AN EXTENSION OF SCHWARZ'S LEMMA*

BY

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I. The fundamental inequality

1. To every neighborhood on a Riemann surface there is given a map onto a region of the complex plane. For any two overlapping neighborhoods the corresponding maps are directly conformal.† We agree to denote points on the surface by $\mathfrak{w}$, corresponding values of the local complex parameter by w.

We introduce a Riemannian metric of the form

$$ds = \lambda \mid dw \mid, \tag{1}$$

where the positive function λ is supposed to depend on the particular parameter chosen, in such a way that ds becomes invariant. The metric is regular if λ is of class C_2. In this paper we shall, without mentioning it further, allow λ to become zero, although such points are of course singularities of the metric.

It is well known that the Gaussian curvature of the metric (1) is given by

$$K = -\lambda^{-2} \Delta \log \lambda, \tag{2}$$

and that this expression remains invariant under conformal mappings of the w-plane. We are interested in the case of a metric with negative curvature, bounded away from zero. It is convenient to choose the upper bound of the curvature equal to -4. From (2) it follows that the corresponding λ satisfies the condition

$$\Delta \log \lambda \geqq 4\lambda^2. \tag{3}$$

When we set $u = \log \lambda$ this is equivalent to

$$\Delta u \geqq 4e^{2u}. \tag{4}$$

The hyperbolic metric of the unit circle $|z| < 1$ is defined by

$$d\sigma = (1 - | z |^2)^{-1} | dz | \tag{5}$$

and has the constant curvature -4.

2. Consider now an analytic function $\mathfrak{w} = f(z)$ from the circle $|z| < 1$ to a Riemann surface W. The analyticity is expressed by the fact that every local parameter w is an analytic function of z. To a differential element dz corresponds an element dw whose length does not depend on the direction of dz. The corresponding value of $ds = \lambda |dw| = \lambda_z |dz|$ is therefore uniquely de-

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† For the definition of a Riemann surface see T. Radó, *Über den Begriff der Riemannschen Fläche*, Acta Szeged, vol. 2 (1925).

termined, and we have $\lambda_z = \lambda |w'(z)|$. It is also seen that $u = \log \lambda_z$ satisfies the condition (4) whenever the given metric has a curvature $\leqq -4$. An exception has to be made for the possible zeros of λ_z, corresponding to the zeros of λ and $w'(z)$.

THEOREM A. *If the function $\mathfrak{w} = f(z)$ is analytic in $|z| < 1$, and if the metric* (1) *of W has a negative curvature $\leqq -4$ at every point, then the inequality*

$$ds \leqq d\sigma \tag{6}$$

will hold throughout the circle.

Proof: Choose an arbitrary $R < 1$ and set $v = \log R(R^2 - |z|^2)^{-1}$ for $|z| < R$. We note that $\Delta v = 4e^{2v}$ and consequently

$$\Delta(u - v) \geqq 4(e^{2u} - e^{2v}). \tag{7}$$

Let us denote by E the open point set in $|z| < R$ for which $u > v$. It is clear that E cannot contain any zeros of λ_z. Hence (7) is valid and shows that $u - v$ is subharmonic in E. It follows that $u - v$ can have no maximum in E and must approach its least upper bound on a sequence tending to the boundary of E. But E can have no boundary points on $|z| = R$, for v becomes positively infinite as z tends to that circle, and at interior boundary points we must have $u - v = 0$, by continuity. A contradiction is thus obtained, unless E is vacuous. The inequality $u \leqq v$ consequently subsists for all points with $|z| < R$, and letting R tend to 1 we find $u \leqq -\log(1 - |z|^2)$ at all points. This is equivalent to (6).

If W is the unit circle and ds its hyperbolic metric, Theorem A is simply the differential form of Schwarz's lemma given by Pick.*

3. Several generalizations of the theorem just proved suggest themselves at once. Since the only thing we need is to prevent the function $u - v$ from having a maximum in E, it is obvious that the assumptions on λ can be considerably weakened, without affecting the validity of the argument. We shall give below two such generalizations which are found to be particularly useful for the applications.

THEOREM A1. *Let λ be continuous and such that at every point, either* (a) *the second derivatives of $u = \log \lambda$ are continuous and satisfy* (4), *or* (b) *it is possible to find two opposite directions n', n'' for which $\partial u/\partial n' + \partial u/\partial n'' > 0$. Then the statement of the previous theorem is still true.*

Opposite directions in the w-plane correspond to opposite directions in the z-plane. At a maximum of $u - v$ we have $\partial u/\partial n \leqq \partial v/\partial n$ in any direction, when-

* An account of all questions related to Schwarz's lemma will be found in R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer, 1936, pp. 45–58.

ever the directional derivative exists. For opposite directions $\partial v/\partial n' + \partial v/\partial n'' = 0$; hence $\partial u/\partial n' + \partial u/\partial n'' \leqq 0$ in case of a maximum. It follows that no maximum can be attained in points satisfying condition (b).

We shall call $ds' = \lambda' |dw|$ a supporting metric of $ds = \lambda |dw|$ at the point w_0 if: (1) $\lambda' = \lambda$ at w_0, (2) λ' is defined and $\leqq \lambda$ in a neighborhood of w_0.

THEOREM A2. *Suppose that λ is continuous, and that it is possible to find a supporting metric, satisfying* (4), *at every point of W. Then the inequality* (6) *still holds.*

If $u - v > 0$ at z_0, then $u' - v$ will also be positive, and consequently subharmonic, in a neighborhood of z_0.* A maximum of $u - v$ will *a fortiori* be a maximum of $u' - v$. Hence $u - v$ can have no maximum in E.

II. SCHOTTKY'S THEOREM

4. As a first application we prove Schottky's theorem with definite numerical bounds.

THEOREM B. *If $f(z)$ is analytic and different from* 0 *and* 1 *in $|z| < 1$, then*

$$\log |f(z)| < \frac{1+\theta}{1-\theta}(7 + \overset{+}{\log} |f(0)|) \tag{8}$$

for $|z| \leqq \theta < 1$.†

Let $\zeta_1 = \zeta_1(w)$ map the region outside of the segment (0, 1) onto the exterior of the unit circle, so that $w = \infty$ corresponds to $\zeta_1 = \infty$, $w = 1$ to $\zeta_1 = 1$, and $w = 0$ to $\zeta_1 = -1$. We also set $\zeta_2(w) = \zeta_1(w^{-1})$ and $\zeta_3(w) = \zeta_2(1 - w)$. Clearly these functions define similar maps of the regions outside of the segments $(1, \infty)$ and $(-\infty, 0)$. Explicitly, $\zeta_1(w)$ is obtained from the equation

$$\zeta_1 + \zeta_1^{-1} = 4w - 2. \tag{9}$$

We introduce the coordinates $\rho_1 = |w|$, $\rho_2 = |w - 1|$ and divide the plane into regions Ω_1: $\rho_1 \geqq 1$, $\rho_2 \geqq 1$; Ω_2: $\rho_1 \leqq 1$, $\rho_1 \leqq \rho_2$; Ω_3: $\rho_2 \leqq 1$, $\rho_2 \leqq \rho_1$. The metric

$$ds_i = \frac{|d \log \zeta_i|}{2(4 + \log |\zeta_i|)} = \lambda_i |dw| \tag{10}$$

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* u' corresponds to λ' as u to λ.

† Schottky's original theorem was purely qualitative. Numerical relations have been studied at great length, notably by Ostrowski (*Studien über den Schottky'schen Satz*, Basel, 1931, and *Asymptotische Abschätzung des absoluten Betrags einer Funktion, die die Werte* 0 *und* 1 *nicht annimmt*, Commentarii Mathematici Helvetici, vol. 5 (1933)), but no simple inequality comparable with (8) has ever been proved.

Added in proof: Numerical bounds of the same order of magnitude are found by A. Pfluger, *Über numerische Schranken im Schottky'schen Satz*, Commentarii Mathematici Helvetici, vol. 7 (1935). His proof depends on the use of modular functions, while ours is strictly elementary.

is readily recognized as the hyperbolic metric of a half-plane with the constant curvature -4. Computing the derivatives $\zeta_i'(w)$ we find

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\begin{aligned}
\lambda_1^{-1} &= 2(\rho_1\rho_2)^{1/2}(4 + \log |\zeta_1|), \\
\lambda_2^{-1} &= 2\rho_1\rho_2^{1/2}(4 + \log |\zeta_2|), \qquad (11)\\
\lambda_3^{-1} &= 2\rho_2\rho_1^{1/2}(4 + \log |\zeta_3|).
\end{aligned}
$$

We now set $ds = \lambda |dw|$ with $\lambda = \lambda_i$ in Ω_i. This metric is regular and satisfies condition (3) except at the singular points $0, 1, \infty$ and on the lines separating the regions Ω_i. On these lines λ is still continuous, as seen from (11) and the relations between ζ_1, ζ_2, and ζ_3.

Next we wish to show that condition (b) in Theorem A1 holds on the singular lines. We consider the arc $\rho_1 = 1$, $\rho_2 > 1$ and choose n', n'' as the outer and inner normals of the circle. The required condition is

$$\frac{\partial}{\partial n'} \log \lambda_1 + \frac{\partial}{\partial n''} \log \lambda_2 = \frac{\partial}{\partial n'} \log \frac{\lambda_1}{\lambda_2} > 0.$$

From (11) we obtain

$$\frac{\partial}{\partial n'} \log \frac{\lambda_1}{\lambda_2} = \frac{1}{2} - \frac{\dfrac{\partial}{\partial n'} \log \left| \dfrac{\zeta_1}{\zeta_2} \right|}{4 + \log |\zeta_1|},$$

which is also equal to

$$\frac{1}{2} - 2(4 + \log |\zeta_1|)^{-1} \frac{\partial \Phi_1}{\partial \phi},$$

where $\Phi_1 = \arg \zeta_1$, $\phi = \arg w$. For Φ_1 we have the simple relation $\cos \Phi_1 = \rho_1 - \rho_2$, which for $\rho_1 = 1$ becomes $\cos \Phi_1 = 1 - 2 \sin \phi/2$. Differentiating we find

$$\frac{\partial \Phi_1}{\partial \phi} = \frac{1}{2}\left(1 + \csc \frac{\phi}{2}\right)^{1/2},$$

and by use of the inequalities $\pi/3 \leqq \phi \leqq 5\pi/3$, $|\zeta_1| > 1$, we are finally led to the desired result,

$$\frac{\partial}{\partial n'} \log \frac{\lambda_1}{\lambda_2} > \frac{1}{2} - \frac{3^{1/2}}{4} > 0.$$

By symmetry, the same must be true for the arc $\rho_2 = 1$, $\rho_1 > 1$. The transformation $w' = (1 - w)^{-1}$ takes Ω_1 into Ω_2 and Ω_2 into Ω_3. Since the function λ is invariant under the transformation we conclude at once that condition (b) will hold also on the line separating Ω_2 and Ω_3.

From Theorem A1 we can now conclude that $w = f(z)$ satisfies the differential inequality $\lambda|dw| \leqq (1-|z|^2)^{-1}|dz|$. Integrating, we find that the shortest distance between the points $f(0)$ and $f(z)$, $|z|=\theta$, measured in the metric $ds=\lambda|dw|$, cannot exceed $[\log(1+\theta)/(1-\theta)]/2$.

The shortest path between the circles $\rho_1=m$ and $\rho_1=M$, where $M>m\geqq 2$, is a segment of the negative real axis, whose length is found to be

$$\frac{1}{2}\log\frac{4+\log|\zeta_1(-M)|}{4+\log|\zeta_1(-m)|}.$$

To simplify we introduce the lower and upper bounds $|\zeta_1(-M)|\geqq 4M$, $|\zeta_1(-m)|\leqq 5m$. Setting $M=|f(z)|$ and m equal to the greater of the numbers $|f(0)|$ and 2 we obtain

$$4+\log 4M \leqq \frac{1+\theta}{1-\theta}(4+\log 5m).$$

Here $\log 5m \leqq \log 10+\overset{+}{\log}|f(0)| < 3+\overset{+}{\log}|f(0)|$ and we find

$$4+\log 4M < \frac{1+\theta}{1-\theta}(7+\overset{+}{\log}|f(0)|)$$

which is stronger than (8).

III. Bloch's theorem

5. Let $w=f(z)$ be analytic in $|z|<1$ with $|f'(0)|=1$. Let $B'=B'(f)$ be the l.u.b. of the radii of all simple (*schlicht*) circles contained in the Riemann surface W generated by $f(z)$. Bloch's theorem is $B=\min B'>0$. Landau has proved $B>.396$.* Grunsky and Ahlfors proved in a recent paper $B<.472$.†

We show that the method developed in this paper gives an immediate proof of Bloch's theorem with a better lower bound for B. For an arbitrary point $\mathfrak{w}$ on W let $\rho(\mathfrak{w})$ denote the radius of the largest simple circle of center $\mathfrak{w}$ contained in W. It is clear that $\rho(\mathfrak{w})$ is continuous, and equal to zero only at the branch-points. We introduce the metric $ds=\lambda|dw|$ with

$$\lambda = \frac{A}{2\rho^{1/2}(A^2-\rho)} \qquad (\rho=\rho(\mathfrak{w})) \tag{12}$$

and w denoting the variable of the function plane (not the uniformizing variable). A is a constant satisfying the preliminary condition $A^2>B'$.

In the neighborhood of a branch-point $\mathfrak{a}$ we have $\rho=|w-a|$. Let n be the multiplicity of $\mathfrak{a}$; then $w_1=(w-a)^{1/n}$ is a uniformizing variable, and

* E. Landau, *Über den Blochschen Satz und zwei verwandte Weltkonstanten*, Mathematische Zeitschrift, vol. 30 (1929).

† L. V. Ahlfors and H. Grunsky, *Über die Blochsche Konstante*, Mathematische Zeitschrift, vol. 42 (1937). The result was found independently by R. M. Robertson.

the corresponding λ_1 is determined from $\lambda_1|dw_1| = \lambda|dw|$. We obtain $\lambda_1 = n\rho^{1/2-1/n}/2(A^2-\rho)$, and it is seen at once that the metric is regular in case $n=2$ and that λ_1 becomes zero in case $n>2$.

We wish to apply Theorem A2 and therefore look for a supporting metric satisfying the requirements of that theorem. For a regular point $\mathfrak{w}_0$ the surrounding circle of radius $\rho(\mathfrak{w}_0)$ must pass through at least one singularity $\mathfrak{b}$ which is either a branch-point or a boundary point for the surface. We set $\rho' = |w-b|$ and define $\lambda' = A/[2\rho'^{1/2}(A^2-\rho')]$. This metric has the curvature -4 for it is obtained from the hyperbolic metric of a circle by means of the transformation $w' = w^{1/2}$. In all points of our circle we have $\rho \leqq \rho'$ by the definition of ρ. The inequality $\lambda' \leqq \lambda$ is therefore satisfied in a neighborhood of $\mathfrak{w}_0$ if the function $t^{1/2}(A^2-t)$ increases for $t \leqq \rho(\mathfrak{w}_0)$. Under this condition λ' will be a supporting function of λ, for at the center w_0 we have $\lambda' = \lambda$. The function $t^{1/2}(A^2-t)$ is increasing as long as $t < A^2/3$. Consequently all the conditions in Theorem A2 are fulfilled if we suppose that $A^2 > 3B'$.

Apply the theorem with $z=0$. Using the condition $|dw/dz|_{z=0} = 1$ we get

$$A \leqq 2\rho_0^{1/2}(A^2 - \rho_0), \tag{13}$$

where ρ_0 is the radius of the largest simple circle with center at the image of $z=0$. The function in the right member of (13) is increasing, and we can replace ρ_0 by B' obtaining $A \leqq 2B'^{1/2}(A^2-B')$. Letting A tend to $(3B')^{1/2}$ we finally get $B' \geqq 3^{1/2}/4$. This implies that Bloch's constant $B \geqq 3^{1/2}/4 > .433$.

On the other side, if we insert $A^2 = (3B')^{1/2}$ in (13), lower and upper bounds for ρ_0 in terms of B' can be found.

6. Landau has considered a closely related constant L. Let $L' = L'(f)$ be the l.u.b. of the radii of all circles in the w-plane contained in the projection of W, that is, whose values are taken by the function $w = f(z)$, $|f'(0)| = 1$. L is defined as the minimum of all such L'. Clearly, $L \geqq B$.

The method employed above is immediately applicable if we choose $\lambda = (2\rho \log C/\rho)^{-1}$. This metric is regular at all branch-points, and when we replace ρ by the distance ρ' from a fixed boundary point, the curvature becomes -4. In order that the function λ' thus obtained be a supporting function it is sufficient that $t \log C/t$ is increasing. This is true for $t < Ce^{-1}$. We therefore choose $C > eL'$, obtaining the inequality $1 \leqq 2L' \log C/L'$ as above. Letting C tend to eL' we find $L' \geqq 1/2$ and hence $L \geqq 1/2$.

This lower bound is the best known. It shows in particular that $L > B$.*

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* In the other direction R. M. Robinson has proved $L. < .544$. This result has not been published.