D_\mu^{(1)}$ be any one of the sets $D_j^{(1)}$. Suppose that $N_{\phi_m}\cdot D_\mu^{(1)}\neq 0$. Then $N_{\phi_m}\cdot\overline{D_\mu^{(1)}}$ is an open set with respect to $\overline{D_\mu^{(1)}}$. Since $D_\mu^{(1)}$ is strongly homeomorphic with N_α, Lemma 8 is true if $D_\mu^{(1)}$ and the sets $N_{\phi_m}\cdot\overline{D_\mu^{(1)}}$ are substituted for N_α and the sets P_k respectively. Then $D_\mu^{(1)}=\sum_{i=1}^{e}\overline{D_{\mu i}^{(1)}}$, and the sets $D_{\mu i}^{(1)}$ and $\lambda[D_{\mu i}^{(1)}]$ have the properties and the relations among themselves ascribed to the corresponding sets in Lemma 8. Corresponding to $D_\mu^{(1)}$ is the set $K_\mu^{(1)}$. Then $\overline{K_\mu^{(1)}}=\sum_{i=1}^{e}\overline{K_{\mu i}^{(1)}}$. $K_{\mu i}^{(1)}$ is an n-cell and there exists a homeomorphism, $H_{\rho_1}(\sum_{i=1}^{e}\lambda[D_{\mu i}^{(1)}])=\sum_{i=1}^{e}\lambda[K_{\mu i}^{(1)}]$, such that $H_{\rho_1}(\lambda[D_{\mu i}^{(1)}])=\lambda[K_{\mu i}^{(1)}]$, $i=1, 2, \cdots, e$, and $H_{\rho_1}(\lambda[D_\mu^{(1)}])=\lambda[K_\mu^{(1)}]$. $H_{\psi_1}H_{\rho_1}^{-1}(\lambda[K_\mu^{(1)}])=\lambda[K_\mu^{(1)}]$. Let H_{ρ_2} denote the transformation $H_{\psi_1}H_{\rho_1}^{-1}$. Then by Corollary, Theorem P_1 there is a homeomorphism, $H_{\rho_3}(\overline{K_\mu^{(1)}})=\overline{K_\mu^{(1)}}$, such that $H_{\rho_3}(\lambda[K_\mu^{(1)}])=H_{\rho_2}(\lambda[K_\mu^{(1)}])$. Let $H_{\rho_3}(\overline{K_{\mu i}^{(1)}})=\overline{K_{\mu i}^{(1,1)}}$. $H_{\rho_3}(\lambda[K_{\mu i}^{(1)}])=\lambda[K_{\mu i}^{(1,1)}]$. If H_δ is used to denote the transformation $H_{\rho_3}H_{\rho_1}$,

$$H_\delta\left(\sum_{i=1}^{e}\lambda[D_{\mu i}^{(1)}]\right) = H_{\rho_2}H_{\rho_1}\left(\sum_{i=1}^{e}\lambda[D_{\mu i}^{(1)}]\right) = H_{\rho_2}\left(\sum_{i=1}^{e}\lambda[K_{\mu i}^{(1)}]\right)$$
$$= \sum_{i=1}^{e}\lambda[K_{\mu i}^{(1,1)}],$$
$$H_\delta(\lambda[D_{\mu i}^{(1)}]) = \lambda[K_{\mu i}^{(1,1)}],$$
$$H_\delta(\lambda[D_{\mu}^{(1)}]) = H_{\rho_2}H_{\rho_1}(\lambda[D_{\mu}^{(1)}]) = H_{\rho_3}(\lambda[K_{\mu}^{(1)}]) = H_{\rho_2}(\lambda[K_{\mu}^{(1)}])$$
$$= H_{\psi_1}H_{\rho_1}^{-1}(\lambda[K_{\mu}^{(1)}]) = H_{\psi_1}(\lambda[D_{\mu}^{(1)}]).$$

Similar results can be obtained for any two corresponding sets $D_j^{(1)}$ and $K_j^{(1)}$.

By the process described above, we can obtain the following

$$\overline{N}_\alpha = \sum_{j=1}^{r_2}\overline{D_j^{(2)}}, \qquad \overline{K} = \sum_{j=1}^{r_2}\overline{K_j^{(2)}},$$

$D_j^{(2)}$ is strongly homeomorphic with N_α, $K_j^{(2)}$ is an n-cell, and $d(K_j^{(2)})<\epsilon_2$, $j=1, 2, \cdots, r_2$. $\sum_{j=1}^{r_1}\lambda[D_j^{(1)}]\subset\sum_{j=1}^{r_2}\lambda[D_j^{(2)}]$ and $\sum_{j=1}^{r_1}\lambda[K_j^{(1)}]\subset\sum_{j=1}^{r_2}\lambda[K_j^{(2)}]$. There exists a homeomorphism, $H_{\psi_2}(\sum_{j=1}^{r_2}\lambda[D_j^{(2)}])=\sum_{j=1}^{r_2}\lambda[K_j^{(2)}]$, such that $H_{\psi_2}(\lambda[D_j^{(2)}])=\lambda[K_j^{(2)}]$ and $H_{\psi_2}(\sum_{j=1}^{r_1}\lambda[D_j^{(1)}])=H_{\psi_1}(\sum_{j=1}^{r_1}\lambda[D_j^{(1)}])$.

Continuing the process, we can obtain the sets $F_1=\sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\lambda[D_j^{(k)}]$ and $F_2=\sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\lambda[K_j^{(k)}]$, such that

$$\overline{N}_\alpha = \sum_{j=1}^{r_k}\overline{D_j^{(k)}}, \qquad \overline{K} = \sum_{j=1}^{r_k}\overline{K_j^{(k)}},$$

$D_j^{(k)}$ is strongly homeomorphic with N_α, $K_j^{(k)}$ is an n-cell, $d(K_j^{(k)})<\epsilon_k$, $\sum_{j=1}^{r_{k-1}}\lambda[D_j^{(k-1)}]\subset\sum_{j=1}^{r_k}\lambda[D_j^{(k)}]$, $\sum_{j=1}^{r_{k-1}}\lambda[K_j^{(k-1)}]\subset\sum_{j=1}^{r_k}\lambda[K_j^{(k)}]$ and, for every value of k, there is a homeomorphism, $H_{\psi_k}(\sum_{j=1}^{r_k}\lambda[D_j^{(k)}])=\sum_{j=1}^{r_k}\lambda[K_j^{(k)}]$, such that $H_{\psi_k}(\lambda[D_j^{(k)}])=\lambda[K_j^{(k)}]$ and $H_{\psi_k}(\sum_{j=1}^{r_{k-1}}\lambda[D_j^{(k-1)}])=H_{\psi_{k-1}}(\sum_{j=1}^{r_{k-1}}\lambda[D_j^{(k-1)}])$.

Corresponding to T_2, the set of neighborhoods used in obtaining the sets $D_j^{(2)}$, there is a set of neighborhoods T_k employed in a similar manner to obtain the sets $D_j^{(k)}$. The neighborhoods of T_k are determined as follows: Assign to each point p of $\overline{N}_\alpha$ the neighborhood N_{c_p} such that N_{c_p} is the neighborhood of $G(p)$ of least subscript greater than k having the following properties: $\overline{N}_{c_p}$ belongs to every neighborhood of T_j, $j=1, 2, \cdots, k-1$, which contains p; if N_η belongs to T_j, $j=1, 2, \cdots, k-1$, and $N_\eta \not\supset p$, then $\overline{N}_{c_p}\cdot\overline{N}_\eta=0$; $N_{c_p}\not\supset D_j^{(k-1)}$ for all values of j. There exists a finite subset of the set of all such neighborhoods: $T_k=\{N_{\kappa_j}\}$, $j=1, 2, \cdots, g$, such that $\sum_{j=1}^{g}N_{\kappa_j}\supset\overline{N}_\alpha$ and $N_{\kappa_a}\not\supset N_{\kappa_b}$, $a\neq b$.

We can now conclude without difficulty that, if p is a point of $\overline{N}_\alpha$, there exists a sequence of neighborhoods $\{N_{\delta_j}\}$, $j=1, 2, \cdots$, such that $N_{\delta_j}\supset p$,

$N_{\delta_a} \neq N_{\delta_b}$, $a \neq b$, N_{δ_j} belongs to T_{k_j}, $k_j < k_{j+1}$, and $N_{\delta_{j+1}} \subset N_{\delta_j}$. By Lemma 1, $\prod_{j=1}^{\infty} \overline{N}_{\delta_j} = p$.

Every neighborhood of T_k which contains a given point p of $\overline{N}_\alpha$ contains all the sets $D_j^{(k)}$ for which $\overline{D_j^{(k)}} \supset p$. If the sets $D_{\mu_k}^{(k)}$, $k=1, 2, \cdots$, are such that $D_{\mu_k}^{(k)} \subset D_{\mu_{k-1}}^{k-1}$, $k > 1$, then $\prod_{k=1}^{\infty} \overline{D_{\mu_k}^{(k)}}$ contains a point p, since $\overline{D_{\mu_k}^{(k)}}$ is a compact set. Then, by means of the result stated in the preceding paragraph, we can infer that $\prod_{k=1}^{\infty} \overline{D_{\mu_k}^{(k)}} = p$.

Let p be a point of $\overline{N}_\alpha - F_1$ and $D_{\mu_k}^{(k)}$, the set $D_j^{(k)}$ which contains p. It is evident that $p = \prod_{k=1}^{\infty} D_{\mu_k}^{(k)} = \prod_{k=1}^{\infty} \overline{D_{\mu_k}^{(k)}}$. Let $K_{\mu_k}^{(k)}$ be the set which corresponds to $D_{\mu_k}^{(k)}$. $\prod_{k=1}^{\infty} \overline{K}_{\mu_k}^{(k)}$ consists of one point p'. The point p' cannot belong to F_2. For, if $F_2 \supset p'$, the point p' appears in F_2 for the first time in the set $\lambda[K_{\mu_m}^{(m)}]$, $m \geqq 1$. Then $p' = \prod_{k=m}^{\infty} \lambda[K_{\mu_k}^{(k)}]$. Under the circumstances, however, $\prod_{k=m}^{\infty} \lambda[D_{\mu_k}^{(k)}]$ would be non-vacuous. This is impossible, since $\prod_{k=m}^{\infty} \lambda[D_{\mu_k}^{(k)}] = 0$. Then $\overline{K} - F_2 \supset p'$.

Make every point p of $\overline{N}_\alpha - F_1$ to correspond to the point p' of $\overline{K} - F_2$ obtained in the manner just described. The process used in obtaining the sets F_1 and F_2 establishes a (1, 1) correspondence between the points of F_1 and the points of F_2. We now have a (1, 1) correspondence between the points of $\overline{N}_\alpha$ and the points of $\overline{K}$. This correspondence is continuous both ways. Then $\overline{N}_\alpha$ is homeomorphic with $\overline{K}$. Therefore Z^n, which is homeomorphic with $\overline{N}_\alpha$ is a closed n-cell and I^n is an n-cell.

6. **Proof that the conditions in Theorem I′ are necessary.** The proof which follows is applicable when $n > 1$. The slight modifications necessary when $n = 1$ are obvious.

In euclidean n-space let K be the set of all points whose coordinates x_j satisfy the condition $0 < x_j < 1$, $j = 1, 2, \cdots, n$. K is an n-cell and $\lambda[K]$ is an $(n-1)$-sphere. Let $N_k^{(9)}$ be the irrational number $(.99 \cdots 9)^{1/2}$ in which the symbol 9 occurs k times and $N_k^{(8)}$, the irrational number similarly defined in which the symbol 8 takes the place of 9. The neighborhoods N_i to be defined belong to certain sets L_i, $i = 1, 2, 3, \cdots$.

The neighborhoods belonging to the set L_1 are obtained as follows: Denote by e_1 the number $1 - N_1^{(9)}$. Let h_1 be the smallest integer greater than $1/e_1$ and m_1, the number $1/h_1$. Then $m_1 < e_1$. The $(n-1)$-dimensional planes $x_j = tm_1$ $(j = 1, 2, \cdots, n;\ t = 1, 2, \cdots, h_1 - 1)$ separate K into h_1^n sets K_{y_1} such that $\overline{K} = \sum_{y_1=1}^{h_1^n} \overline{K}_{y_1}$. Each set K_{y_1} is the interior of an $(n-1)$-dimensional cube whose edges are equal in length to m_1. Denote by r_{y_1} the smallest positive integer such that $N_{y_1}^{(8)}/r_{y_1} < m_1/2$, and by g_{y_1} the number $N_{y_1}^{(8)}/r_{y_1}$. Let the point $(x_1^{y_1}, x_2^{y_1}, \cdots, x_n^{y_1})$ be the point of $\overline{K}_{y_1}$ which is equidistant from the vertices of $\overline{K}_{y_1}$. Denote by M_{y_1} the set whose points have coordinates

x_j which satisfy the condition $x_j^{y_1}+g_{y_1}-e_1<x_j<x_j^{y_1}+g_{y_1}+e_1$. Then $\overline{K}_{y_1}\subset M_{y_1}$. The neighborhoods of L_1 are the sets $\overline{K}\cdot M_{y_1}$, $y_1=1, 2, \cdots, h_1^n$.

The neighborhoods belonging to the set L_2 are obtained as follows: Let E_{21}^{n-1} be an $(n-1)$-dimensional plane containing an $(n-1)$-dimensional face of $\overline{K}$ or of a set $\overline{M}_{y_1}$ and E_{22}^{n-1}, an $(n-1)$-dimensional plane containing an $(n-1)$-dimensional face of a set $\overline{M}_{y_1}$ such that $E_{21}^{n-1}\cdot E_{22}^{n-1}=0$. Designate by δ_2 the lower bound of the set of numbers which give the distances between all such pairs of $(n-1)$-dimensional planes. Denote by e_2 the number $1-N_{q_2}^{(9)}$, where q_2 is the smallest integer such that $6e_2<\delta_2$. Define the numbers h_2, m_2, r_{y_2} and g_{y_2} by the method used in obtaining the corresponding numbers in the preceding paragraph and obtain the sets K_{y_2} and M_{y_2}, $y_2=1, 2, \cdots, h_2^n$, which correspond to the sets K_{y_1} and M_{y_1}. The neighborhoods which belong to L_2 are the sets $\overline{K}\cdot M_{y_2}$, $y_2=1, 2, \cdots, h_2^n$.

In general, the neighborhoods belonging to the set L_i are obtained as follows:

Let E_{i1}^{n-1} be an $(n-1)$-dimensional plane containing an $(n-1)$-dimensional face of $\overline{K}$ or of a set $\overline{M}_{y_j}$, $j=1, 2, \cdots, i-1$, and E_{i2}^{n-1}, an $(n-1)$-dimensional plane containing an $(n-1)$-dimensional face of a set $\overline{M}_{yk}$, $k=1, 2, \cdots, i-1$, such that $E_{i1}^{n-1}\cdot E_{i2}^{n-1}=0$. Denote by δ_i the lower bound of the set of numbers which give the distances between all such pairs of $(n-1)$-dimensional planes. Let e_i be the number $1-N_{q_i}^{(9)}$, where q_i is the smallest integer such that $6e_i<\delta_i$. Define the numbers h_i, m_i, r_{y_i}, and g_{y_i} by the method used in obtaining the corresponding numbers in the preceding paragraph and obtain the sets K_{y_i} and M_{y_i}. The neighborhoods belonging to L_i are the sets $\overline{K}\cdot M_{y_i}$, $y_i=1, 2, \cdots, h_i^n$.

If a is a given value of i, no $(n-1)$-dimensional face of one set $\overline{M}_{y_a}$ is in the same $(n-1)$-dimensional plane with an $(n-1)$-dimensional face of a second set $\overline{M}_{y_a}$. No $(n-1)$-dimensional face of a set $\overline{M}_{y_b}$ is in the same $(n-1)$-dimensional plane with an $(n-1)$-dimensional face of a set $\overline{M}_{y_c}$, $b\neq c$.

When $\overline{K}$ and K are substituted for Z^n and I^n respectively in Theorem I′, all of the conditions of the theorem are satisfied.

7. **Proof of Theorem I.** We can conclude from the result obtained in (5.9) that, *if Z^{n-1} is a space such that I^{n-1} is an $(n-1)$-cell, Z^n is a closed n-cell and I^n is an n-cell.* By Theorem I (4.4), Z^0 is a closed 0-cell and I^0 is a 0-cell. The proof by induction that the conditions in Theorem I are sufficient follows immediately. The proof that the conditions are necessary is found in §6.

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