# GEOMETRY IN AN $n$-DIMENSIONAL SPACE WITH THE ARC LENGTH

$$s = \int \{A_i(x, x')x''^i + B(x, x')\}^{1/p} dt^*$$

BY

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Geometry in the manifold $K_n^{(m)}$ of line elements of a higher order, that is, $(x, dx, \cdots, d^m x)$ or $(x, x', \cdots, x^{(m)})$, in a space of $n$ dimensions with a point coordinate system $x^i$, $(i = 1, 2, \cdots, n)$, was first furnished with its foundation by the present author.† In close connection with this theory, there are two important problems. The first is the geometry of paths in this space, that is, the geometrical study of the system of paths $x^i = x^i(t)$ which are defined by the ordinary differential equations of the form

$$x^{(m+1)i} + \Gamma^i(t, x, x', \cdots, x^{(m)}) = 0,$$

where $x^{(\lambda)i}$ denotes $d^\lambda x^i/dt^\lambda$ and $t$ is a parameter. The second is a case of metric geometry in this space, that is, the study of the geometrical properties characterized by a metric which gives as the arc length $s$ of a curve $x^i = x^i(t)$

$$s = \int F(t, x, x', \cdots, x^{(m)}) dt.$$

The first problem is a generalization of the so-called geometry of paths in a space with a generalized affine connection‡ $x''^i + \Gamma^i(t, x, x') = 0$, and was

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† A. Kawaguchi, *Die Differentialgeometrie in der verallgemeinerten Mannigfaltigkeit*, Rendiconti del Circolo Matematico di Palermo, vol. 56 (1932), pp. 245–276; *Theory of connections in the generalized Finsler manifold*, Proceedings of the Imperial Academy, vol. 7 (1931), pp. 211–214; ibid., vol. 8 (1932), pp. 340–343; ibid., vol. 9 (1933), pp. 347–350.

‡ L. Berwald, *Untersuchung über die Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Mathematische Zeitschrift, vol. 25 (1926), pp. 40–73; J. Douglas, *The general geometry of paths*, Annals of Mathematics, (2), vol. 29 (1927), pp. 143–168; D. D. Kosambi, *Parallelism and path-space*, Mathematische Zeitschrift, vol. 37 (1933), pp. 608–618, and E. Cartan, *Observations sur le memoire précédent*, ibid., pp. 619–622; D. D. Kosambi, *Systems of differential equations of the second order*, Quarterly Journal of Mathematics (Oxford), vol. 6 (1935), pp. 1–12; W. Ślebodziński, *Sur deux connections affines généralisées*, Prace Matematyczno-Fizyczne, vol. 43 (1935), pp. 1–39.

first propounded by D. D. Kosambi.* Although he derived many interesting results, his theory seems to be systematically not yet complete, and there are many irregularities. The present writer and H. Hombu have reconstructed this theory from another and more general standpoint, by which the irregularities of the theory of Kosambi exhibit themselves.†

On the other hand, the second problem is a generalization of a Finsler space and there have been many attempts to solve it,‡ but it remains yet unsolved. Recently E. Cartan§ has introduced a connection, which is related intrinsically with an integral $\int F(x, y, y', y'')dx$, in a plane under the group of all contact transformations by use of several invariant Pfaffians and has discussed in detail a special case where the integral has the form $\int (Ay''+B)^p dx$. This theory of Cartan has been extended by H. Hombu‖ to the case where the integral possesses the form $\int F(x, y, y', y'', y''')dx$. These theories give a foundation to the invariant theory of the integral under contact transformations and take the first step towards the general problem. But unfortunately, by reason of its restricted method which is available only for a plane, these theories cannot be extended to an $n$-dimensional space as they stand. Furthermore it is more desirable to set up the theory under the group of all point transformations rather than contact transformations.

In the present work it is proposed to establish the foundation of the geometry in an $n$-dimensional space with arc length given by the integral $\int \{A_i(x, x')x''^i+B(x, x')\}^{1/p}dt$, by introducing several connections under the group of all point transformations. Although the integral has a special form, this would be the second step towards the general problem; and in this special case many concrete results are obtainable, which may be of interest. The integrand does not contain the parameter $t$ explicitly, and I shall investigate mainly only those properties that are invariant under any (analytic) transformation of the parameter $t$, for the reason that only these properties have geometrical meaning, that is, are related intrinsically to the base curves. This

* D. D. Kosambi, *Path-spaces of higher order*, Quarterly Journal of Mathematics (Oxford), vol. 7 (1936), pp. 97–104.

† A. Kawaguchi and H. Hombu, *Die Geometrie des Systems der partiellen Differentialgleichungen höherer Ordnung*, Journal of the Faculty of Science, Hokkaido Imperial University, vol. 6 (1937), pp. 21–62.

‡ H. V. Craig, *On a generalized tangent vector*, American Journal of Mathematics, vol. 57 (1935), pp. 457–462; J. L. Synge, *Some intrinsic and derived vectors in a Kawaguchi space*, ibid., pp. 679–691; A. Kawaguchi, *Some intrinsic derivations in a generalized space*, Proceedings of the Imperial Academy, vol. 12 (1936), pp. 149–151; *Certain identities in a generalized space*, ibid., pp. 152–155.

§ E. Cartan, *La géométrie de l'intégrale* $\int F(x, y, y', y'')dx$, Journal de Mathématiques, (9), vol. 15 (1936), pp. 42–69.

‖ H. Hombu, *Invariantentheorie des Integrals* $\int F(x, y, y', y'', y''')dx$, Proceedings of the Imperial Academy, vol. 12 (1936), pp. 156–158; *Geometrie des Integrals* $\int (Ly'''+M)dx$, ibid., pp. 159–161.

restriction makes the theory somewhat complicated. The theory can be generalized without difficulty to the case where the integral is of the form

$$\int \{A_i(x, x', x'')x'''^i + B(x, x', x'')\}^{1/p}dt^*$$

or

$$\int \{a_i(x, x')a_j(x, x')x''^i x''^j + 2b(x, x')a_i x''^i + c(x, x')\}^{1/p}dt.\dagger$$

Chapter 1 is devoted to discussion of the general case, where $p \neq 3/2$. In this case there exists a system of paths having the equations of the form $x''^i + \Gamma^i(x, x') = 0$, which leads to the definition not only of the base connection but also of a connection $C$, by recalling the geometry of paths in the manifold $K_n^{(1)}$. There is another connection $C'$ which is stated in Chapter 2 and which has a form similar to that of Cartan in a Finsler space.‡

## Chapter 1. Base connection and connection $C$

1. Let us consider an $n$-dimensional metric space such that the arc length of a curve $x^i = x^i(t)$ is given by the integral

$$s = \int \{A_i(x, x')x''^i + B(x, x')\}^{1/p}dt, \tag{1.1}$$

where $x'^i = dx^i/dt$, $x''^i = d^2x^i/dt^2$, and $A_i$ and $B$ are differentiable functions of $x^i$ and $x'^i$. In order that the arc length should be related intrinsically to the curve, that is, that it should remain unaltered by a transformation of the parameter $t$, we must have

$$A_i x'^i = 0, \tag{1.2}$$

$$2A_i x''^i + (A_{k(i)}x''^k + B_{(i)})x'^i = p(A_i x''^i + B), \tag{1.3}$$

where

$$A_{k(i)} = \frac{\partial A_k}{\partial x'^i}, \qquad B_{(i)} = \frac{\partial B}{\partial x'^i}\cdot$$

We shall denote partial differentiation by $x'^i$ with $(i)$ and that by $x^i$ with $i$ throughout this paper. Equation (1.3) implies

* T. Ohkubo, *Base connections in a special Kawaguchi space*, Journal of the Faculty of Science, Hokkaido Imperial University, vol. 5 (1936), pp. 167–188.

† H. Hokari, *Die Geometrie des Integrals* $\int(a_ia_jx''^ix''^j+2ba_ix''^i+c)^{1/p}dt$, Proceedings of the Imperial Academy, vol. 12 (1936), pp. 209–212.

‡ E. Cartan, *Les espaces de Finsler*, Actualités Scientifiques et Industrielles, vol. 79, 1934.

(1.4) $$A_{k(i)}x'^i = (p-2)A_k, \qquad B_{(i)}x'^i = pB.$$

Thus, the $A_i$ are homogeneous of degree $p-2$ with regard to the $x'^i$, and $B$ is homogeneous of degree $p$. From (1.2) it follows, on differentiating by $x'^j$, that

(1.5) $$A_{i(j)}x'^i = -A_j.$$

2. The point transformation $y^i = y^i(x^j)$ gives rise to the following relations:

(1.6) $$y'^r = P_s^r x'^s, \qquad y''^r = P_s^r x''^s + P_{sq}^r x'^s x'^q,$$

where

$$P_s^r = \frac{\partial y^r}{\partial x^s}, \qquad Q_s^r = \frac{\partial x^r}{\partial y^s} \qquad \text{and} \qquad P_{sq}^r = \frac{\partial^2 y^r}{\partial x^s \partial x^q}.$$

Since the arc length (1.1) is a scalar, it can be seen at once that $A_i$ is a vector. But $B$ is not a scalar and becomes, by the point transformation,

(1.7) $$B(y, y') = B(x, x') - A_q(x, x')Q_i^q P_{jk}^i x'^j x'^k.$$

Besides $A_i$ we get two other intrinsic vectors

(1.8) $$T_i = (A_{k(i)} - 2A_{i(k)})x''^k - 2A_{il}x'^l + B_{(i)},$$

(1.9) $$\begin{aligned} E_i = {} & (A_{i(k)} - A_{k(i)})x'''^k + (A_{i(k)(l)} - A_{k(i)(l)})x''^l x''^k \\ & + (A_{i(k)l}x'^l - A_{k(i)l}x'^l + A_{ki} + A_{ik} - B_{(i)(k)} + A_{ij(k)}x'^j)x''^k \\ & + A_{ijk}x'^j x'^k - B_{(i)l}x'^l + B_i. \end{aligned}$$

The vector $T_i$ is the Craig vector*

$$T_i = -2\frac{d}{dt}\frac{\partial F}{\partial x''^i} + \frac{\partial F}{\partial x'^i},$$

and $E_i$ is the Euler vector

$$E_i = \frac{d^2}{dt^2}\frac{\partial F}{\partial x''^i} - \frac{d}{dt}\frac{\partial F}{\partial x'^i} + \frac{\partial F}{\partial x^i}$$

of the function $F = A_i x''^i + B$. It can be verified easily that

(1.10) $$T_i x'^i = p(A_i x''^i + B), \qquad E_i x'^i = (1-p)\frac{d}{dt}(A_i x''^i + B).$$

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* See H. V. Craig, *On a generalized tangent vector*, loc. cit.

When $s$ stands for $t$, we get the normalized vectors $T_i^*$ and $E_i^*$ for $T_i$ and $E_i$. Then we have

$$T_i^* \dot{x}^i = p, \qquad E_i^* \dot{x}^i = 0, \tag{1.10'}$$

where the dot denotes differentiation by $s$. Among these vectors there are the relations

$$\begin{aligned} T_i^* &= T_i \left(\frac{dt}{ds}\right)^{p-1} - (2p-3)A_i \left(\frac{dt}{ds}\right)^{p-3} \frac{d^2t}{ds^2}, \\ E_i^* &= E_i \left(\frac{dt}{ds}\right)^{p} - (p-1)T_i \left(\frac{dt}{ds}\right)^{p-2} \frac{d^2t}{ds^2} \\ &\quad + (p-1)A_i \left\{(p-3)\left(\frac{d^2t}{ds^2}\right)^2 + \frac{dt}{ds}\frac{d^3t}{ds^3}\right\}\left(\frac{dt}{ds}\right)^{p-4}. \end{aligned} \tag{1.11}$$

3. If $2p=3$, the determinant of the tensor

$$G_{ik} = 2A_{i(k)} - A_{k(i)} \tag{1.12}$$

vanishes, identically, because

$$G_{ik}x'^k = 0.$$

Now we assume that $p \neq 3/2$, and that the determinant does not vanish identically; then we have from (1.8)

$$x^{[2]l} \equiv - T_i G^{il} = x''^l + 2\Gamma^l, \tag{1.13}$$

where

$$2\Gamma^l = (2A_{ik}x'^k - B_{(i)})G^{il} \qquad \text{and} \qquad G_{ik}G^{il} = \delta_k^l. \tag{1.14}$$

The functions $\Gamma^l$ are homogeneous of degree two with regard to the $x'^i$. As the relation $G_{ik}x'^k = (2p-3)A_i$ holds, the last equation of (1.14) shows that

$$A_i G^{il} = \frac{1}{2p-3} x'^l. \tag{1.15}$$

From $G_{ki}G^{kl} = \delta_i^l$ we get also

$$A_i G^{il} = -\frac{1}{p} x'^i, \tag{1.15'}$$

which leads to

$$F = A_i x^{[2]i}. \tag{1.16}$$

From (1.14) we obtain further

$$2\Gamma^l A_l = B. \tag{1.17}$$

Now we shall introduce, as in a Finsler space, the manifold $K_n^{(1)}$ of line elements $(x, x')$ in the space; and for this manifold (1.13) gives the base connection in our space.*

4. Let $v^i$ be a contravariant vector field in $K_n^{(1)}$, homogeneous of degree zero with regard to the $x'^j$, that is, independent of a choice of the parameter $t$. Then a new vector can be derived from $v^i$ in connection with the base connection:

$$\frac{\delta v^i}{dt} = \frac{dv^i}{dt} + \Gamma^i_{(j)} v^j, \dagger \tag{1.18}$$

which is an absolute derivative of the vector $v^i$ along the curve and is independent of transformation of $t$. From (1.18) we can define further the absolute differential corresponding to a displacement from a line element $(x, x')$ to a neighboring line element $(x+dx, x'+dx')$

$$\delta v^i = dv^i + \Gamma^i_{(j)(k)} v^j dx^k. \tag{1.19}$$

This absolute differential remains unaltered by any change of $t$. By any transformation of coordinates $\Gamma^i_{(j)(k)}$ are transformed in the same way as the parameters of an affine connection and are homogeneous of degree zero with regard to the $x'^h$, that is, they are invariant under transformation of $t$. For this reason the connection $C$, defined by (1.19), characterizes the geometrical properties of our space. It is not necessary, therefore, to take the line elements $(x, x')$ along any curve; they can be taken arbitrarily at any point.

On account of the base connection (1.13) we have for a displacement of a line element

$$\delta x'^i = dx'^i + \Gamma^i_{(j)} dx^j. \tag{1.20}$$

On account of this, we can write (1.19) in the form

$$\begin{aligned} \delta v^i &= \frac{\partial v^i}{\partial x^j} dx^j + \frac{\partial v^i}{\partial x'^j} dx'^j + \Gamma^i_{(k)(j)} v^k dx^j \\ &= dx^j \nabla_j v^i + \delta x'^j \nabla'_j v^i, \end{aligned} \tag{1.21}$$

where

* For the base connection reference may be made to A. Kawaguchi, *Die Differentialgeometrie in der verallgemeinerten Mannigfaltigkeit*, loc. cit.

† As $x^{[2]i}$ is a vector, we know from (1.13) that $\Gamma^i$ behave under a coordinate transformation in the same way as affine connection parameters $\gamma^i_{jk}$ multiplied by $x'^j x'^k$. From this fact it can be concluded that $\delta v^i/dt$ is a contravariant vector.

$$(1.22)\qquad \nabla_j v^i = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^k}\Gamma^k_{(j)} + \Gamma^i_{(k)(j)}v^k, \qquad \nabla'_j v^i = \frac{\partial v^i}{\partial x'^j},$$

which are defined as the covariant derivatives of a vector $v^i$. $\nabla'_j v^i$ are multiplied by a factor $dt/d\bar{t}$ for a change of the parameter $\bar{t}=\bar{t}(t)$. For a covariant vector or a tensor we can define covariant derivatives in the same manner as in usual tensor calculus, for example,

$$(1.23)\qquad \delta v_i = dv_i - \Gamma^j_{(i)(k)}v_j dx^k.$$

In particular we obtain for $A_i$

$$(1.24)\qquad \nabla_j A_i = A_{ij} - A_{i(k)}\Gamma^k_{(j)} - \Gamma^k_{(i)(j)}A_k, \qquad \nabla'_j A_i = A_{i(j)},$$

for which the following relations hold:

$$(1.25)\qquad x'^j\nabla_j A_i = 0, \qquad x'^i\nabla_j A_i = 0, \qquad \frac{\delta A_i}{dt} = A_{i(j)}x^{[2]j}.$$

These can be verified without difficulty from (1.4), (1.5), and (1.14).

The covariant differential of a relative tensor $\mathfrak{v}^i$ of weight $k$ is also given as follows:

$$(1.26)\qquad \delta\mathfrak{v}^i = d\mathfrak{v}^i + \Gamma^i_{(j)(k)}\mathfrak{v}^j dx^k - k\mathfrak{v}^i\Gamma_{(j)}dx^j,$$

where $\Gamma_{(j)} \equiv \Gamma^t_{(t)(j)}$. Hence

$$(1.27)\qquad \begin{aligned} \nabla_j\mathfrak{v}^i &= \frac{\partial\mathfrak{v}^i}{\partial x^j} - \frac{\partial\mathfrak{v}^i}{\partial x'^h}\Gamma^h_{(j)} + \Gamma^i_{(h)(j)}\mathfrak{v}^h - k\mathfrak{v}^i\Gamma_{(j)}, \\ \nabla'_j\mathfrak{v}^i &= \frac{\partial\mathfrak{v}^i}{\partial x'^j}. \end{aligned}$$

5. If the components of a vector $v^i$ are homogeneous of degree $h$ with regard to the $x'^i$, then they are components of a geometrical object in some sense. But the $\delta v^i$ in the sense of (1.19) have no geometrical meaning. That is, the $\delta v^i$ are not independent of the parameter $t$ but defined when and only when a choice of a parameter $t$ is held fixed. In fact, by transformation of the parameter $t$ one obtains

$$\delta\bar{v}^i = \left(\frac{dt}{d\bar{t}}\right)^h\delta v^i + hv^i\left(\frac{dt}{d\bar{t}}\right)^{h-1} d\,\frac{dt}{d\bar{t}},$$

which carries no geometrical meaning. Although for such a vector the absolute differential having geometrical meaning cannot be defined, it is possible to define it when the values of $x''^i$ in the considered line element $(x, x')$ are given by any reason,† as we shall now show. Nevertheless it must be noticed that the covariant derivatives (1.22) of such a vector are surely homogeneous with regard to the $x'^i$.

Since $x''^i$ are known in the considered line element $(x, x')$, the values of $F$ and $x^{[2]i}$ are also determined. For displacement from a line element $(x, x')$ to a neighboring line element $(x+dx, x'+dx')$ it is known that $\delta A_i x^{[2]i}$ and $(\delta A_i/dt)\delta x'^i$ are both scalars. They are varied by transformation of $t$ as follows:

$$(\delta A_i x^{[2]i})_{\bar{t}} = \sigma^p \delta A_i x^{[2]i} + (p-2)\sigma^{p-1}F d\sigma - \sigma^{p-2}\frac{d\sigma}{d\bar{t}} A_i \delta x'^i,$$

$$\left(\frac{\delta A_i}{dt}\delta x'^i\right)_{\bar{t}} = \sigma^p \frac{\delta A_i}{dt}\delta x'^i + (p-2)\sigma^{p-2}\frac{d\sigma}{d\bar{t}} A_i \delta x'^i - \sigma^{p-1}F d\sigma,$$

where $\sigma = dt/d\bar{t}$. Accordingly one gets

$$(\lambda_i dx^i + \mu_i \delta x'^i)^- = \lambda_i dx^i + \mu_i \delta x'^i - \frac{d\sigma}{\sigma},$$

where

$$\lambda_i = \frac{1}{(p-1)(p-3)F} x^{[2]k} \nabla_i A_k,$$

$$\mu_i = \frac{1}{(p-1)(p-3)F} x^{[2]k}\left\{(p-2)A_{k(i)} + A_{i(k)}\right\},$$

since

$$\lambda_i dx^i + \mu_i \delta x'^i = \frac{1}{(p-1)(p-3)F}\left\{(p-2)x^{[2]i}\delta A_i + \frac{\delta A_i}{dt}\delta x'^i\right\},$$

when $p \neq 1, 3$. Then one obtains as the required absolute differential

$$\delta^* v^i = \delta v^i + h v^i(\lambda_j dx^j + \mu_j \delta x'^j), \tag{1.28}$$

accompanied by the covariant derivatives

$$\begin{aligned} \nabla_j^* v^i &= \nabla_j v^i + \frac{h}{(p-1)(p-3)F} v^i x^{[2]k} \nabla_j A_k, \\ \nabla_j^{*\prime} v^i &= \nabla_j' v^i + \frac{h}{(p-1)(p-3)F} v^i x^{[2]k}\left\{(p-2)A_{k(j)} + A_{j(k)}\right\}, \end{aligned} \tag{1.29}$$

since

$$x'^i \nabla_k A_i = 0, \qquad \left\{(p-2)A_{k(j)} + A_{j(k)}\right\} x'^k = 0. \tag{1.30}$$

† For example, if one considers an absolute differential adjoint to a curve touching the line element $(x, x')$, one can take $x''^i$ along the curve as the associated one.

$F^{h/p}\delta v^i$ are independent of a change of the parameter $t$. For $x'^i$ equation (1.29) yields

$$\frac{\delta^* x'^i}{dt} = x^{[2]i} + \frac{1}{(p-3)F} x'^i A_{k(j)} x^{[2]k} x^{[2]j}. \tag{1.31}$$

If $v^i$ be defined in the manifold $K_n^{(2)}$ of line elements of the second order $(x, x', x'')$ and go into $\sigma^h v^i$ by transformation

$$\bar{x}'^i = \sigma x'^i,$$

$$\bar{x}''^i = \sigma^2 x''^i + \sigma \frac{d\sigma}{dt} x'^i,$$

then it is very easy to derive $\delta v^i$, which goes into $\sigma^h \delta v^i$ for the same transformations:

$$\delta v^i = dv^i + \Gamma^i_{(j)(k)} v^j dx^k - \frac{h}{p} v^i d \log F. \tag{1.32}$$

6. Displace a line element $(x, x')$ parallel to itself in the sense of the base connection on its direction. A curve is obtained, which satisfies the differential equations

$$x^{[2]i} = 0, \tag{1.33}$$

that is,

$$x''^i + 2\Gamma^i = 0.$$

These differential equations define a system of paths in our space, but these paths are not extremal curves of the integral $\int F^{1/p} dt$, unlike those in the Finsler space. They are minimal curves, for along them $F=0$ holds. But it is to be noticed that minimal curves do not always satisfy (1.33).

The equations in (1.33) in any parameter $t$ become

$$x''^i + 2\Gamma^i = \rho x'^i,$$

where $\rho$ is a function of $t$. We shall refer to a parameter $t$ as *affine length,* if it makes $\rho$ equal zero. It is determined except for a linear transformation $\bar{t} = at + b$. If we confine ourselves to affine length, the discussion in §6 is superfluous.

Considering the covariant derivatives of a vector $v^i$ along any path, we obtain from (1.22)

$$\nabla_0 v^i = \frac{\partial v^i}{\partial x^j} x'^j - 2 \frac{\partial v^i}{\partial x'^j} \Gamma^j + \Gamma^i_{(j)} v^j, \tag{1.34}$$

as $\delta x'^i = 0$. Particularly it is known from (1.25) that

$$\nabla_0 A_i = 0. \tag{1.35}$$

The paths are formally the same ones as those in the space with a generalized affine connection. There is, however, a difference in that in our space there is an intrinsic covariant vector $A_i$ besides the functions $\Gamma^i$, and our space is characterized by both quantities $A_i$ and $\Gamma^i$. We can develop the theory of curvatures, treat the equivalence problem, and so forth, by slight modifications of the methods employed in the geometry of generalized paths.

7. Computing the parenthesis of Poisson for the covariant derivatives, we find the curvature tensors as follows:

$$\begin{aligned}(\nabla_j\nabla_k - \nabla_k\nabla_j)v^i &= -R_{jkl}^{\cdot\cdot\cdot i}v^l + K_{jk}^{\cdot\cdot l}\nabla_l' v^i,\\ (\nabla_j\nabla_k' - \nabla_k'\nabla_j)v^i &= -B_{jkl}^{\cdot\cdot\cdot i}v^l,\end{aligned} \tag{1.36}$$

where we put

$$\begin{aligned}B_{jkl}^{\cdot\cdot\cdot i} &= \Gamma^i_{(j)(k)(l)},\\ R_{jkl}^{\cdot\cdot\cdot i} &= \frac{\partial\Gamma^i_{(l)(j)}}{\partial x^k} - \frac{\partial\Gamma^i_{(l)(k)}}{\partial x^j} + \Gamma^h_{(l)(j)}\Gamma^i_{(k)(h)} - \Gamma^h_{(l)(k)}\Gamma^i_{(j)(h)}\\ &\quad + \Gamma^h_{(j)}\Gamma^i_{(l)(k)(h)} - \Gamma^h_{(k)}\Gamma^i_{(l)(j)(h)},\\ K_{jk}^{\cdot\cdot i} &= \frac{\partial\Gamma^i_{(j)}}{\partial x^k} - \frac{\partial\Gamma^i_{(k)}}{\partial x^j} + \Gamma^h_{(j)}\Gamma^i_{(k)(h)} - \Gamma^h_{(k)}\Gamma^i_{(j)(h)}.\end{aligned} \tag{1.37}$$

It is not difficult to find the relations satisfied by these curvature tensors. They are

$$\begin{aligned}&R_{jkl}^{\cdot\cdot\cdot i} + R_{kjl}^{\cdot\cdot\cdot i} = 0, \qquad R_{jkl}^{\cdot\cdot\cdot i} + R_{klj}^{\cdot\cdot\cdot i} + R_{ljk}^{\cdot\cdot\cdot i} = 0,\\ &K_{jk}^{\cdot\cdot i} = R_{jkl}^{\cdot\cdot\cdot i}x'^l, \qquad B_{jkl}^{\cdot\cdot\cdot i}x'^l = 0,\\ &R_{jkl}^{\cdot\cdot\cdot i} = K_{jk\cdot(l)}^{\cdot\cdot i} = \nabla_l' K_{jk}^{\cdot\cdot i}.\end{aligned} \tag{1.38}$$

The identities of Bianchi are now expressed by

$$\begin{aligned}\nabla_{[h}R_{jk]l}^{\cdot\cdot\cdot i} + K_{[hj}^{\cdot\cdot r}B_{k]lr}^{\cdot\cdot\cdot i} &= 0,\\ 2\nabla_{[h}B_{j]kl}^{\cdot\cdot\cdot i} + \nabla_l' R_{hjk}^{\cdot\cdot\cdot i} &= 0,\\ \nabla_{[h}K_{jk]}^{\cdot\cdot i} &= 0.\end{aligned} \tag{1.39}$$

The invariant theory under transformation of the parameter $t$ can also be developed by the fundamental descriptive invariants of Douglas*

* J. Douglas, loc. cit.

$$\Pi^{\cdot\, i}_{\cdot\, jk} = \Gamma^{i}_{(j)(k)} - \frac{\delta_j^{\,i}}{n+1}\Gamma^{h}_{(h)(k)} - \frac{\delta_k^{\,i}}{n+1}\Gamma^{h}_{(j)(h)} - \frac{1}{n+1}x'^{i}B_{jkh}^{\cdots h},$$

$$(1.40)\qquad \mathfrak{H}_{jkl}^{\cdots i} = \Pi^{\cdot\, i}_{\cdot\, jk(l)},$$

$$\mathfrak{R}_{jkl}^{\cdots i} = R^{*\cdots i}_{\ jkl} - \frac{\delta_k^{\,i}}{n-1}R^{*\cdots h}_{\ hlj} - \frac{\delta_l^{\,i}}{n-1}R^{*\cdots h}_{\ khj},$$

where $R^{*\cdots i}_{jkl}$ denotes $R^{\cdots i}_{jkl}$ formed with the $\Pi$'s in place of the $\Gamma$'s.

8. The conditions

$$(1.41)\qquad B_{jkl}^{\cdots i} = 0, \qquad R_{jkl}^{\cdots i} = 0$$

are necessary and sufficient in order that it may be possible by transformation of coordinates to reduce the differential equations of the paths in the space to the form $x''^{i}=0$, accordingly to reduce the curve length to

$$s = \int (A_i x''^{i})^{1/p} dt.$$

If moreover $A_{i(j)(k)}=0$, that is, if the $A_i$ are linear homogeneous functions of $x'^{j}$,

$$A_{i(j)}x'^{j} = A_i, \qquad A_{i(j)} = -A_{j(i)}, \qquad p = 3.$$

The $A_{i(j)}$ are functions only of the $x$'s. Then we have the curve length

$$s = \int (A_{i(j)}x''^{i}x'^{j})^{1/3} dt.$$

For the two-dimensional affine space, $A_{i(j)}$ are all constants.

The conditions

$$(1.42)\qquad \mathfrak{H}_{jkl}^{\cdots i} = 0, \qquad \mathfrak{R}_{jkl}^{\cdots i} = 0$$

are necessary and sufficient in order that there may exist a coordinate system together with a parametrization such that the differential equations of the paths have the form $x''^{i}=0$.

For the equivalence problem we have without difficulty the theorem:

THEOREM. *A necessary and sufficient condition for the equivalence of two spaces is that there exist an integer* $N$ ($\geqq 1$) *such that the first* $N$ *sets of following equations are compatible considered as equations for the determination of* $y^i$ *and* $P_j^{\,i}$ *as functions of the independent variables* $x^i$, *and all their solutions*

$$y^i = y^i(x), \qquad P_j^{\,i} = \phi_j^{\,i}(x)$$

*satisfy the* $(N+1)$*st set of the equations*:

$$1.\quad \begin{cases} \overline{A}_i P^i_j = A_j, \\ \overline{B}^{\cdots i}_{jkl} P^j_r P^k_s P^l_t = B^{\cdots u}_{rst} P^i_u, \\ \overline{R}^{\cdots i}_{jkl} P^j_r P^k_s P^l_t = R^{\cdots u}_{rst} P^i_u, \end{cases}$$

$$2.\quad \begin{cases} \overline{\nabla}_j \overline{A}_i P^j_r P^i_s = \nabla_r A_s, \quad \overline{\nabla}'_j \overline{A}_i P^j_r P^i_s = \nabla'_r A_s, \\ \overline{\nabla}_h \overline{B}^{\cdots i}_{jkl} P^h_r P^j_s P^k_t P^l_u = \nabla_r B^{\cdots v}_{stu} P^i_v, \\ \overline{\nabla}'_h \overline{B}^{\cdots i}_{jkl} P^h_r P^j_s P^k_t P^l_u = \nabla'_r B^{\cdots v}_{stu} P^i_v, \\ \overline{\nabla}_h \overline{R}^{\cdots i}_{jkl} P^h_r P^j_s P^k_t P^l_u = \nabla_r R^{\cdots v}_{stu} P^i_v, \\ \overline{\nabla}'_h \overline{R}^{\cdots i}_{jkl} P^h_r P^j_s P^k_t P^l_u = \nabla'_r R^{\cdots v}_{stu} P^i_v, \\ \cdots\cdots\cdots\cdots\cdots\cdots . \end{cases}$$

Furthermore normal coordinates can be defined, and the replacement theorem also can be proved.

## Chapter 2. The connection $C'$

9. We shall now proceed to define another connection in our space. In a previous paper* the author has proved the theorem:

*Let $\Phi_i$ be a covariant tensor of order $m$, that is, dependent upon $x^i$, $x'^i, \cdots, x^{(m)i}$; then $n$ quantities*

$$D^{[m-\rho]}_{ij}(\Phi)v^j = \sum_{\alpha=\rho}^{m} \binom{\alpha}{\rho} \frac{\partial \Phi_i}{\partial x^{(\alpha)j}} \frac{d^{\alpha-\rho} v^j}{dt^{\alpha-\rho}}$$

*for a fixed positive integer $\rho$ ($\leqq m$) are the components of a covariant vector, provided that $v^i$ is a contravariant vector.*

In consequence of this theorem one can deduce from a vector $v^i$ the following vectors:

$$(2.1)\qquad -D^{[1]}_{ij}(T)v^j = 2G_{ij}\frac{dv^j}{dt} + G_{ik(l)}x''^k v^l + 2\Gamma_{i(l)}v^l,$$

$$(2.2)\qquad D^{[1]}_{ij}(E)v^j = 3H_{ik}\frac{dv^k}{dt} + L_{ilk}x''^l v^k + M_{ik}v^k,$$

$$(2.3)\qquad D^{[2]}_{ij}(E)v^j = 3H_{ik}\frac{d^2v^k}{dt^2} + 2(L_{ilk}x''^l + M_{ik})\frac{dv^k}{dt}$$
$$+ (H_{il(k)}x'''^l + H_{il(j)(k)}x''^l x''^j + M_{ij(k)}x''^j + P_{i(k)})v^k,$$

where

* A. Kawaguchi, *Some intrinsic derivations in a generalized space*, loc. cit.

$$\Gamma_i = G_{ik}\Gamma^k = \frac{1}{2(p-1)} M_{ik}x'^k, \tag{2.4}$$

$$H_{ik} = A_{i(k)} - A_{k(i)}, \qquad L_{ilk} = H_{il(k)} + H_{ik(l)}, \tag{2.5}$$

$$M_{ik} = A_{i(k)l}x'^l - A_{k(i)l}x'^l + A_{ki} + A_{ik} - B_{(i)(k)} + A_{ij(k)}x'^j, \tag{2.6}$$

$$P_i = A_{ijl}x'^j x'^l - B_{(i)l}x'^l + B_i. \tag{2.7}$$

It can be verified easily that the following relations hold:

$$\begin{aligned} -\overset{[1]}{D}_{ij}(T)x'^j &= (p-1)T_i, \\ -\overset{[1]}{D}_{ij}(E)x'^j &= (p-1)T_i, \\ \overset{[2]}{D}_{ij}(E)x'^j &= (p-1)E_i. \end{aligned} \tag{2.8}$$

10. Suppose now once more that $p \neq 3/2$. Then we have from (2.1) an absolute derivative along a curve

$$\overset{[1]}{D}_t v^i = \frac{dv^i}{dt} + G^{hi}(\tfrac{1}{2}G_{hk(l)}x''^k + \Gamma_{h(l)})v^l.$$

Since

$$G^{hi}(\tfrac{1}{2}G_{hk(l)}x''^k + \Gamma_{h(l)}) = \tfrac{1}{2}G^{hi}G_{hk(l)}(x''^k + 2\Gamma^k) + \Gamma^i_{(l)},$$

the absolute differential of a vector corresponding to a displacement from a line element $(x, x')$ to a neighboring line element $(x+dx, x'+dx')$ is obtained without difficulty. We have

$$D^{[1]}v^i = dv^i + \Gamma^i_{(k)(j)}v^k dx^j + \tfrac{1}{2}G^{hi}G_{hj(k)}v^k \delta x'^j, \tag{2.9}$$

which reduces to (1.19), when $\delta x'^j = 0$.

The absolute differential (2.9) is, however, not invariant under transformation of $t$, because

$$G^{hi}G_{hk(l)}x'^k = (2p-3)G^{hi}A_{h(l)} - \delta_l^i \tag{2.10}$$

does not vanish in general. When and only when

$$(p-1)A_{i(k)} = A_{k(i)}, \tag{2.11}$$

(2.10) does vanish identically, and (2.9) is independent of transformation of $t$.

11. If we adopt the notations

$$\begin{aligned} I_{lji} &= A_{l(j)(i)} + A_{j(l)(i)} + (p-3)A_{i(j)(l)}, \\ J_{li} &= A_{l(i)} + (p-2)A_{i(l)}; \end{aligned} \tag{2.12}$$

it will be easily inferred from (1.4) and (1.5) that

$$I_{lji}x'^i = 0, \qquad J_{li}x'^i = 0, \qquad I_{lji}x'^l = (p-4)J_{li}.$$

Hence the tensor

$$\tilde{C}^{\cdot k}_{\cdot ij} = (p-3)G^{lk}I_{lji} + G^{lk}_{\cdot\cdot(j)}J_{li} \tag{2.13}$$

satisfies the relations

$$\tilde{C}^{\cdot k}_{\cdot ij}x'^i = 0, \qquad \tilde{C}^{\cdot k}_{\cdot ij}x'^j = 0. \tag{2.14}$$

We shall now put

$$\begin{aligned} C^{\cdot k}_{\cdot ij} &= \frac{1}{(p-3)^2}\tilde{C}^{\cdot k}_{\cdot ij}, & p &\neq 3, \\ &= \tilde{C}^{\cdot k}_{\cdot ij}, & p &= 3. \end{aligned} \tag{2.15}$$

The tensor $C^{\cdot k}_{\cdot ij}$ is constructed from only $A_i$ and is independent of $B$. With the help of the tensor $C^{\cdot k}_{\cdot ij}$ an absolute differential can be defined

$$\delta v^i = dv^i + \Gamma^{\cdot i}_{(j)(k)}v^j dx^k + C^{\cdot i}_{\cdot jk}v^j\delta x'^k, \tag{2.16}$$

which is invariant under transformation of the parameter $t$. The connection characterized by $A_i$, $\Gamma^i$, and $C^{\cdot i}_{\cdot jk}$ will be referred to as the connection $C'$. The torsion* of this connection is $C^{\cdot i}_{\cdot jk}$, and for $v^i = x'^i$ (2.16) gives us the base connection (1.20).

12. The covariant derivatives referred to the connection $C'$ possess the form

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^k}\Gamma^{k}_{(j)} + \Gamma^{\cdot i}_{(k)(j)}v^k, \quad \tilde{\nabla}'_j v^i = \frac{\partial v^i}{\partial x'^j} + C^{\cdot i}_{\cdot kj}v^k, \tag{2.17}$$

and we have

$$\begin{aligned} (\nabla_j\nabla_k - \nabla_k\nabla_j)v^i &= -\tilde{R}^{\cdot\cdot\cdot i}_{jkl}v^l + K^{\cdot\cdot l}_{jk}\tilde{\nabla}'_l v^i, \\ (\nabla_j\tilde{\nabla}'_k - \tilde{\nabla}'_k\nabla_j)v^i &= -\tilde{B}^{\cdot\cdot\cdot i}_{jkl}v^l + C^{\cdot l}_{\cdot jk}\nabla_l v^i, \\ (\tilde{\nabla}'_j\tilde{\nabla}'_k - \tilde{\nabla}'_k\tilde{\nabla}'_j)v^i &= -P^{\cdot\cdot\cdot i}_{jkl}v^l - 2C^{\cdot h}_{\cdot[kj]}\tilde{\nabla}'_h v^i, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} \tilde{R}^{\cdot\cdot\cdot i}_{jkl} &= R^{\cdot\cdot\cdot i}_{jkl} + K^{\cdot\cdot h}_{jk}C^{\cdot i}_{\cdot lh}, \\ \tilde{B}^{\cdot\cdot\cdot i}_{jkl} &= \Gamma^{\cdot i}_{(l)(j)(k)} + C^{\cdot i}_{\cdot hk}\Gamma^{\cdot h}_{(l)(j)} - \frac{\partial C^{\cdot i}_{\cdot lk}}{\partial x^j} + C^{\cdot i}_{\cdot l(kh)}\Gamma^{h}_{(j)} \\ &\quad - \Gamma^{\cdot i}_{(h)(j)}C^{\cdot h}_{\cdot lk} + \Gamma^{\cdot h}_{(j)(k)}C^{\cdot i}_{\cdot lh}, \\ P^{\cdot\cdot\cdot i}_{jkl} &= C^{\cdot i}_{\cdot lj(k)} - C^{\cdot i}_{\cdot lk(j)} + C^{\cdot i}_{\cdot hk}C^{\cdot h}_{\cdot lj} - C^{\cdot i}_{\cdot hj}C^{\cdot h}_{\cdot lk} \end{aligned} \tag{2.19}$$

are the curvature tensors.

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* See Cartan, *Les espaces de Finsler*, loc. cit., p. 32.

Consider the differential forms

$$
\begin{aligned}
(2.20)\qquad & dx^{i}, \qquad \omega^{i} = dx'^{i} + \Gamma^{i}_{(j)} dx^{j}, \\
& \omega^{i}_{j} = \Gamma^{i}_{(j)(k)} dx^{k} + C^{i}_{\cdot jk}\omega^{k};
\end{aligned}
$$

then we have the equations of structure* of the connection $C'$:

$$
\begin{aligned}
& [dx^{j}\omega^{i}_{j}] = C^{i}_{\cdot jk}[dx^{j}\omega^{k}], \\
(2.21)\qquad & (\omega^{i})' + [\omega^{i}_{j}\omega^{j}] = -\tfrac{1}{2}K^{\cdot\cdot i}_{jk}[dx^{j}dx^{k}] - C^{i}_{\cdot[jk]}[\omega^{j}\omega^{k}], \\
& (\omega_{j}{}^{i})' + [\omega^{i}_{k}\omega^{k}_{j}] = -\tfrac{1}{2}\tilde{R}^{\cdot\cdot\cdot i}_{klj}[dx^{k}dx^{l}] - B^{\cdot\cdot\cdot i}_{klj}[dx^{k}\omega^{l}] - \tfrac{1}{2}P^{\cdot\cdot\cdot i}_{klj}[\omega^{k}\omega^{l}].
\end{aligned}
$$

The identities of Bianchi are now expressed by

$$
\begin{aligned}
& \nabla_{[h}\tilde{R}^{\cdot\cdot\cdot i}_{jk]l} + K^{\cdot\cdot r}_{[hj}\tilde{B}^{\cdot\cdot\cdot i}_{k]lr} = 0, \\
& 2\nabla_{[h}\tilde{B}^{\cdot\cdot\cdot i}_{j]kl} + \tilde{\nabla}'_{l} R^{\cdot\cdot\cdot i}_{hjk} + P^{\cdot\cdot\cdot i}_{lrk}K^{\cdot\cdot r}_{hj} = 0, \\
(2.22)\qquad & 2\tilde{\nabla}'_{[m}\tilde{B}^{\cdot\cdot\cdot i}_{|k|l]j} + \nabla_{k}P^{\cdot\cdot\cdot i}_{lmj} = 0, \\
& \tilde{\nabla}'_{[h}P^{\cdot\cdot\cdot i}_{jk]l} = 0.
\end{aligned}
$$

The equivalence theorem can be proved, but we shall forego the details here.

* See Cartan, loc. cit., pp. 32–36.

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