EQUIVALENCE OF PAIRS OF MATRICES*

BY

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1. **Introduction.** Two pairs of matrices, $[A_1, A_2]$ and $[B_1, B_2]$, with elements in a commutative field F, are said to be *equivalent*† if and only if there exist two non-singular matrices P and Q, with elements in F, such that $A_1 = PB_1Q$ and $A_2 = PB_2Q$.

The totality of pairs of matrices may be separated into different classes in such a way that all pairs in one class are equivalent to one another while pairs in different classes are not equivalent. The problem which arises naturally is to determine a set of invariants which will characterize the pairs in each class and to select from each class a *canonical* pair defined uniquely in terms of these invariants. A *rational* solution of the problem is one which is carried out completely in the field F; the invariants and canonical pairs obtained in such a solution will be *rational*.

The *rank* of a pair $[A_1, A_2]$ is the maximum rank of the matrices of the *matric pencil* $A = A_1x_1 + A_2x_2$, where x_1 and x_2 are indeterminates in F. A matric pencil is said to be *non-singular* if it is square and of rank equal to its order (otherwise it is called *singular*), and it is said to be *regular* if the rank of either one of its coefficients is the same as the rank of the pencil.

Non-singular matric pencils were first classified by Weierstrass‡ who constructed an irrational canonical form defined by means of the elementary divisors of the pencil. Frobenius§ later gave a rational treatment of the non-singular case. Kronecker‖ treated the singular case and gave an irrational canonical form. Muth¶ gave a full account of the theory of pairs of bilinear forms as it stood at the turn of the century. De Séguier** seems to have been the first to give a rational treatment of the singular case. More recently, it has received the attention of Dickson,†† Turnbull and Aitken,‡‡ Wedderburn,§§

* Presented to the Society, March 27, 1937; received by the editors May 25, 1937.

† C. C. MacDuffee, *The Theory of Matrices*, Berlin, 1933, p. 48.

‡ K. Weierstrass, Monatsberichte, Preussische Akademie der Wissenschaften, 1868, pp. 310–338.

§ G. Frobenius, Sitzungsberichte, Preussische Akademie der Wissenschaften, 1894, pp. 31–44.

‖ L. Kronecker, Sitzungsberichte, Preussische Akademie der Wissenschaften, 1890, pp. 1225–1237.

¶ P. Muth, *Theorie und Anwendung der Elementartheiler*, Leipzig, 1899.

** J. A. de Séguier, Bulletin de la Société Mathématique de France, vol. 36 (1908), pp. 20–40.

†† L. E. Dickson, these Transactions, vol. 29 (1927), pp. 239–253.

‡‡ Turnbull and Aitken, *Canonical Matrices*, Glasgow, 1932, chap. 9.

§§ J. H. M. Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, 1934, chap. 4.

Turnbull,* Ledermann,† Williamson,‡ and others.

In this paper the problem of constructing a rational canonical form in the singular case is reduced to the consideration of the non-singular case. The proofs are completely rational, quite elementary, and relatively short. The canonical form which is obtained is defined essentially in terms of the set of invariants shown by Williamson‡ to characterize the classes of equivalent matrices. The method of proof is very similar to that used by Ingraham§ in his treatment of the equivalence of singular pencils of Hermitian matrices.

2. **Preliminary remarks.** Consider a singular pencil $A=A_1x_1+A_2x_2$ of rank $\rho(A)=r$ and order $[\theta, \theta']$. Set $R_1=A_1t_{11}+A_2t_{21}$ and $R_2=A_1t_{12}+A_2t_{22}$, where the t_{ij} are quantities of F such that $|t_{ij}|\neq 0$. If $R=R_1x_1+R_2x_2$ and $T=\|t_{ij}\|$, then the relations above may be written $R=AT$, and the pencils A and R are said to be *transformable.* If B is a second matric pencil, it follows easily that A is equivalent to B $(A\sim B)$ if and only if $AT\sim BT$. In particular, there exist two quantities t_{11}^0 and t_{21}^0 of F not both zero and such that $\rho(A_1t_{11}^0+A_2t_{21}^0)=r$, and in this case the pencil $R=AT$ is said to be regular.

If it is desired only to obtain necessary and sufficient conditions for the equivalence of two pencils, then there is no loss of generality in considering only regular pencils. However, if a canonical form in the most strict sense is required, it is necessary to start with the original pencils rather than their regular transforms, as has been pointed out by Ledermann.|| Canonical forms will be constructed only for regular pencils, but the invariants used will be shown to afford a satisfactory classification for all pencils. It is felt that this solves the important part of the problem.

3. **Rational canonical form for regular matric pencils.** Constant non-singular matrices P and Q exist¶ such that

$$PR_1Q = e = \left\| \begin{matrix} 1_r & 0 \\ 0 & 0 \end{matrix} \right\|;$$

hence $R\sim ex_1+PR_2Qx_2=ex_1+ax_2=R_0$. If we set

$$R_0 = \left\| \begin{matrix} 1_r & 0 \\ 0 & 0 \end{matrix} \right\| x_1 + \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\| x_2,$$

* H. W. Turnbull, Proceedings of the Edinburgh Mathematical Society, (2), vol. 4 (1935) pp. 67–76.

† W. Ledermann, ibid., (2), vol. 4 (1935), pp. 92–105.

‡ J. Williamson, ibid., (2), vol. 4 (1936), pp. 224–231.

§ Ingraham and Wegner, these Transactions, vol. 38 (1935), pp. 145–162.

|| Loc. cit.

¶ MacDuffee, loc. cit., p. 43.

it follows immediately that $a_{22}=0$, for otherwise the rank of R_0 would be greater than r, which is impossible since R_0 has been assumed to be regular.

Since $1_r x_1+a_{11}x_2$ is non-singular, the rows of $(a_{21}\ 0)$ must be linearly dependent on the rows of $(1_r x+a_{11}\ a_{12})$; thus there exists a matrix X_{21} such that $X_{21}(1_r x+a_{11})=a_{21}$ and $X_{21}a_{12}=0$. Necessarily then, $X_{21}=a_{21}(1_r x+a_{11})^{-1}$, and $a_{21}(1_r x+a_{11})^{-1}a_{12}=0$. For x sufficiently large

$$(1_r x + a_{11})^{-1} = \frac{1}{x}\sum_{k=0}^{\infty}\left(\frac{-a_{11}}{x}\right)^k;$$

hence $a_{21}a_{11}^k a_{12}=0$ for $k=0, 1, 2, \cdots$.

Conversely, if $a_{21}a_{11}^k a_{12}=0$ for $k=0, 1, 2, \cdots$, then $a_{21}(1_r x+a_{11})^{-1}a_{12}=0$. Hence, if $X_{21}=a_{21}(1_r x+a_{11})^{-1}$, it follows that $X_{21}(1_r x+a_{11})=a_{21}$ and $X_{21}a_{12}=0$, so that the last $\theta-r$ rows of $ex+a$ are dependent on the first r rows. This proves the following lemma:

LEMMA A. *The rank of the matric pencil*

$$\left\|\begin{matrix} 1_r & 0 \\ 0 & 0 \end{matrix}\right\| x_1 + \left\|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| x_2$$

is r if and only if $a_{22}=0$ and $a_{21}a_{11}^k a_{12}=0$ for $k=0, 1, 2, \cdots$.

Lemma A holds true only if the coefficient field F is commutative and has characteristic zero. For a field of characteristic $p\neq 0$ it is necessary to alter the treatment slightly. The present author has shown (Annals of Mathematics, (2), vol. 36 (1935), p. 865) that the matric pencil of Lemma A is equivalent to a pencil

$$\left\|\begin{matrix} 1_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right\| x_1 + \left\|\begin{matrix} b_{11} & 0 & b_{13} \\ 0 & 1_s & 0 \\ b_{31} & 0 & 0 \end{matrix}\right\| x_2,$$

where $s=\rho(a_{22})$ and $b_{31}b_{11}^k b_{13}=0$ for $k=0, 1, 2, \cdots$. It follows easily that there is no loss of generality in considering pencils satisfying the conditions of Lemma A; and in this way the proofs given in this paper may be extended to include pencils with coefficients in an arbitrary field. This more general method of proof has been used by the present author in a recent paper (*Strict equivalence of matric pencils*, presented to the Society December 29, 1937, but not yet published) treating the problem of equivalence of matric pencils, singular and non-singular.

We now proceed with the construction of a canonical form for the regular pencil $S=ex+a$. If α and β are non-singular matrices of orders θ and θ' with elements in F, then $\alpha e\beta=e$ if

$$\alpha = \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{Vmatrix} \quad \text{and} \quad \beta = \begin{Vmatrix} \alpha_{11}^{-1} & 0 \\ \beta_{21} & \beta_{22} \end{Vmatrix},$$

where α_{22} and β_{22} are non-singular, and so

$$\alpha a \beta = \begin{Vmatrix} \alpha_{11}a_{11}\alpha_{11}^{-1} + \alpha_{12}a_{21}\alpha_{11}^{-1} + \alpha_{11}a_{12}\beta_{21} & \alpha_{11}a_{12}\beta_{22} \\ \alpha_{22}a_{21}\alpha_{11}^{-1} & 0 \end{Vmatrix}.$$

Now if the rank of a_{21} is r_1, α_{22} and α_{11} may be chosen so that

$$\alpha_{22}a_{21}\alpha_{11}^{-1} = \begin{Vmatrix} 1_{r_1} & 0 \\ 0 & 0 \end{Vmatrix},$$

and it follows from Lemma A that the first r_1 rows of $\alpha_{11}a_{12}\beta_{22}$ must be zero. Furthermore, if the rank of a_{12} is c_1, it is clear that β_{22} may be chosen so that

$$\alpha_{11}a_{12}\beta_{22} = \begin{Vmatrix} 0 & 0 \\ 0 & 1_{c_1} \end{Vmatrix}.$$

With this choice of α_{jj} and β_{kk} the pencil $T = \alpha S \beta$ takes the form

$$T = \begin{Vmatrix} 1_{r_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_h & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{c_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{r_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{jk} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{c_1} \end{Vmatrix} x + \begin{Vmatrix} x_1 & a_1 & a_4 & 0 & 0 & 0 \\ x_2 & a_2 & a_3 & 0 & 0 & 0 \\ x_3 & x_4 & x_5 & 0 & 0 & 1_{c_1} \\ 1_{r_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix},$$

where $h = r - r_1 - c_1$, $j = \theta - r - r_1 - c_1$, and $k = \theta' - r - r_1 - c_1$. Finally α_{12} and β_{21} may clearly be chosen so that the x_i are all zero; then

$$T = \begin{Vmatrix} x & a_1 & a_4 & 0 & 0 & 0 \\ 0 & x + a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

The rank of T is r, since α and β were chosen non-singular, therefore

$$\begin{Vmatrix} a_1 & a_4 \\ x + a_2 & a_3 \end{Vmatrix}$$

is regular and of rank $r-r_1-c_1$. Hence, by Lemma A, $a_4=0$ and $a_1a_2^k a_3=0$ for $k=0, 1, 2, \cdots$.

Consider a second regular pencil $U=U_1x_1+U_2x_2$ which is equivalent to R. The rank of U_1 is necessarily r, therefore, constant non-singular matrices P_0 and Q_0 exist such that $P_0U_1Q_0=e$. Then if

$$V = P_0UQ_0 = ex_1 + bx_2 = ex_1 + \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix} x_2,$$

it follows from Lemma A that $b_{22}=0$. Since $V\sim S$ there exist constant non-singular matrices x and y such that $xV=Sy$, which equation is equivalent to the relations $xe=ey$ and $xb=ay$. From $xe=ey$ it follows that $x_{11}=y_{11}$, $x_{21}=0$, $y_{12}=0$, and that x_{11}, x_{22}, and y_{22} are non-singular. Then from $xb=ay$ it follows that $x_{22}b_{21}=a_{21}x_{11}$ and $x_{11}b_{12}=a_{12}y_{22}$. Since x_{22}, x_{11}, and y_{22} are necessarily non-singular this shows that the ranks of b_{21} and b_{12} are the same as the ranks r_1 and c_1 of a_{21} and a_{12}. r_1 will be called the first "row invariant subrank" and c_1 the first "column invariant subrank" of R or of any pencil equivalent to R. It follows that constant non-singular matrices α_0 and β_0 can be chosen so that $W=\alpha_0V\beta_0$ takes a form analogous to that of T but with a_k replaced by b_k.

The pencils U and R are equivalent if and only if there exist constant non-singular matrices p and q such that $pT=Wq$. From $pe=eq$ it is clear that p and q must be of the forms

$$p = \begin{Vmatrix} z & * \\ 0 & * \end{Vmatrix} \quad \text{and} \quad q = \begin{Vmatrix} z & 0 \\ * & * \end{Vmatrix},$$

hence $pT=Wq$ may be replaced by the equation

$$\begin{Vmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & p_{46} \\ 0 & 0 & 0 & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & p_{64} & p_{65} & p_{66} \end{Vmatrix} \begin{Vmatrix} 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix} = \begin{Vmatrix} 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix} \begin{Vmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & 0 & 0 & 0 \\ p_{31} & p_{32} & p_{33} & 0 & 0 & 0 \\ q_{41} & q_{42} & q_{43} & q_{44} & q_{45} & q_{46} \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} & q_{56} \\ q_{61} & q_{62} & q_{63} & q_{64} & q_{65} & q_{66} \end{Vmatrix}.$$

This equation is equivalent to the set of relations:

$$\begin{aligned} &p_{54} = p_{64} = p_{12} = p_{13} = q_{64} = q_{65} = p_{23} = 0, \\ &p_{11} = p_{44}, \quad p_{33} = q_{66}, \quad p_{34} = q_{61}, \\ &p_{14} = b_1p_{21}, \qquad p_{32}a_3 = q_{63}, \qquad p_{12}a_3 = b_1p_{23}, \\ &p_{24} = b_2p_{21} + b_3p_{31}, \qquad p_{11}a_1 + p_{12}a_2 = b_1p_{22}, \\ &p_{31}a_1 + p_{32}a_2 = q_{62}, \qquad p_{22}a_3 = b_2p_{23} + b_3p_{33}, \\ &p_{21}a_1 + p_{22}a_2 = b_2p_{22} + b_3p_{32}. \end{aligned}$$

It follows easily that these equations have a solution such that p is non-singular if and only if there exist constant non-singular matrices p_{11}, p_{22}, and p_{33} which satisfy the relations:

$$p_{11}a_1 = b_1p_{22}, \qquad p_{22}a_3 = b_3p_{33},$$
$$p_{21}a_1 + p_{22}a_2 = b_2p_{22} + b_3p_{32}.$$

These equations may be rewritten in the form

$$\left\| \begin{matrix} p_{22} & p_{21} \\ 0 & p_{11} \end{matrix} \right\| \left\| \begin{matrix} a_2 & a_3 \\ a_1 & 0 \end{matrix} \right\| = \left\| \begin{matrix} b_2 & b_3 \\ b_1 & 0 \end{matrix} \right\| \left\| \begin{matrix} p_{22} & 0 \\ p_{32} & p_{33} \end{matrix} \right\|,$$

and this is simply the condition for the equivalence of the two regular pencils

$$T^1 = \left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right\| x + \left\| \begin{matrix} a_2 & a_3 \\ a_1 & 0 \end{matrix} \right\|, \qquad W^1 = \left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right\| x + \left\| \begin{matrix} b_2 & b_3 \\ b_1 & 0 \end{matrix} \right\|.$$

The pencils T^1 and W^1 will be called "first kernels" of the pencils R and U. Thus the problem of classifying singular pencils of rank r has been reduced to that of classifying singular pencils of rank $r-r_1-c_1$, or else to that of classifying non-singular pencils if a_1 and a_3 happen to be zero.

If r_{j+1} and c_{j+1} are the first invariant subranks of T^j, and T^{j+1} is a first kernel of T^j for $j=1, 2, 3, \cdots, n$, and if T^{n+1} is non-singular or zero; then r_j and c_j for $j=1, 2, \cdots, n+1$ will be called the "invariant subranks" of R, and T^{n+1} a "kernel" of R. This proves the following:

THEOREM 1. *Two regular matric pencils are equivalent if and only if they have identical sets of invariant subranks and equivalent kernels.*

It is clear that the construction which leads to T can be extended until a rational canonical form for R is obtained. This canonical form would display the invariant subranks and invariant factors of R. The invariant factors of R are clearly the same as those of any kernel of R except for $\sum_{j=1}^{n+1}(r_j+c_j)$ units which would appear in the normal form of R but not in the normal form of any kernel of R. This demonstrates the corollary:

COROLLARY 1.1. *Two regular matric pencils are equivalent if and only if they have the same invariant subranks and the same elementary divisors.*

4. **Transformable matric pencils.** Let $A = A_1x_1 + A_2x_2$ be an arbitrary matric pencil, and define matrices $M_k(A)$ and $N_k(A)$, for $k=1$ 2, 3, $\cdots$, by the relations

$$M_1(A) = \| A_1 \quad A_2 \|, \qquad M_2(A) = \left\| \begin{matrix} A_1 & A_2 & 0 \\ 0 & A_1 & A_2 \end{matrix} \right\|, \cdots,$$

$$N_1(A) = \left\| \begin{matrix} A_1 \\ A_2 \end{matrix} \right\|, \qquad N_2(A) = \left\| \begin{matrix} A_1 & 0 \\ A_2 & A_1 \\ 0 & A_2 \end{matrix} \right\|, \cdots .$$

It is convenient to denote by $m_k(A)$ and $n_k(A)$ the ranks of $M_k(A)$ and $N_k(A)$ and to call $m_k(A)$ the "row singularities" and $n_k(A)$ the "column singularities" of A.

It is obvious that equivalent matric pencils have the same row and column singularities. We now proceed to prove the following:

THEOREM 2. *Transformable matric pencils have the same singularities.*

Proof.* Consider a matric pencil $A = A_1x_1 + A_2x_2$ and a non-singular transformation of indeterminates $x = tx'$, or more explicitly

$$x_1 = t_{11}x_1' + t_{12}x_2', \qquad x_2 = t_{21}x_1' + t_{22}x_2'.$$

Under this transformation, the pencil A is carried into the pencil

$$A' = At = (t_{11}A_1 + t_{21}A_2)x_1' + (t_{12}A_1 + t_{22}A_2)x_2' = A_1'x_1' + A_2'x_2',$$

and the theorem states that $m_k(A) = m_k(A')$ and $n_k(A) = n_k(A')$. The first of these equalities will be demonstrated by constructing non-singular matrices T^k such that

$$M_k(A)T^k = T^{k-1}M_k(A') \qquad \text{for} \qquad k = 1, 2, 3, \cdots,$$

and the second can be shown by a similar construction.

If $u_0, u_1, u_2, \cdots, u_k$ are $k+1$ indeterminates, then the identity

$$\begin{aligned} &u_0x_1^k + C_{k,1}u_1x_1^{k-1}x_2 + C_{k,2}u_2x_1^{k-2}x_2^2 + \cdots + u_kx_2^k \\ &\qquad \equiv u_0'x_1'^k + C_{k,1}u_1'x_1'^{\,-1}x_2' + C_{k,2}u_2'x_1'^{k-2}x_2'^2 + \cdots + u_k'x_2'^k \end{aligned} \tag{1}$$

defines $u_0', u_1', \cdots, u_k'$ as linear combinations of $u_0, u_1, \cdots, u_k$, and these may be written in either of the forms

$$u_i' = \sum_{j=0}^{k} T_{ij}^k u_j \qquad \text{or} \qquad u' = T^k u.$$

Now if $T^k = \| T_{ij}^k \|$, then $t \to T^k$ is a representation of the full linear group of all non-singular matrices of order two, and hence T^k is non-singular since t is.

If (1) is differentiated with respect to x_1' and x_2', there results:

$$\begin{aligned} &(u_0t_{11} + u_1t_{21})x_1^{k-1} + C_{k-1,1}(u_1t_{11} + u_2t_{21})x_1^{k\,2}x_2 + \cdots + (u_{k-1}t_{11} + u_kt_{21})x_2^{k-1} \\ &\qquad \equiv u_0'x_1'^{k-1} + C_{k-1,1}u_1'x_1'^{\,-2}x_2' + \cdots + u_{k-1}'x_2'^{k-1}, \end{aligned} \tag{2a}$$

* I am indebted to Dr. A. H. Clifford for this proof.

$$(2b)\quad \begin{aligned}&(u_0t_{12}+u_1t_{22})x_1^{k-1}+C_{k-1,1}(u_1t_{12}+u_2t_{22})x_1^{k-2}x_2+\cdots+(u_{k-1}t_{12}+u_kt_{22})x_2^{k-1}\\ &\qquad\equiv u_1'x_1'^{k-1}+C_{k-1,1}u_2'x_1'^{k-2}x_2'+\cdots+u_k'x_2'^{k-1}.\end{aligned}$$

If (2a) is multiplied by A_1, and (2b) by A_2, and the resulting equations are added, it follows that

$$\begin{aligned}&(u_0A_1'+u_1A_2')x_1^{k-1}+C_{k-1,1}(u_1A_1'+u_2A_2')x_1^{k-2}x_2+\cdots+(u_{k-1}A_1'+u_kA_2')x_2^{k-1}\\ &\equiv(u_0'A_1+u_1'A_2)x_1'^{k-1}+C_{k-1,1}(u_1'A_1+u_2'A_2)x_1'^{k-2}x_2'+\cdots+(u_{k-1}'A_1+u_2'A_2)x_2'^{k-1},\end{aligned}$$

and from this identity that

$$u_i'A_1+u_{i+1}'A_2\equiv\sum_{j=0}^{k-1}T_{ij}^{k-1}(u_jA_1'+u_{j+1}A_2')\quad\text{for}\quad i=0,1,\cdots,k-1.$$

These identities may be written in the form

$$M_k(A)u'\equiv T^{k-1}M_k(A')u$$

or, since $u'=T^ku$, in the form

$$M_k(A)T^ku\equiv T^{k-1}M_k(A')u.$$

The indeterminate vector u may be cancelled in this equation and so

$$M_k(A)T^k=T^{k-1}M_k(A')$$

as was to be shown.

It is convenient, at this point, to state the following:

LEMMA B.* *The invariant factors of transformable matric pencils are connected by the same transformation of the indeterminates x_1 and x_2 as the pencils themselves.*

5. **Equivalence of general matric pencils.** Williamson† has shown that the minimal numbers of a matric pencil can be expressed in terms of its singularities, from which follows the theorem:

THEOREM 3. *Two matric pencils are equivalent if and only if they have the same singularities and invariant factors.*

This theorem may also be proved with the help of Theorem 2 and Lemma B by showing that the invariant subranks of a regular matric pencil can be expressed in terms of its singularities. This will now be done for the row subranks, and an analogous treatment of the column subranks would complete the proof. There is no loss of generality if the pencil is taken to be in canonical form.

* See MacDuffee, loc. cit.

† Loc. cit.

Consider the regular canonical pencil $W=e_0x+a$ of rank r, with row subranks r_k and column subranks c_k. Set

$$E_k = \begin{Vmatrix} 1_{r_k} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad F_k = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_{c_k} \end{Vmatrix}, \quad \text{and} \quad a = \begin{Vmatrix} E_1 & 0 \\ A_1 & F_1 \end{Vmatrix}.$$

Then square matrices A_k may be defined by the relations

$$A_k = \begin{Vmatrix} 0 & E_k & 0 \\ 0 & A_k & F_k \\ 0 & 0 & 0 \end{Vmatrix} \quad \text{for} \quad k = 1, 2, \cdots, \phi,$$

where $A_{\phi+1}$ is the canonical kernel of S and $E_kA_k^jF_k=0$ for $j=1, 2, \cdots$. Of course r_k is the rank of E_k, and c_k is the rank of F_k.

Now, by definition,

$$m_2(W) = \rho \begin{Vmatrix} e_0 & a & 0 \\ 0 & e_0 & a \end{Vmatrix} \doteq \rho \begin{Vmatrix} 0 & 0 & E_1 & 0 & 0 & 0 \\ 1 & 0 & A_1 & F_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_1 & 0 \\ 0 & 0 & 1 & 0 & A_1 & F_1 \end{Vmatrix} = r + \rho \begin{Vmatrix} E_1 & 0 & 0 \\ 0 & E_1 & 0 \\ 1 & A_1 & F_1 \end{Vmatrix}.$$

If the first column of this matrix is multiplied by $-F_1$ and added to the last column, and then the third row is multiplied by $-E_1$ and added to the first row; since $E_1F_1=0$, it follows that

$$m_2(W) = r + \rho \begin{Vmatrix} 0 & -E_1A_1 \\ 0 & E_1 \\ 1 & A_1 \end{Vmatrix} = 2r + \rho \begin{Vmatrix} E_1 \\ E_1A_1 \end{Vmatrix}.$$

In similar fashion, it is easily shown that

$$m_k(W) = kr + \rho \begin{Vmatrix} E_1 \\ E_1A_1 \\ E_1A_1^2 \\ \cdot\ \cdot\ \cdot\ \cdot \\ E_1A_1^{k-1} \end{Vmatrix} \quad \text{for} \quad k = 1, 2, 3, \cdots.$$

Since $E_2A_2^jF_2=0$, it follows that

$$A_1^j = \begin{Vmatrix} 0 & E_2A_2^{j-1} & 0 \\ 0 & A_2^j & A_2^{j-1}F_2 \\ 0 & 0 & 0 \end{Vmatrix},$$

and hence that

$$\rho \begin{Vmatrix} E_1 \\ E_1 A_1 \\ \cdot\ \cdot\ \cdot \\ E_1 A_1^{k-1} \end{Vmatrix} = r_1 + \rho \begin{Vmatrix} E_2 \\ E_2 A_2 \\ \cdot\ \cdot\ \cdot \\ E_2 A_2^{k-1} \end{Vmatrix};$$

and a simple induction now shows that

$$m_k(W) = kr + \sum_{j=1}^{k-1} r_j.$$

This equation provides the necessary relationship between the row singularities and subranks of W and the proof of Theorem 3 is complete. Of course,

$$r_k = m_k - r - \sum_{j=1}^{k-1} m_j \qquad \text{and} \qquad c_k = n_k - r - \sum_{j=1}^{k-1} n_j$$

are the inverse equations which express the subranks in terms of the singularities.

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