POLYNOMIAL APPROXIMATIONS FOR ELLIPTIC FUNCTIONS*

BY

E. T. BELL

1. **Equivalences modulo y^{r+1}.** The approximations to be obtained are with respect to k^2, where k is the modulus of the Jacobian elliptic functions. It will be convenient to use the arithmetical concept of congruence, applied to absolutely convergent power series in two independent variables x, y.

All the power series $F(x, y)$, $G(x, y)$, $H(x, y)$, $\cdots$, $P(x, y)$, $\cdots$ considered have a common maximum domain R different from $|xy| = 0$, of absolute convergence; that is, R is the only domain of absolute convergence, different from $|xy| = 0$, of all the series considered, containing R.

The series are given initially in the forms

$$F(x, y) = \sum_{s=0}^{\infty} x^s \left(\sum_{t=0}^{s} f_t(s) y^t \right), \cdots, P(x, y) = \sum_{s=0}^{\infty} x^s \left(\sum_{t=0}^{s} p_t(s) y^t \right), \cdots.$$

they may be rearranged with respect to ascending powers of y; thus, for example,

$$F(x, y) = \sum_{s=0}^{\infty} y^s F^{(s)}(x), \tag{1.1}$$

where

$$F^{(s)}(x) = \sum_{t=0}^{\infty} f_s(s + t) x^{s+t}.$$

Let r be a constant integer $\geqq 0$. Then (1.1) may be written as

$$F(x, y) = F_r(x, y) + y^{r+1} F^{(r+1)}(x, y), \tag{1.2}$$

in which

$$F_r(x, y) = \sum_{s=0}^{r} y^s F^{(s)}(x),$$

$$F^{(r+1)}(x, y) = \sum_{s=0}^{\infty} y^s F^{(r+s+1)}(x).$$

As a function of y, $F_r(x, y)$ is a polynomial of degree $\leqq r$. The degree of the lowest power of y occurring in $F^{(r+1)}(x, y)$ is $\geqq 0$. In analogy with arithmetic, we write (1.2) as

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$$F(x, y) \equiv F_r(x, y) \pmod{y^{r+1}}; \tag{1.3}$$

$F_r(x, y)$ is the *residue* of $F(x, y)$ modulo y^{r+1}. Such residues are the polynomials (in y) with which we shall be concerned. As in arithmetic, we might now pass from congruence as in (1.3) to equality between residue classes. It is more convenient however to proceed as follows: (1.3) is written

$$F(x, y) \sim F_r(x, y), \tag{1.4}$$

which is read, "$F(x, y)$ is *equivalent* to $F_r(x, y)$." The properties of this equivalence follow from (1.2) or (1.3), the latter of which has the usual properties of a congruence relation; the conclusions are restated in the form (1.4). The equivalence in (1.4) has the usual properties of an equivalence relation in algebra. In each of (1.5)–(1.7) the first relation implies the second:

$$\begin{aligned} aF(x, y) + bG(x, y) &= H(x, y), \\ aF_r(x, y) + bG_r(x, y) &\sim H_r(x, y), \end{aligned} \tag{1.5}$$

where a, b are constants;

$$\begin{aligned} F(x, y)G(x, y) &= H(x, y), \\ F_r(x, y)G_r(x, y) &\sim H_r(x, y); \end{aligned} \tag{1.6}$$

$$\begin{aligned} F(x, y)/G(x, y) &= H(x, y), \\ F_r(x, y)/G_r(x, y) &\sim H_r(x, y), \end{aligned} \tag{1.7}$$

provided $G^{(0)}(x)$, in the notation of (1.1), is not zero.

From (1.5)–(1.7) we obtain the equivalence corresponding to any rational relation between $F(x, y), G(x, y), \cdots, P(x, y), \cdots$. Irrational relations also occur, for example $(F(x, y))^\sigma$, written as $F^\sigma(x, y)$, where σ is a positive rational number, may be expanded in the form (1.1), say $F^\sigma(x, y) = P(x, y)$; we then have $F_r^\sigma(x, y) \sim P_r(x, y)$. If $\sigma < 0$, the equivalence holds provided $F^{(0)}(x) \neq 0$.

It will be seen that the residues $F_r(x, y), \cdots$ obtained from elliptic functions can be constructed entirely by finite processes for $r = 0, 1, 2, \cdots$, so that we are operating essentially in the finite domain. The elliptic functions are the infinite series, as $r \to \infty$ through integer values, of the polynomials.

The functions and polynomials introduced in the following section are indispensable for our purpose; they do not seem to have been noticed before.

2. **Functions** U, V; **polynomials** A, B, C, D. Let a be a non-negative integer. The functions U, V are defined by

$$U_0(x) = \sin x, \qquad U_a(x) = \sum_{s=0}^{\infty} (-1)^s s^a \frac{x^{2s+1}}{(2s+1)!}, \qquad a > 0; \tag{2.1}$$

$$(2.2)\qquad V_0(x) = \cos x, \qquad V_a(x) = \sum_{s=0}^{\infty} (-1)^s s^a \frac{x^{2s}}{(2s)!}, \qquad a > 0.$$

If Δ is the operator of finite differences, and m, n are integers >0, we define $v_n(m)$ by

$$(2.3)\qquad m! v_n(m) = (\Delta^m x^n)_{x=0} = \sum_{i=0}^{m} (-1)^i (m, i)(m-i)^n,$$

where (m, i) is the binomial coefficient $m!/i!(m-i)!$. For $m \leqq n$, $v_n(m)$ is a positive integer; $v_n(m)=0$ for $m>n$. We shall write

$$s^{[0]} = 1, \; s^{[m]} = s(s-1)\cdots(s-m+1);$$

$$(2.4)\quad u_a(0) = 1; u_a(i) = 0, i > a; u_a(i) = \sum_{j=0}^{a-i} (-1)^j (a, i+j) v_{i+j}(i), 0 < i \leqq a.$$

The $u_a(i)$ are further considered in (2.27)–(2.30).

From the binomial expansion of $(-1)^a 2^a s^a = [1-(2s+1)]^a$, (2.1), (2.4), and the known expansion

$$(2.5)\qquad s^a = \sum_{i=1}^{a} v_a(i) s^{[i]}, \qquad a > 0,$$

we obtain

$$(-1)^a 2^a U_a(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+1}}{(2s+1)!} \left(1 + \sum_{i=1}^{a} (-1)^i u_a(i)(2s+1)^{[i]}\right).$$

If D_x^i denotes the operator d^i/dx^i, then

$$x^i D_x^i \sin x = \sum_{s=0}^{\infty} (-1)^s (2s+1)^{[i]} \frac{x^{2s+1}}{(2s+1)!};$$

hence, if $a>0$,

$$(-1)^a 2^a U_a(x) = \sin x + \sum_{i=1}^{a} (-1)^i u_a(i) x^i D_x^i \sin x;$$

therefore, if A, B are defined by

$$(2.6)\qquad A_a(x) = \sum_{i=0}^{\leqq a/2} (-1)^i u_a(2i) x^{2i}, \qquad a > 0,$$

$$(2.7)\qquad B_a(x) = \sum_{i=0}^{\leqq (a-1)/2} (-1)^i u_a(2i+1) x^{2i+1}, \qquad a > 0,$$

we have

$$(2.8)\qquad (-1)^a 2^a U_a(x) = A_a(x)\sin x - B_a(x)\cos x, \qquad a > 0.$$

To (2.6), (2.7) we add the initial values

$$(2.9)\qquad A_0(x) = 1, \qquad B_0(x) = 0,$$

so that (2.8) holds for $a \geqq 0$.

The next equations follow similarly from (2.2) and

$$(2s)^a = \sum_{i=1}^{a} v_a(i)(2s)^{[i]}, \qquad a > 0:$$

$$(2.10)\qquad 2^a V_a(x) = C_a(x)\sin x + D_a(x)\cos x, \qquad a \geqq 0;$$

$$(2.11)\qquad C_0(x) = 0, \qquad C_1(x) = -x, \qquad D_0(x) = 1, \qquad D_1(x) = 0;$$

$$(2.12)\qquad C_a(x) = \sum_{i=1}^{\leqq (a+1)/2} (-1)^i v_a(2i-1)x^{2i-1}, \qquad a > 0;$$

$$(2.13)\qquad D_a(x) = \sum_{i=1}^{\leqq a/2} (-1)^i v_a(2i)x^{2i}, \qquad a > 1.$$

The values of the polynomials for $a=0, 1, \cdots$ can be calculated directly from (2.7), (2.9), (2.11)–(2.13), but it is easier to obtain them by recurrence. Where the argument x is understood, we shall suppress it, and write $U_a, V_a, \cdots, D_a$ for $U_a(x), \cdots, D_a(x)$. Primes indicate derivatives with respect to x. From (2.1), (2.2) we have at once

$$(2.14)\qquad 2U_{a+1} = xU_a' - U_a, \qquad 2V_{a+1} = xV_a', \qquad a \geqq 0;$$

and by reducing $U_a + xV_a$,

$$(2.15)\qquad 2U_{a+1} + U_a - xV_a = 0, \qquad a \geqq 0.$$

Combining (2.8) with the first of (2.14), and (2.10) with the second, we find that

$$(2.16)\qquad A_{a+1} = A_a - xA_a' - xB_a, \qquad B_{a+1} = B_a - xB_a' + xA_a;$$

$$(2.17)\qquad C_{a+1} = x(C_a' - D_a), \qquad D_{a+1} = x(C_a + D_a').$$

Similarly, from $U_a' = V_a$ we get

$$(2.18)\qquad C_a = (-1)^a(A_a' + B_a), \qquad D_a = (-1)^a(A_a - B_a');$$

and from (2.15),

$$(2.19)\qquad A_{a+1} - A_a + (-1)^a xC_a = 0, \qquad B_{a+1} - B_a - (-1)^a xD_a = 0.$$

The last give

$$\begin{aligned} A_a &= 1 + (-1)^a x(C_{a-1} - C_{a-2} + C_{a-3} - \cdots + (-1)^{a-1}C_0), \\ B &= -(-1)^a x(D_{a-1} - D_{a-2} + D_{a-3} - \cdots + (-1)^{a-1}D_0), \end{aligned} \tag{2.20}$$

for $a>0$, or for $a\geqq 0$ if we define $C_s=D_s=0$ for $s<0$.

The polynomials satisfy no linear differential equation of order independent of a. Elimination of A or of B from (2.16) gives

$$X_{a+2} - 3X_{a+1} + 2xX'_{a+1} + (x^2+2)X_a - 2xX'_a + x^2X''_a = 0, \tag{2.21}$$

satisfied by $X=A$, $X=B$; and in the same way

$$Y_{a+2} + Y_{a+1} - 2xY'_{a+1} + x^2Y_a + x^2Y''_a = 0, \tag{2.22}$$

satisfied by $Y=C$, $Y=D$.

The preceding relations give recurrences for the coefficients u, v in the polynomials. Thus, substituting from (2.12), (2.13) into (2.17) we get the recurrence, equivalent to that for the numbers $\Delta^m0^n/m!$,

$$v_{a+1}(i) - iv_a(i) - v_a(i-1) = 0; \tag{2.23}$$

and similarly from (2.16), (2.6), (2.7),

$$u_{a+1}(i) + (i-1)u_a(i) - u_a(i-1) = 0, \qquad a>0. \tag{2.24}$$

The u are expressed in terms of the v from (2.18),

$$iu_a(i) - u_a(i-1) = (-1)^{a+i}v_a(i-1), \qquad a>0; \tag{2.25}$$

or from (2.20),

$$u_a(i) = (-1)^a \sum_{j=1}^{\leqq a-i+1} (-1)^j v_{a-j}(i-1), \qquad 1<i\leqq a. \tag{2.26}$$

For i fixed, (2.24) is in the standard form of the linear difference equation of the first order, with the restriction that $u_0(i)$ is not defined. Solving (2.24) we find

$$u_{a+1}(i) = (1-i)^a\left[\frac{u_2(i)}{1-i} + \sum_{j=2}^{a}\frac{u_j(i-1)}{(1-i)^j}\right], \qquad a>1, \qquad i>1;$$

and hence, since $u_2(2)=1$, $u_2(i)=0$, $i>2$, we have

$$u_{a+1}(2) = (-1)^a\left[-1+\sum_{j=2}^{a}(-1)^j\right] = \tfrac{1}{2}[1+(-1)^{a+1}], \qquad a>1,$$

which also is seen directly to hold for $a=0, 1$, and

$$u_{a+1}(i) = (1-i)^a\sum_{j=2}^{a}\frac{u_j(i-1)}{(1-i)^j}, \qquad a>1, \qquad i>2. \tag{2.27}$$

At once from (2.24),

$$u_a(0) = u_a(1) = 1. \tag{2.28}$$

Taking $i=2$ in (2.27), and noting that $u_1(3)=0$, we get

$$6u_a(3) = -(-1)^a[2^a - (-1)^a - 3], \qquad a > 0; \tag{2.29}$$

whence, with $i=4$ in (2.27),

$$24u_a(4) = (-1)^a[3^a - 2^{a+2} + (-1)^a + 6], \qquad a > 0, \tag{2.30}$$

and so on.

The polynomials are easily calculated recursively from (2.11), (2.16)–(2.19). The first 9 in each set follow. The argument in all is x. These suffice for obtaining approximations to the Jacobian elliptic functions with modulus k up to terms of order k^{16}.

$$\begin{array}{ll}
A_0 = 1 & B_0 = 0 \\
A_1 = 1 & B_1 = x \\
A_2 = 1 - x^2 & B_2 = x \\
A_3 = 1 & B_3 = x - x^3 \\
A_4 = 1 - x^2 + x^4 & B_4 = x + 2x^3 \\
A_5 = 1 - 5x^4 & B_5 = x - 5x^3 + x^5 \\
A_6 = 1 - x^2 + 20x^4 - x^6 & B_6 = x + 10x^3 - 9x^5 \\
A_7 = 1 - 70x^4 + 14x^6 & B_7 = x - 21x^3 + 56x^5 - x^7 \\
A_8 = 1 - x^2 + 231x^4 - 126x^6 + x^8 & B_8 = x + 42x^3 - 294x^5 + 20x^7
\end{array}$$

$$\begin{array}{ll}
C_0 = 0 & D_0 = 1 \\
C_1 = -x & D_1 = 0 \\
C_2 = -x & D_2 = -x^2 \\
C_3 = -x + x^3 & D_3 = -3x^2 \\
C_4 = -x + 6x^3 & D_4 = -7x^2 + x^4 \\
C_5 = -x + 25x^3 - x^5 & D_5 = -15x^2 + 10x^4 \\
C_6 = -x + 90x^3 - 15x^5 & D_6 = -31x^2 + 65x^4 - x^6 \\
C_7 = -x + 301x^3 - 140x^5 + x^7 & D_7 = -63x^2 + 350x^4 - 21x^6 \\
C_8 = -x + 966x^3 - 1050x^5 + 28x^7 & D_8 = -127x^2 + 1701x^4 - 266x^6 + x^8
\end{array}$$

Although they will not be required here, it may be mentioned that generalizations of $U_a(x)$, $V_a(x)$ to non-integral values of a, by means of contour integrals, have been investigated by Professor H. Bateman. These lead to expansions of the functions (for any a) in terms of Bessel functions; such expansions offer a point of departure for approximating the values of the polynomials in $y(=k^2)$ next obtained as approximations in y to $\mathrm{sn}(x, k)$, $\mathrm{cn}(x, k)$,

$\mathrm{dn}(x, k)$, for large values of the x occurring in the coefficients of the polynomials.

3. **The polynomials** $\mathrm{sn}_r x$. The modulus of sn x being k, we write $\mathrm{sn}(x, k) = \mathrm{sn}(x|k^2)$, in Milne-Thomson's notation,* as it is usually k^2, not k, that is given in applications, and only even powers of k appear in the power series expansion of $\mathrm{sn}(x, k)$. Finally we write $k^2=y$, $1-y=y_1=k'^2$, and consider $\mathrm{sn}(x, k) = \mathrm{sn}(x|y) = \mathrm{sn}\, x$ as a function of x, y. Similarly for cn x, dn x. For the meaning of $\mathrm{sn}_r x, \cdots$, see §1. We shall use the forms of the power series obtained in a previous paper.† For sn x we found

$$\mathrm{sn} x = \sum_{s=0}^{\infty} \left[\sum_{r=0}^{s} p_r(s) y^r \right] \frac{(-1)^s x^{2s+1}}{(2s+1)!} \cdot$$

$$\begin{aligned} p_0(s) &= 1, \ 2^4 p_1(s) = 3^{2s+1} - 8s - 3, \\ 2^8 p_2(s) &= 5^{2s+1} - 4(2s-1)3^{2s+1} + 32s^2 - 32s - 17, \\ 2^{12} p_3(s) &= 7^{2s+1} - 4(2s-3)5^{2s+1} + (32s^2 - 88s + 30)3^{2s+1} \\ &\quad - \tfrac{1}{3}(256s^3 - 1056s^2 + 752s + 471), \end{aligned}$$

the general $p_j(s)$ being of the form

$$(3.1) \qquad 2^{4j} p_j(s) = (2j+1)^{2s+1} + P_{j1}(s)(2j-1)^{2s+1} + \cdots + P_{jj}(s)1^{2s+1},$$

in which $P_{jt}(s)$ is a polynomial in s of degree t with rational coefficients. Moreover it was shown (loc. cit., p. 846, (8)) that the $p_j(s)$ can be calculated by linear recurrence, and in a subsequent paper‡ numerous linear recurrences were given for the calculation of the coefficients appearing in the recurrences for the $p_j(s)$. We may therefore consider the $p_j(s)$ for $j=0, 1, 2, \cdots$ known, as it is straight-forward elementary algebra to obtain a particular $p_j(s)$ by the means indicated.

For the meaning of $\mathrm{sn}_r x$ see (1.2). The sx occurring in the sin sx, cos sx appearing in $\mathrm{sn}_r x$ expresses a number of radians. We have

$$(3.2) \qquad \mathrm{sn}_0\, x = \sum_{s=0}^{\infty} p_0(s) \frac{(-1)^s x^{2s+1}}{(2s+1)!} = \sin x;$$

$$\mathrm{sn}_1\, x = \sum_{s=0}^{\infty} [p_0(s) + p_1(s) y] \frac{(-1)^s x^{2s+1}}{(2s+1)!} \cdot$$

Reducing the coefficient of y in the last we get (see §2)

* L. M. Milne-Thomson, *Die elliptischen Funktionen von Jacobi* (5-figure tables), Berlin, 1931.

† These Transactions, vol. 36 (1934), pp. 841–852, in which note the following misprints: p. 842 (1), for x^2 read x^{2s}; p. 843, last line, for 2^{2s} read 3^{2s}; p. 844, in the expression for $q_3(s)$, for -297 read $+297$; p. 844 (7), all exponents on the right should be $2s$, not s.

‡ American Journal of Mathematics, vol. 48 (1936), pp. 759–768.

$$
\begin{aligned}
2^4\sum_{s=0}^{\infty} p_1(s)\frac{(-1)^s x^{2s+1}}{(2s+1)!} &= \sum_{s=0}^{\infty}(3^{2s+1}-8s-3)\frac{(-1)^s x^{2s+1}}{(2s+1)!}\\
&= \sin 3x - 8U_1(x) - 3\sin x\\
&= \sin 3x - 3\sin x + 4(A_1\sin x - B_1\cos x)\\
&= \sin 3x + \sin x - 4x\cos x;
\end{aligned}
$$

$$
\text{sn}_1\, x = \sin x + 2^{-4}y(\sin 3x + \sin x - 4x\cos x). \tag{3.21}
$$

Proceeding to $\text{sn}_2 x$ we have the new term in y^2. The coefficient of $2^{-8}y^2$ is

$$
\begin{aligned}
&\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s+1}}{(2s+1)!}\left[5^{2s+1}-(8s-4)3^{2s+1}+32s^2-32s-17\right],\\
&\qquad = \sin 5x + 4\sin 3x - 17\sin x - 8U_1(3x) + 32U_2(x) - 32U_1(x).
\end{aligned}
$$

The new detail $U_1(3x)$ is typical of the like in subsequent calculations. By (2.8),

$$
\begin{aligned}
-8U_1(3x) &= 4\left[A_1(3x)\sin 3x - B_1(3x)\cos 3x\right],\\
&= 4(\sin 3x - 3x\cos 3x).
\end{aligned}
$$

The remaining terms are evaluated as before, and we get

$$
\begin{aligned}
\text{sn}_2\, x &= \sin x + 2^{-4}y(\sin 3x + \sin x - 4x\cos x)\\
&\quad + 2^{-8}y^2\left[\sin 5x + 8\sin 3x + (7-8x^2)\sin x - 12\cos 3x - 24x\cos x\right].
\end{aligned} \tag{3.3}
$$

These are enough to show the process.

By (1.2) and the cited expansion of sn x we have, for $r>0$,

$$
\text{sn}_r\, x = \text{sn}_{r-1}\, x + y^r\sum_{s=0}^{\infty} p_r(s)(-1)^s\frac{x^{2s+1}}{(2s+1)!},
$$

and hence by (3.1) the coefficient of $2^{-4r}y^r$ in $\text{sn}_r\, x$ is

$$
\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s+1}}{(2s+1)!}\left[(2r+1)^{2s+1}+\sum_{t=1}^{r}P_{rt}(s)(2r+1-2t)^{2s+1}\right], \tag{3.4}
$$

in which

$$
P_{rt}(s) = p_{t0}(r)s^t + p_{t1}(r)s^{t-1} + \cdots + p_{tt}(r), \qquad t>0, \tag{3.5}
$$

the $p_{tj}(r)$ being rational numbers; by convention we take $p_{00}(r)=1$. From this we can determine the general form of $\text{sn}_r\, x$. The result of substituting from (3.5) into (3.4) and reducing by (2.1) is

$$
\sin(2r+1)x + \sum_{t=1}^{r}\sum_{j=0}^{t} p_{tj}(r)U_{t-j}((2r+1-2t)x). \tag{3.6}
$$

To express this in the same form as (3.3) we apply (2.8), noting that $p_{00}(r)=1$, and defining the polynomials α, β by

$$\alpha_t(r,\,x) = \sum_{j=0}^{t}(-\,2)^{j-t}p_{tj}(r)A_{t-j}((2r+1-2t)x),$$

$$\beta_t(r,\,x) = \sum_{j=0}^{t}(-\,2)^{j-t}p_{tj}(r)B_{t-j}((2r+1-2t)x), \qquad t \geqq 0.$$

Then the whole expression in (3.6) becomes

$$\sum_{t=0}^{r}\left[\alpha_t(r,\,x)\sin\,(2r+1-2t)x - \beta_t(r,\,x)\cos\,(2r+1-2t)x\right].$$

Hence finally, for $r>0$,

$$\begin{aligned}\operatorname{sn}_r x = \sin x + \sum_{i=1}^{r} 2^{-4i}y^i\sum_{t=0}^{i}\big[&\alpha_t(i,\,x)\sin\,(2r+1-2t)x \\ &- \beta_t(i,\,x)\cos\,(2i+1-2t)x\big].\end{aligned} \tag{3.7}$$

For a given r, and t an integer >0, $\operatorname{sn}_r{}^t\, x$ can be calculated directly from $\operatorname{sn}_r x$. Powers of sines and cosines are expressed as sums of sines or cosines of multiple angles. For $t=r=2$ the result is, by (3.3),

$$\begin{aligned}\operatorname{sn}_2^2 x = {} & 2^{-1}(1-\cos 2x) + 2^{-4}y(1-\cos 4x - 4x\sin 2x) \\ & + 2^{-9}y^2[16 + (3+32x^2)\cos 2x - 16\cos 4x - 3\cos 6x \\ & - 32x\sin 4x - 40x\sin 2x].\end{aligned} \tag{3.8}$$

4. **The polynomials** $\operatorname{cn}_r x$. The preliminary expansion is

$$\operatorname{cn} x = 1 + \sum_{s=1}^{\infty}\left[\sum_{r=0}^{s-1}q_r(s)y^r\right]\frac{(-\,1)^s x^{2s}}{(2s)!},$$

$$q_0(s) = 1, \qquad 2^4q_1(s) = 3^{2s} - 8s - 1,$$

$$2^8q_2(s) = 5^{2s} - 8(s-1)3^{2s} + 32s^2 - 48s - 9,$$

$$\begin{aligned}2^{12}q_3(s) = {} & 7^{2s} - 8(s-2)5^{2s} + 2(16s^2 - 60s + 41)3^{2s} \\ & - \tfrac{1}{3}(256s^3 - 1248s^2 + 1280s + 297).\end{aligned}$$

Corresponding to (3.1), (3.5) we have

$$2^{4r}q_r(s) = (2r+1)^{2s} + Q_{r1}(s)(2r-1)^{2s} + \cdots + Q_{rr}(s)1^{2s}, \qquad r > 0,$$

$$Q_{rt}(s) = q_{t0}(r)s^t + q_{t1}(r)s^{t-1} + \cdots + q_{tt}(r),$$

the $q_{tj}(r)$ being rational numbers, and $q_{00}(r)=1$ by convention.

Proceeding as in §3 we find

$$\begin{aligned}\operatorname{cn}_2 x = {} & \cos x + 2^{-4}y(\cos 3x - \cos x + 4x \sin x) \\ & + 2^{-8}y^2[\cos 5x + 8 \cos 3x - (9 + 8x^2) \cos x \\ & + 12x \sin 3x + 16x \sin x].\end{aligned} \tag{4.1}$$

This may also be obtained (see §1) from $\operatorname{cn}_r^2 x + \operatorname{sn}_r^2 x \sim 1$; whence

$$\operatorname{cn}_2 x = (1 - \operatorname{sn}_2^2 x)^{1/2} = 1 - \tfrac{1}{2} \operatorname{sn}_2^2 x - \tfrac{1}{8} \operatorname{sn}_2^4 x - \cdots .$$

The value of $\operatorname{cn}_2^2 x$ can be determined directly from (4.1), as for $\operatorname{sn}_2^2 x$ in §3. The result is (as it should be) $1 - \operatorname{sn}_2^2 x$, where $\operatorname{sn}_2^2 x$ is given in (3.8). To indicate the general form corresponding to (3.6), we define the polynomials γ, δ:

$$\gamma_t(r, x) = \sum_{j=0}^{t} 2^{j-t} q_{tj}(x) C_{t-j}((2r + 1 - 2t)x),$$

$$\delta_t(r, x) = \sum_{j=0}^{t} 2^{j-t} q_{tj}(x) D_{t-j}((2r + 1 - 2t)x), \qquad t \geqq 0,$$

and find, for $r > 0$,

$$\begin{aligned}\operatorname{cn}_r x = \cos x + \sum_{i=1}^{r} 2^{-4i} y^i \sum_{t=0}^{i} [\gamma_t(i, x) \sin (2i + 1 - 2t)x \\ + \delta_t(i, x) \cos (2i + 1 - 2t)x].\end{aligned} \tag{4.2}$$

In getting (4.2) we have used the second of the following formulas, which follows from the first,

$$\operatorname{cn}_r^2 x + \operatorname{sn}_r^2 x \sim 1, \quad \sum_{t=0}^{r} q_{tt}(r) = 0, \qquad r > 0.$$

5. **The polynomials** $\operatorname{dn}_r x$. The preliminary expansion is

$$\operatorname{dn} x = 1 + \sum_{s=1}^{\infty} \left[\sum_{r=0}^{s-1} h_r(s) y^{r+1} \right] \frac{(-1)^s x^{2s}}{(2s)!},$$

$$2^2 h_0(s) = 2^{2s}, \qquad 2^6 h_1(s) = 2^{2s}(2^{2s} - 8s + 4),$$

$$2^{10} h_2(s) = 2^{2s}[3^{2s} - 4(2s - 3)2^{2s} + 32s^2 - 88s + 31],$$

the general coefficient being of the form

$$h_r(s) = 2^{2s-4r-2}[(r + 1)^{2s} + H_{r1}(s) r^{2s} + \cdots + H_{rr}(s) 1^{2s}],$$

$$H_{rt}(s) = h_{t0}(r) s^t + h_{t1}(r) s^{t-1} + \cdots + h_{tt}(r);$$

the $h_{tj}(r)$ are rational numbers, and by convention $h_{00}(r) = 1$. As before we get

$$\begin{aligned}\operatorname{dn}_2 x = {} & 1 - 2^{-2}y(1 - \cos 2x) \\ & + 2^{-6}y^2(-5 + 4 \cos 2x + \cos 4x + 8x \sin 2x).\end{aligned} \tag{5.1}$$

The identity $\mathrm{dn}^2 x = 1 - y\,\mathrm{sn}^2 x$ gives

$$\mathrm{dn}_r^2\, x \sim 1 - y\,\mathrm{sn}_r^2\, x;$$

hence in particular, by (3.8),

$$\mathrm{dn}_2^2\, x = 1 - 2^{-1}y(1 - \cos 2x) + 2^{-4}y^2(-1 + \cos 4x + 4x \sin 2x). \tag{5.2}$$

In the same way as before we find the general form,

$$\mathrm{dn}_0\, x = 1, \qquad \mathrm{dn}_1\, x = 1 + 2^{-2}y(-1 + \cos 2x), \tag{5.3}$$

with ϵ, ζ defined by

$$\epsilon_t(r, x) = \sum_{j=0}^{t} 2^{j-t} h_{tj}(r) C_{t-j}(2(r + 1 - t)x),$$

$$\zeta_t(r, x) = \sum_{j=0}^{t} 2^{j-t} h_{tj}(r) D_{t-j}(2(r + 1 - t)x), \qquad t \geqq 0;$$

and, for $r>0$,

$$\begin{aligned} \mathrm{dn}_{r+1}\, x = {} & 1 + 2^{-2}y(-1 + \cos 2x) \\ & + \sum_{i=1}^{r} 2^{-4i-2}y^{i+1} \sum_{t=0}^{i} [-h_{tt}(i) + \epsilon_t(i, x) \sin (2(i + 1 - t)x) \\ & + \zeta_t(i, x) \cos (2(i + 1 - t)x)]. \end{aligned} \tag{5.4}$$

The approximations in this and preceding sections give approximations to the standard elliptic integrals. Thus from

$$E(x) = \int_0^x \mathrm{dn}^2\, x\, dx$$

and (5.2) we get

$$E_2(x) = x - 2^{-1}y(x - \sin x) - 2^{-6}y^2(4x - 4 \sin 2x - \sin 4x + 8x \cos 2x)$$

for the elliptic integral of the second kind.

CALIFORNIA INSTITUTE OF TECHNOLOGY,
PASADENA, CALIF.