FIXED POINTS UNDER TRANSFORMATIONS OF CONTINUA WHICH ARE NOT CONNECTED IM KLEINEN*

BY

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1. **Introduction.** Ayres† has shown that if M is a compact continuous curve in the plane which does not separate the plane and T is a reversibly continuous transformation of M into a subset of itself, then T leaves some point of M invariant. He also proves similar theorems for special cases of non-planar continuous curves. The purpose of this paper is to extend these results to certain types of continua which are not connected im kleinen.

2. **Preliminary theorem and lemmas.** We prove first the following preliminary theorems and lemmas.

THEOREM I. *If M is a compact continuum in a metric space but is not an indecomposable continuum, and if T is any reversibly continuous transformation of M into a subset of itself, then some proper subcontinuum N of M contains a point of $T(N)$, the image of N under T.*

Proof. If T carries M into a proper subset of itself, then $T(M)$ contains its image $T^2(M)$ and the theorem is true. Suppose then that T carries M into itself. It can easily be shown that if M is not indecomposable, it does not contain two mutually exclusive composants. The transformation T carries a composant of M into a composant of M. It has been shown that a composant‡ of a continuum is the sum of a countable number of continua $N_1, N_2, N_3, \cdots$, where for each integer i, N_i contains N_{i-1}.

Let V be any composant of M, and let $T(V)$ be its image under T. Then V and $T(V)$, by what was said above, must have a point in common. Let V be expressed as $\sum_{i=1}^{\infty} V_i$, where for each integer i, V_i is a continuum which contains V_{i-1}. Then $T(V) = \sum_{i=1}^{\infty} T(V_i)$ where for each integer i, $T(V_i)$ is the image of V_i under T and is a continuum which contains $T(V_{i-1})$. For some integer j, V_j contains a point of $T(V_j)$, for if we suppose the contrary to be true, it is obvious that V contains no point of $T(V)$, contradicting the fact

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† W. L. Ayres, *Some generalizations of the Scherrer fixed-point theorem*, Fundamenta Mathematicae, vol. 16 (1930), pp. 333–336.

‡ See R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, p. 75.

that V and $T(V)$ have a point in common. V_i is therefore a continuum satisfying the conclusions of the theorem.

LEMMA 1. *If M is a compact continuum in a metric space and M is not an indecomposable continuum and contains no continuum which is the sum of two continua whose common part is disconnected, and if T is a reversibly continuous transformation of M into a subset of itself and neither T nor T^{-1} carries a proper subcontinuum of M into itself, then there is a proper subcontinuum L_0 of M such that L_0 contains a point of $T(L_0)$ but such that for $n>1$, the common part of L_0 and $T^n(L_0)$ is vacuous.*

Proof. By Theorem I, some proper subcontinuum K_0 of M contains a point of its image $T(K_0)$. For each positive or negative integer n, let K_n designate $T^n(K_0)$. By a hypothesis of the theorem, K_0 is not identical with K_1, for suppose $K_0=K_1$. Then K_0 is a proper subcontinuum of M which is carried into itself by T. Furthermore $K_0 \cdot K_1$ is not identical with K_0 or with K_1; for suppose $K_0 \cdot K_1$ is identical with K_0. Then T carries K_0 into a subset of itself. Consider the continuum $A=\prod_{n=0}^{\infty} K_n$. Then $T(A)=A$, and T carries A, a proper subcontinuum of M, into itself. Similarly, if $K_0 \cdot K_1$ is identical with K_1, T^{-1} carries a proper subcontinuum of M into itself; thus we have a contradiction of a hypothesis of the theorem.

There exists an integer r such that the $\prod_{i=0}^{r} K_i$ is not vacuous but such that $\prod_{i=0}^{r+1} K_i$ is vacuous. This can be shown as follows. $K_0 \cdot K_1$ is not vacuous. Suppose $\prod_{i=0}^{s} K_i$ contains a point for each integer s. Then $B=\prod_{n=0}^{\infty} K_n$ is not vacuous and is a proper subcontinuum of M which is carried into itself by T, which is, again, a contradiction of a hypothesis of the theorem. Let L_0 be the continuum $\prod_{i=0}^{r-1} K_i$, where r is an integer such that $\prod_{i=0}^{r} K_i$ is not vacuous but $\prod_{i=0}^{r+1} K_i$ is vacuous. For each integer i, let $L_i=T^i(L_0)$. Then L_1 is $\prod_{i=1}^{r} K_i$. $L_0 \cdot L_1$ is not vacuous since $L_0 \cdot L_1=\prod_{i=0}^{r} K_i$ which is not vacuous. $L_0 \cdot L_2$ is vacuous. For suppose L_0 contains a point of L_2. Then $\prod_{i=0}^{r-1} K_i$ contains a point P of $\prod_{i=2}^{r+1} K_i$. It follows that P is in $\prod_{i=0}^{r+1} K_i$, which by hypothesis is vacuous. Furthermore L_0 contains no point of L_r, $r>2$. For suppose the contrary and that s is the smallest integer greater than two such that L_0 contains a point of L_s. Consider the two continua, L_0+L_s and $L_1+L_2+ \cdots +L_{s-1}$. Their common part is the sum of the two mutually exclusive continua, $L_0 \cdot L_1$ and $L_{s-1} \cdot L_s$. Then $\sum_{i=0}^{s} L_i$ is a subcontinuum of M which is the sum of two continua whose common part is disconnected. This is a contradiction of a hypothesis of the theorem. The lemma is therefore true.

LEMMA 2. *If, in a metric space, M is a compact non-degenerate continuum but is not an indecomposable continuum and does not contain a continuum which is the sum of two continua whose common part is disconnected, and if T is* *a reversibly continuous transformation of M into a subset of itself, then T carries some proper subcontinuum of M into itself.*

Proof. Suppose T does not carry a proper subcontinuum of M into itself. Then by Lemma 1, there exists a sequence of subcontinua $L_0, L_1, L_2, \cdots$ such that (1) for each positive and negative integer n, L_n is $T^n(L_0)$, (2) L_n and $L_{n\pm 1}$ have a point in common, but L_n and L_{n+r}, $|r|>1$, do not have a point in common. Let V_0 be an irreducible continuum from $L_{-1}\cdot L_0$ to $L_0\cdot L_1$ which is a subcontinuum of L_0. Let V_n, for each positive or negative integer, designate $T^n(V_0)$. Then since T is reversibly continuous, it can be easily shown that V_n is irreducible from $L_{n-1}\cdot L_n$ to $L_n\cdot L_{n+1}$. Let P_0 be a point of V_0 not in $L_{-1}\cdot L_0$ or $L_0\cdot L_1$. There is such a point for otherwise L_{-1} would contain a point of L_1. For each positive or negative integer n, let P_n designate $T^n(P_0)$. Let M_0 be an irreducible continuum from P_0 to P_1 which is a subset of $V_0+V_1+L_0\cdot L_1$. Then since T is reversibly continuous, it follows that (1) M_n for each integer n is irreducible from P_n to P_{n+1}, (2) $M_n\cdot M_{n\pm 1}$ is not vacuous, but $M_n\cdot M_{n+r}$, $|r|>1$, is vacuous, (3) $(M_i+M_{i-1})\cdot V_i$ is an irreducible continuum from $L_{i-1}\cdot L_i$ to $L_i\cdot L_{i+1}$.

Since M is not indecomposable, M is the sum of two proper subcontinua H and K. One of these continua contains points of infinitely many of the continua $M_0, M_1, M_2, \cdots$. Suppose H contains points of infinitely many of these continua. Let j be the smallest integer greater than or equal to zero such that H contains a point of M_j. Let r be any integer greater than 5 such that H contains a point of M_{j+r}. Let H' designate the common part of H and the continuum $\sum_{i=j}^{j+r} M_i$. But H' is itself a continuum since the common part of two subcontinua of M cannot be disconnected. H' must contain points of $L_{j+1}\cdot L_{j+2}\cdot M_{j+1}$, $L_{j+2}\cdot L_{j+3}\cdot M_{j+2}, \cdots, L_{j+r-2}\cdot L_{j+r-1}\cdot M_{j+r-2}$, since each of these continua separates M_j from M_{j+r} in $\sum_{i=j}^{j+r} M_i$. It follows that H' contains $P_{j+2}, P_{j+3}, \cdots, P_{j+r-2}$, since for each integer s, P_{j+s} lies in a subcontinuum of $\sum_{i=j}^{j+r+1} L_i$ which is irreducible from $L_{j+s-1}\cdot L_{j+s}$ to $L_{j+s}\cdot L_{j+s+1}$. Then H' contains each of the continua $M_{j+2}, M_{j+3}, \cdots, M_{j+r-3}$; since each of these continua is irreducible between two points of $\sum_{i=j+1}^{j+r-2} P_i$. Since r can be taken arbitrarily large, it follows that H' contains M_{j+n} for each integer n greater than 1.

Consider the continuum $C=\overline{\sum_{i=j+2}^{\infty} M_i}$. It is obvious that $T(C)$ is a subset of C. It follows that C is identical with M, for otherwise C contains a proper subcontinuum of M which is carried into itself by T, contrary to hypothesis. But by the argument given above C is a subset of H; therefore H is identical with M. This contradicts the assumption that H is a proper subcontinuum of M. The lemma is therefore true.

3. **A general theorem on fixed points.** We may now prove the general theorem:

Theorem II. *If M is a compact continuum in a metric space, and if M does not contain an indecomposable continuum and does not contain a continuum which is the sum of two continua whose common part is disconnected, then every reversibly continuous transformation of M into a subset of itself leaves some point of M invariant.*

Proof. By Lemma 2, T carries some proper subcontinuum M_1 of M into itself. Let $M_1, M_2, \cdots, M_\omega, M_{\omega+1}, \cdots$ be a well ordered sequence β of subcontinua of M having the following properties: (1) for each ordinal number θ which has an immediate predecessor, M_θ is a proper subcontinuum of $M_{\theta-1}$ which is carried into itself by T; (2) for each ordinal number ϕ which has no immediate predecessor, M_ϕ is a proper subcontinuum of M_α which is carried into itself by T, where α is any ordinal which is less than ϕ; (3) if M_λ is any non-degenerate continuum which belongs to the sequence β, $M_{\lambda+1}$ is a proper subcontinuum of M_λ belonging to the sequence β. Since M is metric, it is completely separable, and it follows that the sequence β thus defined is countable. Therefore there exists a countable simple subsequence $M_{n1}, M_{n2}, \cdots$ running through β. That is, if M_λ is any element of β, there is an integer i such that M_{ni} is a subcontinuum of M_λ. Since M is compact, the point set $P=\prod_{i=1}^{\infty} M_{ni}$ is a compact continuum and is carried into itself by T, since each of the continua M_{ni} is carried into itself by T. But P cannot be a non-degenerate continuum, for if it is, it has a proper subcontinuum which belongs to the sequence β; and this contradicts the definitions of P and of the sequences β and M_{ni}. It follows that P is a point of M, invariant under T.

4. **Applications to continua in the plane.** We have now the following applications:

Theorem III. *If M is a compact continuum in the plane which contains no indecomposable continuum, which does not separate the plane, and which contains no domain, then every reversibly continuous transformation of M into a subset of itself leaves some point invariant.*

Proof. It follows from a theorem proved by S. Janiszewski* and also by Miss Mullikan† that a sufficient condition that a compact continuum M separate the plane is that M be the sum of two continua whose common

* S. Janiszewski, *Sur les coupures du plan faites par les continus*, Prace Matematyczno-fizyczne, vol. 26 (1913), pp. 11–63.

† Anna M. Mullikan, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

part is disconnected. Then if M does not separate the plane and contains no domain, no subcontinuum of M separates the plane, and the hypotheses of Theorem II are satisfied. The conclusions of Theorem III then follow.

THEOREM IV. *If D is a bounded simply connected domain in the plane which, together with its boundary, does not separate the plane and whose outer boundary M contains no indecomposable continuum, then every reversibly continuous transformation of $\bar{D}$ into itself leaves some point of $\bar{D}$ invariant.*

Proof. Without loss of generality we may suppose the boundary of D to be identical with its outer boundary, for the compact complementary domain of the outer boundary M of D is itself a simply connected domain which is a subset of $\bar{D}$ and whose boundary is M. Carathéodory* has shown that there exists a conformal and reversibly continuous transformation T_1 of D into the interior I of a given circle J, that there exists a reversibly one-to-one correspondence, also designated by T_1, between the prime ends of D and the points of J, and that this correspondence is characterized as follows: If $P_1, P_2, P_3, \cdots$ is a sequence of points of D converging to a prime end E_P of D (using Carathéodory's definition of convergence), then the sequence of points $T_1(P_1), T_1(P_2), T_1(P_3), \cdots$ converges (in the usual sense) to the point P of J with which E_P is associated by this correspondence; and conversely, if $Q_1, Q_2, Q_3, \cdots$ is a sequence of points of I converging to a point Q of J, then the sequence of points $T_1^{-1}(Q_1), T_1^{-1}(Q_2), T_1^{-1}(Q_3), \cdots$ converges to the prime end E_Q with which Q is associated by the correspondence T_1. Let T be any reversibly continuous transformation of $\bar{D}$ into itself. It can be easily shown that T carries a prime end of D into a prime end of D. Let T_2 be a transformation of the continuum $J+I$ into itself defined as follows: If P is any point of I, let $T_2(P)$ be the point $T_1[T(T_1^{-1}(P))]$. If P is any point of J, let $P_1, P_2, \cdots$ be a sequence of points of I converging to P. Let E_P be the prime end of D associated with P by T_1. Then the sequence $T_1^{-1}(P_1), T_1^{-1}(P_2), T_1^{-1}(P_3), \cdots$ converges to E_P. Since T is reversibly continuous, the sequence $T(T_1^{-1}(P_1)), T(T_1^{-1}(P_2)), \cdots$ converges to the prime end $T(E_P)$ into which E_P is transformed by T. It follows that the sequence of points of I, $T_1[T(T_1^{-1}(P_1))], T_1[T(T_1^{-1}(P_2))], \cdots$ converges to a point of J. That point is defined as $T_2(P)$. Since $T_1[T(T_1^{-1}(P_i))]$ is defined as $T_2(P_i)$, and since T is reversibly continuous on $\bar{D}$, and T_1 is a reversible continuous transformation of D into I, it follows that T_2 is reversibly continuous on I. Also from the definition of T_2 it follows that if $P_1, P_2, P_3, \cdots$ is a sequence of points of I converging to a point P of J, then the sequence $T_2(P_1), T_2(P_2), T_2(P_3), \cdots$

* C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1912), pp. 323–370.

converges to $T_2(P)$, and the sequence $T_2^{-1}(P_1)$, $T_2^{-1}(P_2)$, $T_2^{-1}(P_3)$, $\cdots$ converges to $T_2^{-1}(P)$. Suppose that $Q_1, Q_2, Q_3, \cdots$ is a sequence of points of J converging to a point Q of J. For each positive integer n, let Z_n be a point of I such that each of the distances $d(T_n, Q_n)$ and $d[T_2(Z_n), T_2(Q_n)]$ is less than $1/n$. That such a point Z_n exists is shown as follows: There exists a sequence of points $Z_{n1}, Z_{n2}, Z_{n3}, \cdots$ converging to Q_n. Then by the discussion above, the sequence $T_2(Z_{n1}), T_2(Z_{n2}), T_2(Z_{n3}), \cdots$ converges to $T_2(Q_n)$. There exists an integer i such that $d(Z_{ni}, Q_n) < 1/n$ and $d[T_2(Z_{ni}), T_2(Q_n)] < 1/n$. Then Z_{ni} has the property required of Z_n. We see then that the convergence of the sequence $Q_1, Q_2, \cdots$ to Q implies the convergence of the sequence $Z_1, Z_2, \cdots$ to Q. This in turn implies the convergence of $T_2(Z_1), T_2(Z_2), \cdots$ to $T_2(Q)$ and therefore the convergence of $T_2(Q_1), T_2(Q_2), \cdots$ to $T_2(Q)$. Similarly the sequence of points $T_2^{-1}(Q_1), T_2^{-1}(Q_2), \cdots$ converges to $T_2^{-1}(Q)$, and T_2 is a reversibly continuous transformation of the continuum $J+I$ into itself. By a fundamental theorem on fixed points, T_2 leaves some point of $J+I$ fixed. If T_2 leaves a point of I fixed, then obviously T leaves a point of D fixed. If T_2 leaves a point P of J fixed, then T carries some prime end E_P of D into itself. Let N_P be the continuum in the boundary of D associated with the prime end E_P in the sense that every sequence of points of D converging to E_P has a subsequence which converges in the usual sense to a point of N_P. Then T carries N_P into itself. N. E. Rutt* has shown that a necessary condition that such a continuum as N_P shall be the whole boundary of D is that the boundary of D shall be indecomposable or the sum of two indecomposable subcontinua. Since by hypothesis the boundary of D contains no indecomposable continuum, N_P is a proper subcontinuum of the boundary of D. N_P then is a compact continuum which does not separate the plane and which contains no domain. It satisfies the hypothesis of Theorem III; therefore T leaves some point of N_P fixed. Thus, in any case, T leaves a point of $\bar{D}$ invariant.

5. **An example.** There exist continua which admit of no continuous transformation into themselves except the identity. We give below an example of a compact acyclic continuous curve in the plane having this property.

Let A_1 be any arc in the plane of length 1. Let A_2 be the sum of A_1 and two arcs of length $1/2^2$ having no point in common with each other or with A_1 except that each has one end point at the midpoint of A_1. For each integer n, let E_{n-1} designate the set of points of A_{n-1} of Menger order greater than two. Let A_n be the sum of A_{n-1} and n arcs of length $1/2^n$ having no point in common with each other or with A_{n-1} except that they all have in common one

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* N. E. Rutt, *Prime ends and indecomposability*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 265–273.

end point which is the midpoint of a component of $A_{n-1}-E_{n-1}$ whose length is not less than the length of any other component of $A_{n-1}-E_{n-1}$. Let $M=\sum_{r=1}^{\infty}A_r$. Then M is obviously an acyclic continuous curve. Let K be the set of points of M of Menger order greater than two. From the definition of M it follows that no two points of K are of the same Menger order and that K is everywhere dense in M. Let T be any continuous transformation of M into itself. T carries a point of given Menger order into a point of the same Menger order and therefore leaves each point of K fixed. Since K is everywhere dense in M, T leaves each point of M fixed.

It should be noted that M is homeomorphic with a proper subset of itself. E. W. Miller* has given an example of an acyclic continuous curve which is not homeomorphic with any proper subset of itself, but which however does not have the property of the example we have given, since his acyclic continuous curve contains an arc which contains no points of Menger order greater than two. I have not been able to find an example of an acyclic continuous curve in the plane whose only transformation into a subset of itself is the identity.

* E. W. Miller, *The Zarankiewicz problem*, Bulletin of the American Mathematical Society, vol. 38 (1932), pp. 831–834.

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