ANALYTICITY OF EQUILIBRIUM FIGURES OF ROTATION*

BY

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INTRODUCTION

The problem of ascertaining the possible forms of relative equilibrium of a homogeneous gravitating mass of liquid, when rotating about a fixed axis with constant angular velocity, had its origin in the investigations on the theory of the earth's figure which began with Newton and MacLaurin. In recent times it has undergone much development especially at the hands of Poincaré, Liapounoff, and Lichtenstein.

We take the axis of rotation as the axis of z and the mass-center, which must evidently lie on the axis, as origin. If ω be the angular velocity of rotation, the component accelerations at (x, y, z) are $-\omega^2 x$, $-\omega^2 y$, $-\omega^2 z$ and the dynamical equations reduce to

$$-\omega^2 x = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial \Omega}{\partial x}, \qquad -\omega^2 y = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial \Omega}{\partial y}, \qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\partial \Omega}{\partial z},$$

where Ω is the potential energy per unit mass, p the pressure, and ρ the density. Hence, integrating, we have

$$p/\rho = \tfrac{1}{2}\omega^2(x^2 + y^2) - \Omega + \text{const.}$$

At the free surface, $p = \text{constant}$ and we have

$$\tfrac{1}{2}\omega^2(x^2 + y^2) - \int_R \frac{dV_Q}{\overline{PQ}} = \text{const.}, \tag{1}$$

where R is the region containing the rotating mass.

Liapounoff and Lichtenstein† have proved that, at all points where the apparent gravity is not zero, the surface possesses continuous derivatives of all orders but the problem of the analyticity has so far defied solution. This problem is equivalent in difficulty to that of the analyticity of the solutions of elliptic differential equations of the second order which was solved by E. Hopf.‡

* Presented to the Society, September 10, 1937; received by the editors July 6, 1936, and in revised form, January 11, 1937.

† Lichtenstein, *Gleichgewichtsfiguren rotierender Flussigkeiten.*

‡ Mathematische Zeitschrift, vol. 34 (1931), p. 194.

In this article I use the method developed by E. Hopf in the above paper to prove that the surface of equilibrium figures of rotation is analytic at all points where the apparent gravity exists, that is the gradient of the pressure is not zero.

The equation of the surface of revolution is given implicitly by equation (1). By a few simple transformations we generalize (1) so that it will have a meaning for complex values of x and y and then we differentiate partially with respect to x and y obtaining equations (10) of Part I.

Knowing that a solution exists for real values of x and y we set up a sequence of approximating non-monogenic functions (cf. equations (14), (15) of Part II) which reduce for x and y real to the known solution. We prove the sequence converges uniformly and that the limit is an analytic function.

I wish to express my thanks and gratitude to Professor E. Hopf who proposed the problem and without whose assistance and encouragement it would not have been solved.

I. Formulation of the Problem

We wish to prove that if R is any 3-dimensional region, B its boundary, which satisfies the following equation:

$$\int_R \frac{dV_Q}{\overline{PQ}} - F(P) = 0 \quad \text{for all } P \text{ on } B, \tag{1}$$

where $F(P)$ is an analytic function of P, dV_Q the element of volume and $\overline{PQ}$ the distance from P to Q, then the surface formed by B is analytic at all points P' where the gradient of (1) is not zero. It will be assumed that surfaces satisfying this equation possess a sufficient number of derivatives. This has been proved by Lichtenstein.*

Take any such point P' as $(0, 0, z_0)$, $z_0>0$, and let the z-axis be normal to B at P'. Let the equation of the surface be $z=z(x, y)$. Then since $z(x, y)$ has partial derivatives of all orders we have $z_x'(0, 0)=z_y'(0, 0)=0$.

Since the gradient of (1) is not zero

$$\frac{\partial}{\partial z}\left[\int_R \frac{dV_Q}{\overline{P'Q}} - F(P')\right] \neq 0, \qquad P' = (0, 0, z_0).$$

Because of the continuity of the surface there then exist positive numbers r, r_2 such that for $x^2+y^2<r_1^2$ we have $z(x, y)>0$, $|z(x, y)-z_0|<r_2$ and

$$\frac{\partial}{\partial z}\left[\int_R \frac{dV_Q}{\overline{PQ}} - F(P)\right] > 4d \quad \text{for} \quad P = (x, y, z), \tag{2}$$

* Loc. cit.

where d is some positive constant.

Let $a<r_1$. Denote the semi-cylinder $\xi^2+\eta^2<a^2, \zeta>0$ by (a). Then (1) can be written as

$$\int_{(a)\cdot R} \frac{dV_Q}{\overline{PQ}} + \int_{R-(a)\cdot R} \frac{dV_Q}{\overline{PQ}} - F(P) = 0, \qquad Q = (\xi, \eta, \zeta). \tag{3}$$

The second integral, call it $G_a(x, y, z)$, is the potential at P of the region $R-(a)\cdot R$. $G_a(x, y, z)$ is known to be an analytic function of x, y, z as long as P is not in $R-(a)\cdot R$ that is if $x^2+y^2<a^2$, $|z-z_0|<r_2$.

Consider*

$$\begin{aligned}\frac{\partial}{\partial z}\int_{(a)\cdot R} \frac{dV_Q}{\overline{PQ}}\bigg|_{z=z(x,y)} &= \frac{\partial}{\partial z}\iint_a d\xi d\eta \int_0^z \frac{d\zeta}{[(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2]^{1/2}}\bigg|_{z=z(x,y)} \\ &= \iint_a \frac{d\xi d\eta}{[\rho^2+z(x,y)^2]^{1/2}},\end{aligned}$$

where a denotes the circle $\xi^2+\eta^2<a^2$, $\rho^2=(\xi-x)^2+(\eta-y)^2$ and so

$$\left|\frac{\partial}{\partial z}\int_{(a)\cdot R} \frac{dV_Q}{\overline{PQ}}\right| < 2\iint_a \frac{d\xi d\eta}{\rho}.$$

By Schmidt's inequality,† we have, however

$$\iint_a \frac{d\xi d\eta}{\rho} \leqq 2\pi\left(\frac{\pi a^2}{\pi}\right)^{1/2} = 2\pi a, \tag{4}$$

so that

$$\left|\frac{\partial}{\partial z}\int_{(a)\cdot R} \frac{dV_Q}{\overline{PQ}}\right| < 4\pi a.$$

Taking $4\pi a<2d$ and using (2) and (3) we have

$$\left|\frac{\partial}{\partial z}(G_a - F)\right| > 2d \quad \text{for} \quad x^2+y^2<a^2, \qquad |z-z_0|<r_2.$$

We can now write (1) as follows:

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* Note that z means the third independent coordinate of the set (x, y, z) but $z(x, y)$ refers to the equation of the surface as a function of x, y.

† Sternberg, *Potentialtheorie*, Sammlung Göschen, p. 99, equation (3).

$$\iint_a d\xi d\eta \int_0^{z(\xi,\eta)} \frac{d\zeta}{[\rho^2 + (\zeta - z(x, y))^2]^{1/2}} + G_a - F = 0. \tag{5}$$

The integral can be split up into

$$\iint_a d\xi d\eta \left[\int_0^{z(x,y)} + \int_{z(x,y)}^{z(\xi,\eta)} \right] \frac{d\zeta}{[\rho^2 + (\zeta - z(x, y))^2]^{1/2}}.$$

After we make the change of variable $\zeta' = z(x, y) - \zeta$ and integrate, it becomes

$$\iint_a d\xi d\eta \left[f\left(\frac{z(x, y)}{\rho}\right) + f\left(\frac{z(x, y) - z(\xi, \eta)}{\rho}\right)\right],$$

where $f(u) = \log (u + (u^2+1)^{1/2})$ is regular for $u = 0$.

Call the integral of the first term $g_a(x, y, z)$, so that

$$g_a(x, y, z) = \iint_a [\log [z + (z^2 + \rho^2)^{1/2}] - \log \rho] d\xi d\eta.$$

$g_a(x, y, z)$ is an analytic function of x, y, z for $|z - z_0| < r_2$, $x^2 + y^2 < a^2$ because $z + (z^2 + \rho^2)^{1/2} > 0$ for all ξ, η and

$$\iint_a \log \rho \, d\xi d\eta = -\frac{\pi}{3}(x^2 + y^2) + \frac{a^2}{2}(\log a - \tfrac{1}{2}).$$

Now by (4)

$$\frac{\partial}{\partial z} g_a(x, y, z) = \iint_a \frac{d\xi d\eta}{(z^2 + \rho^2)^{1/2}} \leqq \iint_a \frac{d\xi d\eta}{\rho} \leqq 2\pi a,$$

so that

$$\left| \frac{\partial}{\partial z} g_a(x, y, z) \right| < d.$$

Call $g_a + G_a - F = H_a(x, y, z)$. Then H_a is an analytic function of x, y, z for $x^2 + y^2 < a^2$, $|z - z_0| < r_2$ and

$$\left| \frac{\partial H_a}{\partial z}(x, y, z) \right| > d. \tag{6}$$

Equation (5) can now be written

$$\iint_a f\left(\frac{z(x, y) - z(\xi, \eta)}{\rho}\right) d\xi d\eta + H_a(x, y, z) = 0. \tag{7}$$

To put (7) in a more easily handled form we differentiate it with respect to x and y. We have

$$
\begin{aligned}
z_x'(x, y)\frac{\partial H_a}{\partial z}(x, y, z(x, y)) + \iint_a f'\left(\frac{z(x, y) - z(\xi, \eta)}{\rho}\right)\Bigg[\frac{z_x'(x, y)}{\rho} \\
- \frac{x - \xi}{\rho^3}(z(x, y) - z(\xi, \eta))\Bigg] d\xi d\eta = -\frac{\partial H_a}{\partial x}
\end{aligned} \tag{8}
$$

and a similar equation for z_y'.

Let

$$
\begin{gathered}
\frac{\partial H_a}{\partial z} = L_a, \qquad \frac{\partial H_a}{\partial x} = M_a, \qquad \frac{\partial H_a}{\partial y} = N_a, \\
z_x'(x, y) = z_1(x, y), \qquad z_y'(x, y) = z_2(x, y).
\end{gathered} \tag{9}
$$

Call the integral in (8) $F_1(x, y)$ and the corresponding integral in the equation for z_y', $F_2(x, y)$.

Then our equations become, omitting for convenience the subscript a,

$$
\begin{aligned}
z_1(x, y)L(x, y, z(x, y)) + F_1(x, y) &= -M(x, y, z(x, y)), \\
z_2(x, y)L(x, y, z(x, y)) + F_2(x, y) &= -N(x, y, z(x, y)),
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&z(x, y) - z(a\cos\phi, a\sin\phi) = \int_{(a\cos\phi, a\sin\phi)}^{(x,y)} z_1(x', y')dx' + z_2(x', y')dy', \\
&z_1(0, 0) = z_2(0, 0) = 0.
\end{aligned} \tag{11}
$$

We wish now to consider (10) for complex values of x and y. To do this we shall extend the meaning of our integrals so that it will take account of complex values of x and y. Then let $z^{(0)}(x, y)$, $z_1^{(0)}(x, y)$, $z_2^{(0)}(x, y)$ be any continuous functions of the complex variables x, y which reduce for real x and real y to $z(x, y)$, $z_1(x, y)$, $z_2(x, y)$ the solutions of (10). We then set up by the method of successive approximations $Z(x, y)$, $Z_1(x, y)$, $Z_2(x, y)$ which satisfy the extended form of (10) also for complex values of x, y and reduce, for real x and real y, to $z(x, y)$, $z_1(x, y)$, $z_2(x, y)$.

Let $x = x' + ix''$, $y = y' + iy''$. We shall restrict x and y to the region R_γ, where R_γ is defined as follows:

$$
x'^2 + y'^2 < a^2, \qquad (x''^2 + y''^2)^{1/2} < \gamma[a - (x'^2 + y'^2)^{1/2}]. \tag{12}
$$

Note that R_γ is convex, for if (x_1, y_1) and (x_2, y_2) are in R_γ so is $(x, y) = t(x_1, y_1) + (1-t)(x_2, y_2)$, since

$$
\begin{aligned}
(x''^2 + y''^2)^{1/2} + \gamma(x'^2 + y'^2)^{1/2} &\leqq t(x_1''^2 + y_1''^2)^{1/2} + (1 - t)(x_2''^2 + y_2''^2)^{1/2} \\
&\quad + \gamma t(x_1'^2 + y_1'^2)^{1/2} + (1 - t)\gamma(x_2'^2 + y_2'^2)^{1/2} \\
&\leqq \gamma ta + \gamma(1 - t)a = \gamma a.
\end{aligned}
$$

We now extend the meaning of the integrals in (10). In

$$(13)\qquad Z(x, y) - Z(a \cos \phi, a \sin \phi) = \int Z_1(x', y')dx' + Z(x', y')dy'$$

let $x' = \tau'(x - a \cos \phi) + a \cos \phi$, $y' = \tau'(y - a \sin \phi) + a \sin \phi$, $0 \leqq \tau' < 1$. Then

$$(14)\quad Z(x, y) - Z(a \cos \phi, a \sin \phi) = \int_0^1 [(x - a \cos \phi)Z_1 + (y - a \sin \phi)Z_2]d\tau'$$

which has a meaning for complex x and y. In

$$Z(x, y) - Z(\xi, \eta) = \int_{\xi,\eta}^{x,y} Z_1(\alpha, \beta)d\alpha + Z_2(\alpha, \beta)d\beta$$

let

$$\alpha = \xi + \tau(x - \xi), \qquad \beta = \eta + \tau(y - \eta), \qquad 0 \leqq \tau \leqq 1.$$

Then

$$(15)\qquad Z(x, y) - Z(\xi, \eta) = \int_0^1 [Z_1(\alpha, \beta)(x - \xi) + Z_2(\alpha, \beta)(y - \eta)]d\tau.$$

In $F_1(x, y)$ let

$$(16)\qquad \begin{aligned} \xi &= x + t(a \cos \phi - x), \qquad 0 \leqq t < 1, \\ \eta &= y + t(a \sin \phi - y), \qquad 0 \leqq \phi < 2\pi. \end{aligned}$$

The substitution is legitimate since $\xi^2 + \eta^2 < a^2$. Then

$$\rho = t[(a \cos \phi - x)^2 + (a \sin \phi - y)^2]^{1/2} = t\Phi(\phi), \qquad \text{say}.$$

Using (15) and (16) we have

$$\begin{aligned} Z(x, y) - Z(\xi, \eta) &= t\int_0^1 [Z_1(\alpha, \beta)(a \cos \phi - x) + Z_2(\alpha, \beta)(a \sin \phi - y)]d\tau \\ &= tZ_4(x, y, t, \phi), \qquad \text{say}, \end{aligned}$$

and then

$$(17)\qquad \begin{aligned} F_1(x, y) &= \int_0^{2\pi}\int_0^1 \left[\left(\frac{z_4}{\Phi}\right)^2 + 1\right]^{-1/2} \left[\frac{z_1(x, y)}{\Phi} + \frac{z_4(a \cos \phi - x)}{\Phi^3}\right] a\,dt \\ &\qquad \cdot (a - x \cos \phi - y \sin \phi)d\phi \\ &\equiv F(x, y, z_1, z_4), \quad \text{say}. \end{aligned}$$

Similarly $F_2(x, y) \equiv F(x, y, z_2, z_4)$. Note that (14), (15), (18), and (19) have meaning for complex x and y. Our equations can now be written as follows:

$$Z_1(x, y)L(x, y, Z(x, y)) + F(x, y, Z_1(x, y), Z_4(x, y)) = -M(x, y, Z(x, y)),$$
$$Z_2(x, y)L(x, y, Z(x, y)) + F(x, y, Z_2(x, y), Z_4(x, y)) = -N(x, y, Z(x, y)),$$
$$Z(x, y) - Z(a \cos \phi, a \sin \phi)$$
$$= \int_0^1 [(x - a \cos \phi)Z_1(x', y') + (y - a \sin \phi)Z_2(x', y')]d\tau'.$$

These are three equations for the unknowns $Z_1(x, y)$, $Z_2(x, y)$, and $Z(x, y)$. Actually by substituting the value of Z from the third equation into the first two, we have two equations in the unknowns $Z_1(x, y)$ and $Z_2(x, y)$. These equations will be considered in the following region:

$$x, y, \text{ in } R_\gamma, \ |z_1|, \ |z_2| < c,$$

where c is a constant to be determined later

We shall now prove that $H(x, y, Z(x, y))$ is analytic for x, y in R_γ and also obtain bounds for its first and second derivatives when a is small. Now in $H_a = g_a + G_a - F$, g_a and F are obviously analytic in R_γ. G_a will be analytic if it can be shown that $(x-\xi)^2+(y-\eta)^2+(z(x, y)-\zeta)^2 \neq 0$, where x, y is in R_γ and ξ, η, ζ is in $R-(a)\cdot R$. This amounts to showing that

$$(x - a \cos \phi)^2 + (y - a \sin \phi)^2 + (Z(x, y) - Z(a \cos \phi, a \sin \phi))^2 \neq 0.$$

Let $x'^2+y'^2=r^2$. Now

$$|(x - a \cos \phi)^2 + (y - a \sin \phi)^2| \geqq \text{real part } [(x - a \cos \phi)^2 + (y - a \sin \phi)^2]$$
$$\geqq (x' - a \cos \phi)^2 + (y' - a \sin \phi)^2 - \gamma^2(a - r)^2$$

because of (12). Also from (14) we have

$$|Z(x, y) - Z(a \cos \phi, a \sin \phi)| \leqq c|x - a \cos \phi| + c|y - a \sin \phi|$$

and

$$|Z(x, y) - Z(a \cos \phi, a \sin \phi)|^2 \leqq c^2\{[(x' - a \cos \phi)^2 + \gamma^2(a - r)^2]^{1/2}$$
$$+ [(y' - a \sin \phi)^2 + \gamma^2(a - r)^2]^{1/2}\}^2,$$

where c is max $[|Z_1(x, y)|, |Z_2(x, y)|]$ for x, y in R_γ; so that

$$|(x - a \cos \phi)^2 + (y - a \sin \phi)^2| > |Z(x, y) - Z(a \cos \phi, a \sin \phi)|^2$$

if

$$(x' - a \cos \phi)^2 + (y' - a \sin \phi)^2 - \gamma^2(a - r)^2$$
$$> c^2\{[(x' - a \cos \phi)^2 + \gamma^2(a - r)^2]^{1/2} + [(y' - a \sin \phi)^2 + \gamma^2(a - r)^2]^{1/2}\}^2.$$

But if $\gamma < \frac{1}{3}$, then

$$\frac{(x'-a\cos\phi)^2+(y'-a\sin\phi)^2+\gamma^2(a-r)^2+\gamma^2(a-r)^2-3\gamma^2(a-r)^2}{\{[(x'-a\cos\phi)^2+\gamma^2(a-r)^2]^{1/2}+[(y'-a\sin\phi)^2+\gamma^2(a-r)^2]^{1/2}\}^2}$$

$$\geqq\frac{1}{2}-\frac{3\gamma^2(a-r)^2}{\{[(x'-a\cos\phi)^2+\gamma^2(a-r)^2]^{1/2}+[(y'-a\sin\phi)^2+\gamma^2(a-r)^2]^{1/2}\}^2}\geqq\frac{1}{6}\,.$$

Therefore if $c^2<\frac{1}{6}$, G_a will be analytic and so will H_a, L_a, M_a, and N_a.

Using the above inequality we have

$$\begin{aligned}&|(x-a\cos\phi)^2+(y-a\sin\phi)^2+[Z(x,y)-Z(a\cos\phi,a\sin\phi)]^2|\\&\geqq(\tfrac{1}{6}-c^2)\{[(x'-a\cos\phi)^2+\gamma^2(a-r)^2]^{1/2}+[(y'-a\sin\phi)^2+\gamma^2(a-r)^2]^{1/2}\}\\&\geqq(\tfrac{1}{6}-c^2)[(x'-a\cos\phi)^2+(y'-a\sin\phi)^2].\end{aligned}$$

Similarly for $a^2<\xi^2+\eta^2<r_3^2$ we have

$$|(x-\xi)^2+(y-\eta)^2|\geqq(x'-\xi)^2+(y'-\eta)^2-\gamma^2(a-r)^2$$

and

$$\begin{aligned}|Z(x,y)-Z(\xi,\eta)|&<|Z(x,y)-Z(a\cos\phi,a\sin\phi)|\\&\qquad+|Z(a\cos\phi,a\sin\phi)-Z(\xi,\eta)|\\&\leqq c[|x-a\cos\phi|+|a\cos\phi-\xi|+|y-a\sin\phi|+|a\sin\phi-\eta|]\\&\leqq 4c[|x-\xi|+|y-\eta|],\end{aligned}$$

and proceeding as above we obtain the similar equality:

$$\begin{aligned}&|(x-\xi)^2+(y-\eta)^2+(Z(x,y)-Z(\xi,\eta))^2|\\&\qquad\geqq(\tfrac{1}{6}-16c^2)[(x'-\xi)^2+(y'-\eta)^2].\end{aligned}\tag{18}$$

To obtain bounds for L, M, N, L_z', M_z', N_z' we proceed as follows: We have*

$$\frac{\partial G}{\partial x}=\iint_{B-(a)\cdot B}\frac{\cos(n,x)}{\overline{PQ}}\,d\omega_Q$$

and similar expressions for the derivatives with respect to y and z where $B-(a)B$ is the boundary of $R-(a)R$ and $d\omega_Q$ is the surface element. Since $z(\xi,\eta)$ has derivatives of all orders there exists $r_4<r_3$ such that for $\xi^2+\eta^2<r_4^2$

$$\cos(n,z)>c_1>0,$$

where n is normal to B at ξ, η, $z(\xi,\eta)$. Let B' denote the part of the surface for which $\xi^2+\eta^2>r_4^2$. Then using (18)

* Ibid., p. 124, equation 35.

$$\left|\frac{\partial G}{\partial x}\right| \leqq \left|\iint_{B'} \frac{\cos(n, x)}{\overline{PQ}}\, d\omega_Q\right| + \frac{1}{c_1}\iint_{a^2<\xi^2+\eta^2<r_4^2} (\tfrac{1}{6} - 16c^2)\frac{d\omega_Q'}{[(\xi - x')^2 + (\eta - y')^2]^{1/2}},$$

where $d\omega' = d\omega \cos(n, z)$ is the projection of $d\omega$ on the (x, y)-plane.

The integral over B' is a constant independent of a, while the second integral is by Schmidt's inequality less than $2\pi r_3$. Therefore,

$$\left|\frac{\partial G}{\partial x}\right| < k_1, \qquad x, y \text{ in } R_\gamma \tag{19}$$

and similar inequalities exist for the derivatives with respect to y and Z, where k_1 is a constant independent of a.

For the second derivatives we have

$$\frac{\partial^2 G}{\partial z \partial x} = \iint_{B-(a)\cdot B} \cos(n, z)\frac{\partial\left(\frac{1}{r}\right)}{\partial x}\, d\omega_Q$$

and similar expressions for $\partial^2 G/\partial z\partial y$, $\partial^2 G/\partial z^2$. As before

$$\left|\frac{\partial^2 G}{\partial z \partial x}\right| \leqq \left|\iint_{B'} \cos(n, z)\frac{\partial\left(\frac{1}{r}\right)}{\partial x}\, d\omega_Q\right| + \frac{1}{c_1}\iint_{a^2<\xi^2+\eta^2<r_3^2} (\tfrac{1}{6} - 16c^2)\frac{d\omega_Q'}{(\xi - x')^2 + (\eta - y')^2}$$

since

$$\left|\frac{\partial\left(\frac{1}{r}\right)}{\partial x}\right| = \left|\frac{x - \xi}{r}\cdot\frac{1}{r^2}\right| \leqq \frac{\frac{1}{6} - 16c^2}{(\xi - x')^2 + (\eta - y')^2}$$

from (18). The first integral is again a constant independent of a. The second integral is of the order of $\log[(a\cos\phi - x')^2 + (a\sin\phi - y')^2]$. Therefore

$$\left|\frac{\partial^2 G}{\partial z \partial x}\right| \leqq k_2 \log[(a\cos\phi - x')^2 + (a\sin\phi - y')^2], \tag{20}$$

where k_2 is independent of a. A similar result holds for the second derivatives with respect to yz and z^2.

Before going any further we must find a better bound for M and N. When

$a=0$, H_a reduces to the left side of (1) and since the z-axis is normal to the surface at $(0, 0, z_0)$ we have

$$L_0(0, 0, z_0) = 4d, \qquad M_0(0, 0, z_0) = 0.$$

Using (7), (19), and Schmidt's inequality

$$| L_0(x, y, Z(x, y)) - L_a(x, y, Z(x, y)) | < k_3 a,$$

with similar inequalities for M and N. Since L_a, M_a, and N_a are analytic we can therefore choose a and γ so small that

$$d/2 < \quad | L_a |, \qquad | M_a |, \qquad | N_a | \quad < \frac{d\iota}{2} \quad \text{for} \quad x, y \text{ in } R_\gamma.$$

Since F and g_a are analytic they are bounded and using (20) we shall have

$$| L_z' | < k \log [(a \cos \phi - x')^2 + (a \sin \phi - y')^2],$$

and the same bound for M_z' and N_z'.

II. Proof of Analyticity

We have to consider the equations

$$(1) \qquad \begin{aligned} &Z_1(x, y)L(x, y, Z) + F(x, y, Z_1, Z_4) = - M(x, y, Z), \\ &Z_2(x, y)L(x, y, Z) + F(x, y, Z_2, Z_4) = - N(x, y, Z), \\ &Z(x, y) - Z(a \cos \phi, a \sin \phi) = \int_0^1 [(a \cos \phi - x)Z_1 + (a \sin \phi - y)Z_2]d\tau', \end{aligned}$$

where L, M, N are analytic functions of x, y, z for x, y in R_γ, $|z_1|$, $|z_2| < c$ and satisfy the following inequalities in that region:

$$(2) \qquad \begin{aligned} &k > | L | > d; \qquad | M |, | N | < \frac{dc}{2}; \\ &| N_z' |, \qquad | M_z' |, \qquad | L_z' | < k \log [(a \cos \phi - x')^2 + (a \sin \phi - y')^2]. \end{aligned}$$

F is defined as follows:

$$(3) \qquad F(x, y, Z_1, Z_4) = \int_0^{2\pi} \int_0^1 \frac{1}{\left(\left(\frac{Z_4}{\Phi}\right)^2 + 1\right)^{1/2}} \left[Z_1(x, y) + \frac{Z_4}{\Phi} \frac{a \cos \phi - x}{\Phi}\right] a\, dt \cdot \frac{a - x \cos \phi - y \sin \phi}{\Phi} d\phi,$$

where

$$(4) \qquad \Phi^2 = (a \cos \phi - x)^2 + (a \sin \phi - y)^2,$$

and

$$(5)\quad Z_4(x, y, t, \phi) = \int_0^1 [Z_1(\alpha, \beta)(a \cos \phi - x) + Z_2(\alpha, \beta)(a \sin \phi - y)]d\tau,$$

where

$$(6)\qquad \alpha = \xi + \tau(x - \xi), \qquad \beta = \eta + \tau(y - \eta), \qquad 0 \leqq \tau \leqq 1$$

and

$$(7)\qquad \begin{aligned} \xi &= x + t(a \cos \phi - x), \qquad 0 \leqq t < 1, \\ \eta &= y + t(a \sin \phi - y), \qquad 0 \leqq \phi < 2\pi. \end{aligned}$$

If we let $x = x' + ix''$, $y = y' + iy''$ when x, y is in R_γ, that is,

$$(8)\qquad x'^2 + y'^2 < a^2, \qquad (x''^2 + y''^2)^{1/2} < \gamma[a - (x'^2 + y'^2)^{1/2}],$$

then $\Phi = 0$ if and only if $x = a \cos \phi$, $y = a \sin \phi$. For

$$\Phi^2 = [ae^{i\phi} - x - iy][ae^{-i\phi} - x + iy].$$

Now if $\Phi = 0$, one of the brackets is zero. Assume $ae^{i\phi} = x + iy$. Then

$$(9)\qquad |a| = |x + iy|$$

or

$$(10)\quad a^2 = (x' - y'')^2 + (x'' - y')^2 = (x'^2 + y'^2) + x''^2 + y''^2 + 2(x''y' - x'y'').$$

By Cauchy's inequality,

$$|x''y' - x'y''| \leqq (x'^2 + y'^2)^{1/2}(x''^2 + y''^2)^{1/2} \leqq \gamma a(x'^2 + y'^2)^{1/2} - \gamma(x'^2 + y'^2)$$

so that

$$\begin{aligned} (x' - y'')^2 + (x'' - y')^2 &\leqq x'^2 + y'^2 + \gamma^2[a - (x'^2 + y'^2)^{1/2}]^2 \\ &\quad + 2\gamma(x'^2 + y'^2)^{1/2}[a - (x'^2 + y'^2)^{1/2}] \\ &= [(x'^2 + y'^2)^{1/2} + \gamma(a - (x'^2 + y'^2)^{1/2})]^2 \\ &= [\gamma a + (1 - \gamma)(x'^2 + y'^2)^{1/2}]^2. \end{aligned}$$

Now this bracket is less than a^2 if $x'^2 + y'^2 < a^2$. So (10) can not be true and $\Phi \neq 0$. Hence for $\Phi = 0$ we must have $x'^2 + y'^2 = a^2$ which implies $x'' = y'' = 0$ and $x = a \cos \phi$, $y = a \sin \phi$. Therefore

$$(11)\qquad \left| \frac{a \cos \phi - x}{\Phi} \right|, \qquad \left| \frac{a \sin \phi - y}{\Phi} \right|$$

are bounded. By taking γ sufficiently small the upper bound of these can be made less than three so that

$$\left|\frac{a\cos\phi - x}{\Phi}\right|, \quad \left|\frac{a\sin\phi - y}{\Phi}\right| < 3 \quad \text{if} \quad x, y \text{ in } R_\gamma. \tag{12}$$

Then

$$\left|\frac{a - x\cos\phi - y\sin\phi}{\Phi}\right| = \left|\frac{\cos\phi(a\cos\phi - x) + \sin\phi(a\sin\phi - y)}{\Phi}\right| \leqq 5,$$

and calling $Z_4/\Phi = Z_5$ we have

$$\begin{aligned} |Z_5| &\leqq \left|\frac{a\cos\phi - x}{\Phi}\right| \int_0^1 |z_1|\, d\tau + \left|\frac{a\sin\phi - y}{\Phi}\right| \int_0^1 |z_2|\, d\tau \\ &\leqq 3\int_0^1 [|z_1| + |z_2|] d\tau. \end{aligned} \tag{13}$$

We now define successive approximations $z^{(\nu)}, z_1^{(\nu)}, z_2^{(\nu)}$ to $Z(x, y), Z_1(x, y), Z_2(x, y)$ as follows:

$$\begin{aligned} z_1^{(\nu+1)} &= -\frac{F(x, y, z_1^{(\nu)}, z_4^{(\nu)})}{L(x, y, z^{(\nu)})} - \frac{M(x, y, z^{(\nu)})}{L(x, y, z^{(\nu)})}, \\ z_2^{(\nu+1)} &= -\frac{F(x, y, z_2^{(\nu)}, z_4^{(\nu)})}{L(x, y, z^{(\nu)})} - \frac{N(x, y, z^{(\nu)})}{L(x, y, z^{(\nu)})}, \end{aligned} \tag{14}$$

$$z^{(\nu)}(x, y) - z(a\cos\phi, a\sin\phi) = \int_0^1 [(a\cos\phi - x)z_1^{(\nu)} + (a\sin\phi - y)z_2^{(\nu)}] d\tau', \tag{15}$$

where

$$z_4^{(\nu)} = \int_0^1 [z_1^{(\nu)}(\alpha, \beta)(a\cos\phi - x) + z_2^{(\nu)}(a\sin\phi - y)] d\tau. \tag{16}$$

The first approximations $z_1^{(0)}(x, y)$ and $z_2^{(0)}(x, y)$ are any continuous functions of the complex variables x, y which reduce, when x, y are real, to $z_1(x, y)$ and $z_2(x, y)$, the solutions of (1) in the real domain. By taking a and γ small enough we shall have

$$|z_1^{(0)}|, \quad |z_2^{(0)}| < c \quad \text{for} \quad x, y \text{ in } R_\gamma. \tag{17}$$

Then from (15) and (16) we have

$$|z^{(0)}(x, y) - z(a\cos\phi, a\sin\phi)| < c[|a\cos\phi - x| + |a\sin\phi - y|] \leqq 3ac \tag{18}$$

and $|z_5^0| < 6c$.

Assume that

$$|z_1^{(\nu)}|, \quad |z_2^{(\nu)}| < c \quad \text{for} \quad x, y \text{ in } R_\gamma. \tag{19}$$

Then from (15) and (16) we have

$$(20)\qquad \left| z^{(\nu)}(x, y) - z(a \cos \phi, a \sin \phi) \right| < 3ac,$$

$$(21)\qquad \left| z_5^{(\nu)} \right| < 6c.$$

But from (14) and (2)

$$\left| z_1^{(\nu+1)} \right| < \frac{1}{d} F(x, y, z_1^{(\nu)}, z_4^{(\nu)}) + \frac{c}{2}.$$

Now

$$|F| \leqq \int_0^{2\pi} \int_0^1 [(z_5^{(\nu)})^2 + 1]^{-1/2} \left[z_1^{(\nu)} + z_5^{(\nu)} \frac{a \cos \phi - x}{\Phi} \right] dt$$
$$\cdot a \cdot \frac{a - x \cos \phi - y \sin \phi}{\Phi} d\phi \leqq 2\pi a(1 - 36c^2)^{-1/2}(5c + 90c).$$

Take $380\pi a < d(1-36c^2)^{1/2}$ so that

$$(22)\qquad |F| < \tfrac{1}{2}cd,$$

and then $\left|z_1^{(\nu+1)}\right| < \frac{1}{2}c + \frac{1}{2}c = c$. Therefore (19) holds for all ν.

We define an operator Δ^ν as follows:

$$\Delta^\nu f(x, y, z_i) = f(x, y, z_i^{(\nu)}) - f(x, y, z_i^{(\nu-1)}) = \int_{z_i^{(\nu-1)}}^{z_i^{(\nu)}} \frac{\partial f}{\partial z_i} dz_i.$$

The integral is to be taken along the straight line joining $z_i^{(\nu-1)}$ to $z_i^{(\nu)}$. Notice that

$$(23)\quad \Delta^\nu f(x, y, z_i) g(x, y, z_j) = g(x, y, z_j^{(\nu)}) \Delta^\nu f(x, y, z_i) + f(x, y, z_i^{(\nu-1)}) \Delta^\nu g(x, y, z_j).$$

By $z_i^{(\nu)}$ we mean the νth approximation to z_i.

Let

$$\max \left[\left| \Delta^\nu z_1 \right|, \left| \Delta^\nu z_2 \right| \right] = \sigma_\nu \quad \text{for} \quad x, y \text{ in } R.$$

From (14), using (23), we have

$$(24)\quad \begin{aligned} \Delta^{\nu+1} z_1 = \Delta^\nu \frac{F}{L} + \Delta^\nu \frac{M}{L} &= \frac{\Delta^\nu F}{L(x, y, z^{(\nu)})} + F(x, y, z_1^{(\nu-1)}, z_4^{(\nu-1)}) \Delta^\nu \frac{1}{L} \\ &\quad + \frac{\Delta^\nu M}{L(x, y, z^{(\nu)})} + M(x, y, z^{(\nu-1)}) \Delta^\nu \frac{1}{L}. \end{aligned}$$

Now

$$(25)\quad \begin{aligned} \Delta^\nu F = \iint \left[\Delta^\nu z_1 (z_5^2 + 1)^{-1/2} + \Delta^\nu z_5 (z_5^2 + 1)^{-1/2} \frac{a \cos \phi - x}{\Phi} \right] a\, dt \\ \cdot \frac{a - x \cos \phi - y \sin \phi}{\Phi} d\phi. \end{aligned}$$

But

$$(26)\qquad \left|\Delta^{\nu} z_1(z_5^{\nu}+1)^{-1/2}\right| \leqq \left|z_1^{(\nu)}\Delta^{\nu}(z_5^2+1)^{-1/2}\right| + \left|\left[(z_5^{(\nu-1)})^2+1\right]^{-1/2}\Delta^{\nu} z_1\right|,$$

and using (21),

$$(27)\qquad \begin{aligned}\left|\Delta^{\nu}(z_5^2+1)^{-1/2}\right| &= \left|\int_{z_5^{(\nu-1)}}^{z_5^{(\nu)}} z_5(z_5^2+1)^{-3/2}dz_5\right| \\ &\leqq 6c(1-36c^2)^{-3/2}\left|z_5^{(\nu)}-z_5^{(\nu-1)}\right|.\end{aligned}$$

Hence in (26)

$$(28)\qquad \begin{aligned}\left|\Delta^{\nu} z_1(z_5^2+1)^{-1/2}\right| &< 6c^2(1-36c^2)^{-3/2}\left|z_5^{(\nu)}-z_5^{(\nu-1)}\right| \\ &\quad + (1-36c^2)^{-1/2}\left|\Delta^{\nu} z_1\right|.\end{aligned}$$

Considering the second term in (25), we have

$$(29)\qquad \begin{aligned}\left|\Delta^{\nu} z_5(z_5^2+1)^{-1/2}\right| &= \left|\int_{z_5^{(\nu-1)}}^{z_5^{(\nu)}} (z_5^2+1)^{-3/2}dz_5\right| \\ &\leqq (1-36c^2)^{-3/2}\left|z_5^{(\nu)}-z_5^{(\nu-1)}\right|.\end{aligned}$$

From (16) we have, using (11),

$$(30)\qquad \left|\Delta^{\nu} z_5\right| \leqq 6\sigma_{\nu}.$$

Hence in (25) using (12), (26), (28), (29), and (30), we have

$$(31)\quad \left|\Delta^{\nu} F\right| \leqq 2\pi a(1-36c^2)^{-1/2}[180c^2(1-36c^2)^{-1}+5+90]\sigma_{\nu} = ad_1\sigma_{\nu}, \quad \text{say}.$$

We also have

$$(32)\qquad \begin{aligned}\left|\Delta^{\nu} L^{-1}\right| &\leqq \left|\int_{z^{(\nu-1)}}^{z^{(\nu)}} L_z' L^{-2}dz\right| \\ &\leqq kd^{-2}\log\left[(a\cos\phi - x')^2 + (a\sin\phi - y')^2\right]\cdot\left|\Delta^{\nu} z\right|,\end{aligned}$$

$$(33)\qquad \begin{aligned}\left|\Delta^{\nu} M\right| &\leqq \left|\int_{z^{(\nu-1)}}^{z^{(\nu)}} M_z' dz\right| \\ &\leqq k\log\left[(a\cos\phi - x')^2 + (a\sin\phi - y')^2\right]\cdot\left|\Delta^{\nu} z\right|.\end{aligned}$$

But from (15)

$$(34)\qquad \left|\Delta^{\nu} z\right| < \sigma_{\nu}\left[\left|a\cos\phi - x\right| + \left|a\sin\phi - y\right|\right],$$

and, using (31), (32), (33), (34), (21), and (2) in (24), we have

$$(35)\qquad \left|\Delta^{\nu+1} z_1\right| < ad^{-1}(d_1 + \tfrac{1}{2}ck\log a + k\log a + k^2d^{-1}\log a)\sigma_{\nu}.$$

Taking a so small that

$$ad^{-1}(d_1 + \tfrac{1}{2}ck \log a + k \log a + k^2 d^{-1} \log a) < \tfrac{1}{2},$$

we have $|\Delta^{\nu+1} z_1| < \frac{1}{2}\sigma_\nu$ and a similar expression for $|\Delta^{\nu+1} z_2|$. Therefore $\sigma_{\nu+1} < \frac{1}{2}\sigma_\nu$ and σ_ν approaches zero since $\sigma_\nu < (\frac{1}{2})^\nu \sigma_0$. From (34) $|\Delta^\nu z|$ approaches zero and hence $z^{(\nu)}$, $z_1^{(\nu)}$, $z_2^{(\nu)}$ approach uniformly limit functions $Z(x, y)$, $Z_1(x, y)$, $Z_2(x, y)$.

We now wish to show that $Z(x, y)$ is an analytic function of x and y. Assume that $z_1^{(0)}(x, y)$ and $z_2^{(0)}(x, y)$ have continuous first partial derivatives with respect to x', x'' and y', y'', e.g., by assuming $Z_1^{(0)}(x, y) = z_1(x', y')$; $Z_2^{(0)}(x, y) = z_2(x', y')$. Then $z_1^{(0)}$ and $z_4^{(0)}$ will also have partial derivatives of the first order. Mathematical induction shows that the same is true for $z_1^{(\nu)}$, $z_2^{(\nu)}$, $z^{(\nu)}$, and $z_4^{(\nu)}$.

Consider the operators

$$\nabla_1 = \frac{\partial}{\partial x'} + i\frac{\partial}{\partial x''}, \qquad \nabla_2 = \frac{\partial}{\partial y'} + i\frac{\partial}{\partial y''}.$$

Applying ∇_1 to (15) we have

$$\nabla_1 z_4^{(\nu)} = \int_0^1 [\nabla_1 z_1^{(\nu)}(a \cos\phi - x) + \nabla_1 z_2^{(\nu)}(a \sin\phi - y)]d\tau. \tag{36}$$

Applying it to (14), we have

$$\nabla_1 z^{(\nu)} = \int [\nabla_1 z_1^{(\nu)}(a \cos\phi - x) + \nabla_1 z_2^{(\nu)}(a \sin\phi - y)]d\tau', \tag{37}$$

since x and y are analytic. Also

$$\begin{aligned}
\nabla_1 z_1^{(\nu+1)} &= -\frac{F(x, y, z_1^{(\nu)}, z_4^{(\nu)})}{L(x, y, z^{(\nu)})^2} L_z'(x, y, z^{(\nu)})\nabla_1 z^{(\nu)} \\
&= -\frac{1}{L}\int\int \Big\{[(z_5^{(\nu)})^2+1]^{-1/2}\nabla_1 z_1^{(\nu)} - z_1^{(\nu)} z_5^{(\nu)}[(z_5^{(\nu)})^2 + 1]^{-3/2}\nabla_1 z_5^{(\nu)} \\
&\quad - \frac{a\cos\phi - x}{\Phi}[(z_5^{(\nu)})^2 + 1]^{-3/2}\nabla_1 z_5^{(\nu)}\Big\} a\,dt \\
&\quad \cdot \frac{a - x\cos\phi - y\sin\phi}{\Phi} d\phi - L^{-2}(LM_z' - L_z' M)\nabla_1 z^{(\nu)}.
\end{aligned} \tag{38}$$

Let $\max[|\nabla_1 z^{(\nu)}|, |\nabla_1 z_2^{(\nu)}|] = \alpha_\nu$ for x, y in R_γ, then in (36) we have

$$|\nabla_1 z_5^{(\nu)}| < 6\alpha_\nu.$$

In (37) we have

$$|\nabla_1 z^{(\nu)}| < \alpha_\nu[|a\cos\phi - x| + |a\sin\phi - y|].$$

Using (21) and (2), we see that the first term on the right of (38) is less than $\frac{1}{2}ck\log[(a\cos\phi - x')^2 + (a\sin\phi - y')^2]\cdot(|a\cos\phi - x| + |a\sin\phi - y|)$. Using (21), (11), and (12), we have that the integral in (38) is less than

$$a(1 - 36c^2)^{-1/2}[5d^{-1} + 108ac^2(1 - 36c^2)^{-1} + 90]\alpha_\nu = d_2 a\alpha_\nu, \text{ say}.$$

The third term is less than $k^2d^{-2}a\log a\cdot\alpha_\nu$.

Hence in (38)

$$|\nabla_1 z_1^{(\nu+1)}| < [\tfrac{1}{2}ck\,a\log a + d_2 a + k^2d^{-2}a\log a]\alpha_\nu.$$

Choose a so that

$$\tfrac{1}{2}ck\,a\log a + d_2 a + k^2d^{-2}a\log a < \tfrac{1}{2}.$$

Then $|\nabla_1 z_1^{(\nu+1)}| < \frac{1}{2}\alpha_\nu$ and $\alpha_{\nu+1} < \frac{1}{2}\alpha_\nu < (\frac{1}{2})^{\nu+1}\alpha_0$ so that α_ν approaches zero. Since $|\nabla_1 z^{(\nu)}| < 4a\alpha_\nu$, $\nabla_1 z^{(\nu)}$ also approaches zero uniformly in x and y and therefore $\nabla_1 Z(x, y)$ exists and equals zero. This proves that $Z(x, y)$ is analytic in x. A similar proof holds for the analyticity in y.

The proof may seem irregular because we have varied our choice of a. But it can be seen that all we have required of it is that it satisfies the following inequalities:

$$380\pi a < d(1 - 36c^2)^{-1/2},$$
$$ad^{-1}(d_1 + \tfrac{1}{2}ck\log a + k\log a + k^2d^{-1}\log a) < \tfrac{1}{2},$$
$$\tfrac{1}{2}ck\,a\log a + d_2 a + k^2d^{-2}a\log a < \tfrac{1}{2},$$

which can be done since the constants c, k, and d were proved to be independent of a.

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