ON FUNCTIONS WITH BOUNDED DERIVATIVES*

BY

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1. The following well known theorem is due to A. Markoff:†

Let $f_n(x)$ be a polynomial of degree n and let M_0 be the maximum of $|f_n(x)|$ in the interval (a, b). One then has for the same interval

$$(1)\qquad |f_n'(x)| \leqq \frac{2M_0 \cdot n^2}{b-a}.$$

The equality sign can only hold for the polynomials

$$(2)\qquad f_n(x) = c \cdot T_n\left(\frac{2x-a-b}{b-a}\right),$$

where $T_n(x)$ is the nth *Tschebyschef* polynomial.

We shall show that this theorem may be formulated in such a manner that it holds for arbitrary functions with a certain number of derivatives. A polynomial of degree n is characterized by the property that its $(n+1)$st derivative vanishes identically. The theorem of Markoff may be considered as a theorem on functions having a bounded $(n+1)$st derivative in a certain interval. One also obtains bounds for all derivatives from the first to the nth. Similar results may be obtained for analytic functions bounded together with some derivative in a part of the complex plane. The proofs are simple and depend upon the polynomial character of the Taylor expansion. It should be remarked that the same extension principle may be applied to several other theorems on polynomials.

2. We shall first prove:

THEOREM 1. *Let $f(x)$ be a function for which derivatives up to the $(n+1)$st exist. Let*

$$(3)\qquad |f(x)| \leqq M_0, \qquad |f^{(n+1)}(x)| \leqq M_{n+1}$$

in the interval (a, b). Then one has in the same interval

$$(4)\qquad |f'(x)| \leqq \frac{2n^2}{b-a}\cdot\left(M_0 + M_{n+1}\cdot\frac{(b-a)^{n+1}}{(n+1)!}\right).$$

* Presented to the Society, March 26, 1937; received by the editors March 20, 1937.

† A. Markoff, *Sur une question posée par Mendeleieff*, Bulletin of the Academy of Sciences of St. Petersburg, vol. 62 (1889), pp. 1–24.

To prove Theorem 1 we apply the Taylor expansion in the following form

$$(5) \qquad f(x+h) = f(x) + h\frac{f'(x)}{1!} + \cdots + h^n\frac{f^{(n)}(x)}{n!} + R_n,$$

where*

$$(6) \qquad R_n = (-1)^n \frac{1}{n!}\int_x^{x+h} (t-x)^n \cdot f^{(n+1)}(t)dt.$$

We now suppose x fixed in the interval (a, b) and let h vary such that $x+h$ belongs to the same interval. Hence

$$(7) \qquad a - x \leqq h \leqq b - x.$$

For the remainder term (6) we then easily find

$$(8) \qquad \left| R_n \right| \leqq M_{n+1}\frac{\left| h \right|^{n+1}}{(n+1)!} \leqq M_{n+1}\cdot\frac{(b-a)^{n+1}}{(n+1)!}.$$

We next consider the polynomial

$$(9) \qquad P(h) = f(x) + h\frac{f'(x)}{1!} + \cdots + h^n\cdot\frac{f^{(n)}(x)}{n!}.$$

From (5) follows

$$P(h) = f(x+h) - R_n,$$

and from (3) and (8) in the interval (7)

$$(10) \qquad \left| P(h) \right| \leqq M_0 + \frac{(b-a)^{n+1}}{(n+1)!}\cdot M_{n+1} = K.$$

By applying Markoff's theorem to the polynomial $P(h)$ we find

$$\left| P'(h) \right| \leqq \frac{2n^2}{b-a}\cdot K$$

and, since $h=0$ belongs to the interval (7), we have

$$\left| P'(0) \right| = \left| f'(x) \right| \leqq \frac{2n^2}{b-a}\cdot K.$$

3. When $f(x)$ is a polynomial, Theorem 1 obviously reduces to the theorem of A. Markoff. One may state Theorem 1 briefly by saying that when $f(x)$ and $f^{(n+1)}(x)$ are bounded in (a, b) then $f'(x)$ has the same property. By repetition one obtains a bound for all intermediate derivatives

* Professor Hille pointed out the advantage in using this form for the remainder term.

$$f^{(i)}(x) \qquad\qquad (i = 1, 2, \cdots, n).$$

A better bound is obtained however by applying the preceding extension principle directly to a more general theorem by W. Markoff:*

Let $f_n(x)$ be a polynomial of degree n and M_0 the maximum of $|f_n(x)|$ in the interval (a, b). Then one has in the same interval

$$|f_n^{(i)}(x)| \leqq K(i, n)\cdot\frac{M_0}{(b-a)^i}, \tag{11}$$

where

$$K(i, n) = \frac{2^i\cdot n^2(n^2-1)\cdots(n^2-(i-1)^2)}{1\cdot 3\cdot 5\cdots(2i-1)} = \frac{n}{n+i}\cdot 2^{2i}\cdot i!C_{n+i,2i}. \tag{12}$$

The equality sign can only hold for the Tschebyschef polynomials (2).

When this result is applied to the derivatives of the polynomial (9) in the interval (7) we obtain:

THEOREM 2. *Let $f(x)$ possess an $(n+1)$st derivative $f^{(n+1)}(x)$ such that in the interval (a, b)*

$$|f(x)| \leqq M_0, \qquad |f^{(n+1)}(x)| \leqq M_{n+1}.$$

Then all intermediate derivatives of $f(x)$ are bounded in the same interval by

$$|f^{(i)}(x)| \leqq \frac{n}{n+i}\cdot 2^{2i}\cdot i!\cdot C_{n+i,2i}\frac{1}{(b-a)^i}\cdot\left(M_0 + M_{n+1}\cdot\frac{(b-a)^{n+1}}{(n+1)!}\right). \tag{13}$$

Let us observe that the results of W. Markoff were proved only under the assumption that the polynomial $f(x)$ has real coefficients. One may however easily extend the results to complex coefficients and hence Theorem 2 to complex valued functions.

If one introduces the notation

$$|f^{(i)}(x)| \leqq M_i, \qquad \left|\frac{f^{(i)}(x)}{i!}\cdot(b-a)^i\right| \leqq \overline{M}_i, \tag{14}$$

then

$$\overline{M}_i = \frac{M_i}{i!}(b-a)^i \tag{15}$$

and the inequality (13) may be written in the simpler form

$$\overline{M}_i \leqq \frac{n}{n+i}\cdot 2^{2i}\cdot C_{n+i,2i}\cdot(\overline{M}_0 + \overline{M}_{n+1}). \tag{16}$$

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* W. Markoff, *Über Polynome die in einem gegebenen Intervalle möglichst wenig von Null abweichen*, Mathematische Annalen, vol. 77 (1916), pp. 213–258.

This relation shows that the constants M_i or $\overline{M}_i$ for a real, differentiable function satisfy certain restricting conditions. It suggests the very interesting problem of determining the necessary and sufficient condition in order that a series of positive numbers

$$M_0, M_1, \cdots$$

be the maxima of the derivatives of a function $f(x)$ in an interval.

4. Let us next turn to the case of analytic functions. Let us suppose that $f(z)$ is analytic and regular in a certain domain D in the complex plane. Furthermore, $f^{(n+1)}(z)$ is bounded in the same domain. Let D be bounded by a Jordan curve C such that one may draw through each point in D a chord of length $d>0$ entirely contained in D. When Theorem 1 is applied to the chords of D, one finds that in D

$$|f'(z)| \leqq \frac{2n^2}{d}\cdot\left(M_0 + \frac{d^{n+1}}{(n+1)!}\cdot M_{n+1}\right).$$

This remark gives extensions of results obtained by Jackson,* Sewell,† and others. For the higher derivatives one finds corresponding to (13)

$$|f^{(i)}(z)| \leqq \frac{1}{d^i}\cdot K(i, n)\cdot K,$$

where $K(i, n)$ is given by (12).

One may however obtain considerably better results through another procedure. For polynomials in the complex plane we have the following theorem of S. Bernstein:‡

Let $f_n(z)$ be a polynomial of degree n and let M_0 denote the maximum of $f_n(z)$ on a circle with radius R. Then we have on the same circle

$$|f_n'(z)| \leqq n\cdot\frac{M_0}{R}\cdot \tag{17}$$

The equality sign holds only for

$$f_n(z) = M_0\cdot e^{i\theta}\cdot\left(\frac{z-a}{R}\right)^n. \tag{18}$$

* D. Jackson, *On the application of Markoff's theorem to problems of approximation in the complex domain*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 883–890.

† W. E. Sewell, *Generalized derivatives and approximation by polynomials*, these Transactions, vol. 41 (1937), pp. 84–123. This paper gives further references particularly to Szegö and Montel.

‡ S. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*, Paris, 1926. See also M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368.

By means of the same extension principle which we used in the preceding this theorem may be extended to arbitrary analytic functions. Let $f(z)$ be a function analytic on the circle C. It is not necessary to assume that $f(z)$ is regular in the interior of C. The results hold even when $f(z)$ is a branch of an analytic function not returning to its original value by a circuit of C.

In all cases there exists a Taylor expansion

$$f(z+h) = f(z) + \frac{h}{1!}f'(z) + \cdots + \frac{h^n}{n!}f^{(n)}(z) + R_n, \tag{19}$$

where

$$R_n = \frac{(-1)^n}{n!}\int_z^{z+h}(t-z-h)^n f^{(n+1)}(t)dt. \tag{20}$$

Here z and $z+h$ are points on C and the path of integration is taken along C in some fixed direction. Furthermore $f(z+h)$ is the value of $f(z)$ determined by the path. Let us now suppose that for the chosen branch of $f(z)$ we have

$$|f(z)| \leqq M_0, \qquad |f^{(n+1)}(z)| \leqq M_{n+1}$$

for the points of C. For the remainder term R_n in (20) one finds the estimate

$$|R_n| \leqq \frac{2}{n!}(2R)^{n+1}\cdot M_{n+1}\cdot \int_0^{\pi/2}\cos^n \phi d\phi.$$

The last integral tends to zero with increasing n. Let us use however only the rough estimate

$$|R_n| < \frac{2}{n!}(2R)^{n+1}\cdot M_{n+1}. \tag{21}$$

Now let z be a fixed point on C. Since $z+h$ is also located on C the point h describes another circle C' with the same radius. Let us write again

$$P(h) = f(z) + \frac{h}{1!}f'(z) + \cdots + \frac{h^n}{n!}f^{(n)}(z). \tag{22}$$

According to (19) and (21) we have for the points on the circle C'

$$|P(h)| \leqq M_0 + \frac{2}{n!}(2R)^{n+1}\cdot M_{n+1} = Q. \tag{23}$$

By applying the theorem of Bernstein, we obtain

$$|P'(h)| \leqq \frac{n}{R}\cdot Q$$

and, since $h=0$ is located on C',

$$|P'(0)| = |f'(z)| \leqq \frac{n}{R}\cdot Q.$$

By specialization to the unit circle this theorem may be stated as follows:

THEOREM 3. *Let $f(z)$ be a function which is analytic on the unit circle C_1. If then*

$$|f(z)| \leqq M_0, \qquad |f^{(n+1)}(z)| \leqq M_{n+1}$$

on C_1, then

$$|f'(z)| \leqq n\left(M_0 + \frac{2^{n+1}}{n!} M_{n+1}\right)$$

on the unit circle.

The generalization of this theorem to higher derivatives implies the following extension of the theorem of Bernstein:

THEOREM 4. *Let $f_n(z)$ be a polynomial of degree n and let M_0 denote its maximum on a circle with radius R. Then one has on the same circle*

$$|f_n^{(i)}(z)| \leqq n(n-1)\cdots(n-i+1)\cdot\frac{M_0}{R^i}\cdot$$

The equality sign can hold only for the polynomials (18).

To prove Theorem 4 it is only necessary to apply the theorem of Bernstein i times.

When Theorem 4 is applied to the polynomial $P(h)$ in (22) we obtain

$$|P^{(i)}(h)| \leqq \frac{n!}{(n-i)!}\cdot\frac{Q}{R^i},$$

where Q is defined by (23). For $h=0$ we find the desired result

$$|f^{(i)}(z)| \leqq \frac{n!}{(n-i)!}\cdot\frac{1}{R^i}\left(M_0 + \frac{2^{n+1}}{n!} M_{n+1}\right).$$

THEOREM 5. *Let $f(z)$ be analytic on the unit circle and*

$$|f(z)| \leqq M_0, \qquad |f^{(n+1)}(z)| \leqq M_{n+1}$$

on the circle. One then also has

$$|f^{(i)}(z)| \leqq \frac{n!}{(n-i)!}\left(M_0 + \frac{2^{n+1}}{n!} M_{n+1}\right), \qquad (i = 1, 2, \cdots, n).$$

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