SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

I. THE DRESDEN CORNER CONDITION*

BY

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The present note is the first of a series, and is the only one which does violence to the general title by failing to exhibit an existence theorem. Instead, we here perform a few rather easy computations for later use, and by them obtain a simple proof of a corner condition for isoperimetric problems. This corner condition was apparently first established by Dresden as a consequence of the Weierstrass condition, so only the method of proof here can qualify as new. However, it will serve as a suggestive guide to further theorems in later papers.

1. **Notation and continuity assumptions.** In order that some of our preliminary calculations and lemmas shall be valid for both parametric and non-parametric problems, we shall recast non-parametric problems in parametric form. Given an integrand

$$f(x, y, y') \equiv f(x, y^1, \cdots, y^q, y^{1'}, \cdots, y^{q'}),$$

we define

$$F(z, z') = F(z^0, z^1, \cdots, z^{\dot{q}}, z^{0'}, z^{1'}, \cdots, z^{q'})$$

$$(1.1) \qquad = z^{0'} f\left(z^0, z^1, \cdots, z^q; \frac{z^{1'}}{z^{0'}}, \cdots, \frac{z^{q'}}{z^{0'}}\right), \qquad z^{0'} > 0,$$

$$F(z, 0) = 0.$$

We use a modification of the tensor summation convention. The repetition of a Greek-letter affix in a term requires the summation of the values of that term over all values of the affix. Thus

$$F_{z^\alpha}(z_n, z_n')z^{\alpha'} = F_{z^0}(z_n, z_n')z^{0'} + \cdots + F_{z^q}(z_n, z_n')z^{q'}$$

is summed over $\alpha = 0, 1, \cdots, q$, but not summed over the values of n. Also, if ν is a vector, its length will be denoted by $|\nu|$. Thus if $z = (z^0, \cdots, z^q)$, then $|z| = [z^\alpha z^\alpha]^{1/2}$.

Throughout the papeis of this series all integrands $f(x, y, y')$ will be assumed (unless specific statement is made to the contrary) to be defined and continuous together with their partial derivatives of first and second order for all (x, y) in a closed set S, and all y'. Furthermore, we assume without further mention the following:

(1.2) *For every bounded subset S_0 of S there is a constant N such that*

$$f(x, y, y') + N[1 + y^{\alpha'} y^{\alpha'}]^{1/2} \geq 0$$

for all (x, y) in S_0 and all y'.

In the parametric notation, $F(z, z')$ being defined by (1.1), this takes the form:

(1.2′) *For every bounded subset S_0 of S there is a constant N such that* $F(z, z') + N|z'| \geq 0$ *for all z in S_0 and all z' with $z^{0'} > 0$.*

Parametric integrands $F(z, z')$ (not those arising by (1.1) from non-parametric integrands) will be assumed to be defined and continuous for all z in a closed set S and all z', to be positively homogeneous of degree 1 in z', and to have continuous partial derivatives of the first and second orders for all z in S and for all $z' \neq (0, \cdots, 0)$.

In order to avoid frequent printing of a complicated symbol we make the definitions

$$f_i(x, y, y') \equiv f_{y^{i'}}(x, y, y'), \qquad i = 1, \cdots, q,$$
$$F_i(z, z') \equiv F_{z^{i'}}(z, z'), \qquad i = 0, 1, \cdots, q.$$

From the assumptions made on integrands $f(x, y, y')$ it follows at once that the function $F(z, z')$ defined by (1.1) is continuous with its partial derivatives of first and second orders for all $z \equiv (x, y)$ in S and all z' with $z^{0'} > 0$, and is positively homogeneous of degree 1 in z'. Whenever we are discussing the integral of such an integrand along a curve $C: z = z(t)$ it will be tacitly assumed that $z^{0'}(t) > 0$ for almost all t, while for all auxiliary curves used in demonstrations we must prove $z^{0'}(t) > 0$ for almost all t.

If the representation $z = z(t)$, $(t_1 \leq t \leq t_2)$, of the curve C satisfies this condition, the function* $F(z(t), \dot{z}(t))$ is measurable. Since by (1.2′) it exceeds a summable function $-N|\dot{z}(t)|$, it has an integral, finite or infinite. Provided that the functions $z^i(t)$ are absolutely continuous, we denote this integral by

$$\mathcal{J}(C) = \int_C F(z, \dot{z})dt = \int_{t_1}^{t_2} F(z(t), \dot{z}(t))dt.$$

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* In analogy with Carathéodory's notation, we define $\dot{z}(t)$ to be the vector $(z^{0'}(t), \cdots, z^{q'}(t))$ if the vector is defined and finite; otherwise $\dot{z}(t)$ is defined to be 0 (i.e., $(0, \cdots, 0)$).

Any two absolutely continuous parametrizations of C (subject to the requirement that $z^{0\prime}(t) > 0$ for almost all t) give the same value to the integral.* In particular, if $z^0(t) \equiv t$, so that C has the form $y = y(x)$, $(a \le x \le b)$, and if furthermore the $y^i(x)$ are absolutely continuous, we write

$$\mathcal{J}[y] = \mathcal{J}(C) = \int_a^b f(x, y(x), \dot{y}(x))dx.$$

If $F(z, z')$ is a parametric integrand, defined for all z in a set S and all z', these questions of integrability do not arise. We write in this case also

$$\mathcal{J}(C) = \int_C F(z, \dot{z})dt = \int_{t_1}^{t_2} F(z(t), \dot{z}(t))dt,$$

provided that the functions $z^i(t)$ are absolutely continuous. The invariance under change of parameter is well known in this case. (It also follows from the theorem cited in the preceding footnote.)

2. **Differentiation formulas.** Let $z = z(\sigma)$, $(\sigma_1 \le \sigma \le \sigma_2)$, be a Lipschitzian representation of a curve C. We shall suppose that $z = \phi(\tau)$ and $z = \psi(\tau)$ are two curves passing through the ends of C, so that

(2.1) $$\phi(\tau_0) = z(\sigma_1), \qquad \psi(\tau_0) = z(\sigma_2),$$

and we shall also suppose that ϕ and ψ are absolutely continuous on an interval (a, b) containing τ_0 and have finite derivatives for $\tau = \tau_0$.

From C we form the curve $C(\tau)$ defined by the equation†

(2.2)
$$C(\tau)\colon z^i = z^i(\sigma, \tau) = z^i(\sigma) + \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}(\phi^i(\tau) - \phi^i(\tau_0))$$
$$+ \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}(\psi^i(\tau) - \psi^i(\tau_0)), \ \sigma_1 \le \sigma \le \sigma_2, i = 0, 1, \cdots, q.$$

This (by (2.1)) joins $\phi(\tau)$ to $\psi(\tau)$, and $z(\sigma, \tau_0) \equiv z(\sigma)$. We wish to calculate the derivative of $\mathcal{J}(C(\tau))$, it being assumed that $C(\tau)$ lies in S for $a \le \tau \le b$.

Since

(2.3) $$\mathcal{J}(C(\tau)) = \int_{\sigma_1}^{\sigma_2} F(z(\sigma, \tau), \dot{z}(\sigma, \tau))d\sigma,$$

by differentiating under the integral sign we obtain

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* E. J. McShane, *Semi-continuity of integrals in the calculus of variations,* Duke Mathematical Journal, vol. 2 (1936), pp. 597–616; in particular, Theorem 2.1 (the first five lines of proof being deleted).

† The curve $C(\tau)$ depends not merely on C and τ, but also on the particular parametrization $z = z(\sigma)$ which we chose for C.

$$\mathcal{J}'(C(\tau))\bigg|_{\tau=\tau_0} = \int_{\sigma_1}^{\sigma_2}\left(F_{z^\alpha}\frac{\partial z^\alpha}{\partial\tau} + F_\alpha\frac{\partial \dot z^\alpha}{\partial\tau}\right)d\sigma\bigg|_{\tau=\tau_0}$$

$$(2.4)\qquad = \int_{\sigma_1}^{\sigma_2}\left\{F_{z^\alpha}(z(\sigma),\dot z(\sigma))\left[\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}\phi^{\alpha\prime}(\tau_0) + \frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}\psi^{\alpha\prime}(\tau_0)\right]\right.$$

$$\left. + F_\alpha(z(\sigma),\dot z(\sigma))\left[\frac{\psi^{\alpha\prime}(\tau_0) - \phi^{\alpha\prime}(\tau_0)}{\sigma_2-\sigma_1}\right]\right\}d\sigma.$$

If $F(z, z')$ arises from a non-parametric integrand by (1.1), equation (2.4) is still valid if we add the hypothesis that $\dot z^0(\sigma)\geq m>0$ for almost all σ.

|Several special cases of this formula will be of use to us. First, suppose $\phi(\tau)-\phi(\tau_0)=\psi(\tau)-\psi(\tau_0)$, so that (2.2) represents merely a translation of the curve C by the (vector) amount $\phi(\tau)-\phi(\tau_0)$. Then

$$(2.5)\qquad \mathcal{J}'(C(\tau_0)) = \phi^{\alpha\prime}(\tau_0)\int_{\sigma_1}^{\sigma_2}F_{z^\alpha}(z,\dot z)d\sigma.$$

(This is independent of the parametric representation of C.)

|Suppose next that C is a line segment, and that $C(\tau)$ is the line segment whose ends are $z(\sigma_1)+\tau\pi_1$ and $z(\sigma_2)+\tau\pi_2$, where π_1 and π_2 are given vectors. If the functions $z=z(\sigma)$ representing C are linear, then (2.2) represents the line segment $C(\tau)$ if we take

$$(2.6)\qquad \phi(\tau) = z(\sigma_1) + \tau\pi_1 \quad\text{and}\quad \psi(\tau) = z(\sigma_2) + \tau\pi_2,$$

so that (2.4) becomes

$$\mathcal{J}'(C(\tau))\bigg|_{\tau=\tau_0} = \int_{\sigma_1}^{\sigma_2}\left\{F_{z^\alpha}(z,\dot z)\left[\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}\pi_1^\alpha + \frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}\pi_2^\alpha\right]\right.$$

$$(2.7)\qquad\left. + F_\alpha(z,\dot z)\left[\frac{\pi_2^\alpha - \pi_1^\alpha}{\sigma_2-\sigma_1}\right]\right\}d\sigma.$$

Applying the mean value theorem to the last term on the right, we obtain

$$\mathcal{J}'(C(\tau))\bigg|_{\tau=\tau_0} = \int_{\sigma_1}^{\sigma_2}F_{z^\alpha}(z(\sigma),\dot z(\sigma))\left\{\pi_1^\alpha\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1} + \pi_2^\alpha\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}\right\}d\sigma$$

$$(2.8)\qquad + F_\alpha(z(\bar\sigma),\dot z(\bar\sigma))[\pi_2^\alpha - \pi_1^\alpha],$$

where $\sigma_1<\bar\sigma<\sigma_2$.

3. **Interchange of arcs.** Let C_1 and C_2 be rectifiable curves

$$(3.1)\qquad C_1: \ z = z_1(\sigma),\ \sigma_1\leq\sigma\leq\sigma_2; \qquad C_2: \ z = z_2(\tau),\ \tau_1\leq\tau\leq\tau_2$$

such that the beginning of C_2 coincides with the end of C_1:

$$(3.2)\qquad z_1(\sigma_2) = z_2(\tau_1).$$

The functions $z_1(\sigma)$ and $z_2(\tau)$ will be supposed to satisfy a Lipschitz condition, and if $F(z, z')$ is defined by (1.1) it will also be supposed that except on a set of measure zero $\dot{z}_1^0(t)$ and $\dot{z}_2^0(t)$ are bounded from zero. We define C_{12} to be the curve obtained by traversing first C_1 and then C_2. Furthermore, we define C_{21} to be the curve obtained by starting at $z_1(\sigma_1)$, traversing a curve C_2^* which is a translation of C_2, and then traversing a curve C_1^* which is a translation of C_1. It is clear that the end of C_{21} is the same as the end of C_{12}, namely $z_2(\tau_2)$. We now assume that the point

$$(3.3) \qquad z_1(\sigma) + z_2(\tau) - z_2(t_1)$$

lies in S whenever $\sigma_1 \leq \sigma \leq \sigma_2$ and $\tau_1 \leq \tau \leq \tau_2$, and we proceed to compute $\mathcal{J}(C_{21}) - \mathcal{J}(C_{12})$.

By the definitions,

$$(3.4) \qquad \mathcal{J}(C_{21}) - \mathcal{J}(C_{12}) = \mathcal{J}(C_1^*) - \mathcal{J}(C_1) + \mathcal{J}(C_2^*) - \mathcal{J}(C_2).$$

We compute $\mathcal{J}(C_1^*) - \mathcal{J}(C_1)$ by (2.5), where we take $\phi(\tau) = z_2(\tau) - z_1(\sigma_2) + z_1(\sigma_1)$, $\psi(\tau) = z_2(\tau)$. If we momentarily let $C(\tau)$ be the curve (which by hypothesis lies in S), $z = z_1(\sigma) + z_2(\tau) - z_2(\tau_1)$, $(\sigma_1 \leq \sigma \leq \sigma_2)$, we obtain by (2.5)

$$(3.5) \qquad \mathcal{J}'(C(\tau)) = \dot{z}_2^\alpha(\tau) \int_{\sigma_1}^{\sigma_2} F_{z^\alpha}(z_1(\sigma) + z_2(\tau) - z_2(\tau_1), \dot{z}_1(\sigma)) d\sigma$$

for all τ such that $z_2'(\tau)$ exists, that is, for almost all τ. Hence, integrating from $\tau = \tau_1$ to $\tau = \tau_2$, we obtain

$$
\begin{aligned}
(3.6) \qquad \mathcal{J}(C_1^*) - \mathcal{J}(C_1) &= \mathcal{J}(C(\tau_2)) - \mathcal{J}(C(\tau_1)) \\
&= \int_{\tau_1}^{\tau_2} \int_{\sigma_1}^{\sigma_2} \dot{z}_2^\alpha(\tau) F_{z^\alpha}(z_1(\sigma) + z_2(\tau) - z_2(\tau_1), \dot{z}_1(\sigma)) d\sigma d\tau.
\end{aligned}
$$

Next we let $C'(\sigma)$ be the curve $z = z_2(\tau) + z_1(\sigma) - z_1(\sigma_2)$, $(\tau_1 \leq \tau \leq \tau_2)$. (Recalling (3.2), we see that this lies by hypothesis in S for $\sigma_1 \leq \sigma \leq \sigma_2$.) Then $C'(\sigma_2)$ is C_2; and $C'(\sigma_1)$ is C_2^*, for it is a translation of C_2 and it starts at $z_2(\tau_1) + z_1(\sigma_1) - z_1(\sigma_2)$, which is $z_1(\sigma_1)$ by (3.2). Again using (2.5), but with σ serving in the role of τ, we find

$$(3.7) \qquad \mathcal{J}'(C'(\sigma)) = \dot{z}_1^\alpha(\sigma) \int_{\tau_1}^{\tau_2} F_{z^\alpha}(z_2(\tau) + z_1(\sigma) - z_1(\sigma_2), \dot{z}_2(\tau)) d\tau.$$

Integrating from σ_1 to σ_2, we have

$$
\begin{aligned}
(3.8) \qquad \mathcal{J}(C_2) - \mathcal{J}(C_2^*) &= \mathcal{J}(C'(\sigma_2)) - \mathcal{J}(C'(\sigma_1)) \\
&= \int_{\sigma_1}^{\sigma_2} \int_{\tau_1}^{\tau_2} \dot{z}_1^\alpha(\sigma) F_{z^\alpha}(z_2(\tau) + z_1(\sigma) - z_1(\sigma_2), \dot{z}_2(\tau)) d\tau d\sigma.
\end{aligned}
$$

With Carathéodory, we now define

$$(3.9) \qquad \Omega(z, p, q) = p^\alpha F_{z^\alpha}(z, q) - q^\alpha F_{z^\alpha}(z, p).$$

Then from (3.4), (3.6), and (3.8) we obtain (recalling (3.2))

$$(3.10) \quad \mathcal{J}(C_{21}) - \mathcal{J}(C_{12}) = -\int_{\tau_1}^{\tau_2}\int_{\sigma_1}^{\sigma_2}\Omega(z_2(\tau) + z_1(\sigma) - z_1(\sigma_2), \dot{z}_1(\sigma), \dot{z}_2(\tau))d\sigma d\tau.$$

We digress for a moment to make some remarks on the Ω-function. It is clear that it is continuous for all z, p, and q, if F is the usual parametric integrand. If F arises by (1.1), then Ω is continuous if $p^0 > 0$ and $q^0 > 0$. From the definition, $\Omega(z, p, q)$ is positively homogeneous of degree 1 in p and in q:

$$(3.11) \qquad \Omega(z, kp, \kappa q) = k\kappa\Omega(z, p, q) \quad \text{if} \quad k > 0 \quad \text{and} \quad \kappa > 0.$$

Also

$$(3.12) \qquad \Omega(z, p, q) = -\Omega(z, q, p).$$

Suppose that $F(z, z')$ arises by (1.1) from $f(x, y, y')$, where $(z^0, z^1, \cdots, z^q) = (x, y^1, \cdots, y^q)$, and that the curves C_1, C_2 are represented in the form

$$C_1: \quad y = y_1(x), \, a \leq x \leq b; \qquad C_2: \quad y = y_2(x), \, b \leq x \leq c,$$

where $y_1(b) = y_2(b)$. We can if we wish revert to the integrand f by simply using x as parameter (it being still assumed that $y_1(x)$ and $y_2(x)$ are Lipschitzian). Then on C_1 we have $\sigma = x$; so $z_1^0(\sigma) = \sigma = x$, and $\dot{z}_1^0(\sigma) \equiv 1$. Likewise $\dot{z}_2^0(\tau) \equiv 1$, so that by (3.10)

$$(3.13) \quad \begin{aligned} &\mathcal{J}(C_{21}) - \mathcal{J}(C_{12}) \\ &= -\int_b^c \left\{ \int_a^b \omega(\sigma + \tau - a, y_1(\sigma) + y_2(\tau) - y_1(b), \dot{y}_1(\sigma), \dot{y}_2(\tau))d\sigma \right\} d\tau, \end{aligned}$$

where

$$(3.14) \quad \begin{aligned} \omega(x, y, y', Y') = &f_x(x, y, Y') - f_x(x, y, y') \\ &+ y^{\alpha'}f_{y^\alpha}(x, y, Y') - Y^{\alpha'}f_{y^\alpha}(x, y, y'); \end{aligned}$$

that is,

$$(3.15) \qquad \omega(x, y, y', Y') = \Omega(x, y; 1, y'; 1, Y').$$

As a special case (useful later) we may assume that C_1 and C_2 are line segments. If $l_i = L(C_i)$ is the length of C_i, $(i = 1, 2)$, we can represent C_1 and C_2 by means of linear functions of arc lengths:

$$(3.16) \qquad C_i: \quad z = z_i(s), \qquad 0 \leq s \leq l_i, \, i = 1, 2.$$

Then the derivative $\dot{z}_i(s)$ is constantly equal to a unit vector p_i, $(i = 1, 2)$,

and the curves C_{12} and C_{21} bound a parallelogram (which may degenerate into a line segment). Formula (3.10) becomes

$$(3.17) \quad \mathcal{J}(C_{21}) - \mathcal{J}(C_{12}) = - \int_0^{l_1} \left\{ \int_0^{l_2} \Omega(z_1(\sigma) + z_2(\tau) - z_2(0), p_1, p_2) d\tau \right\} d\sigma.$$

By the theorem of mean value,

$$(3.18) \qquad \mathcal{J}(C_{21}) - \mathcal{J}(C_{12}) = - l_1 l_2 \Omega(z_1(\bar{\sigma}) + z_2(\bar{\tau}) - z_2(0), p_1, p_2),$$

where $0 < \bar{\sigma} < l_1$ and $0 < \bar{\tau} < l_2$. If we denote $z_1(\bar{\sigma}) + z_2(\bar{\tau}) - z_2(0)$ by ξ, then ξ is a point of the parallelogram bounded by C_{12} and C_{21}, and (3.18) becomes

$$(3.19) \qquad \qquad \mathcal{J}(C_{21}) - \mathcal{J}(C_{12}) = - l_1 l_2 \Omega(\xi, p_1, p_2).$$

4. Dresden's corner condition: the parametric form. In §4 we establish Dresden's corner condition for isoperimetric problems in parametric form.

We shall proceed under the following hypotheses:

(4.1) *The curve* $C: z = z(t)$, $(t_1 \leqq t \leqq t_2)$, *is interior to S and gives a strong relative minimum to the integral*

$$\mathcal{J}(C) = \int_C F(z, \dot{z}) dt$$

in the class of all D' *curves which join* $z(t_1)$ *to* $z(t_2)$ *and give assigned values* γ_j, $(j = 1, \cdots, m)$, *to the integrals*

$$\mathcal{G}^j(C) \equiv \int_C G^j(z, \dot{z}) dt, \qquad \qquad j = 1, \cdots, m.$$

(4.2) *The curve C has a corner at* $z(t_0)$.

(4.3) *The curve C is normal.** (Hence there is a unique set of multipliers $1, l_1, \cdots, l_m$ such that for the function

$$H(z, z') \equiv F(z, z') + l_\alpha G^\alpha(z, z')$$

the first variation of $\int_C H(z, z) dt$ vanishes along C.)

Then we have the theorem:

THEOREM 1. *Under the hypotheses* (4.1), (4.2), *and* (4.3) *the inequality*†

$$(4.4) \qquad \qquad \Omega_H(z(t_0), z'(t_0 - 0), z'(t_0 + 0)) \leqq 0$$

holds.

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* We do not require normality on subarcs of C.

† Here, where several integrands are involved, there are several possible Ω-functions. The subscript indicates the integrand used in defining Ω by (3.9). An analogous notation will be used for the $\mathcal{E}$-function.

For compactness, we define

(4.5) $$p \equiv z'(t_0 - 0), \qquad r \equiv z'(t_0 + 0).$$

Suppose that (4.4) fails to hold; then

(4.6) $$\Omega_H(z(t_0), p, r) > 0.$$

By continuity, there is a positive number δ, so small that C has no corner other than $z(t_0)$ on the arc $z = z(t)$, $t_0 - \delta \leqq t \leqq t_0 + \delta$, for which

(4.7) $$\Omega_H(z(\bar{t}), z'(\bar{t}), r) > 0 \quad \text{if} \quad t_0 - \delta \leqq \bar{t} < t_0.$$

Define $C_{\tau,0}$ to be the curve obtained from C by interchanging the arcs $z = z(t)$, $(t_0 - \delta \leqq t \leqq t_0)$, and $z = z(t)$, $(t_0 \leqq t \leqq t_0 + \tau)$, where $0 \leqq \tau \leqq \delta$. The curve $C_{\tau,0}$ is then defined by equations $z = \zeta_\tau(t)$, $(t_1 \leqq t \leqq t_2)$, where

(4.8)
$$\begin{aligned}
\zeta_\tau(t) &= z(t), & t_1 &\leqq t < t_0 - \delta, \\
\zeta_\tau(t) &= z(t + \delta) - z(t_0) + z(t_0 - \delta), & t_0 - \delta &\leqq t < t_0 - \delta + \tau, \\
\zeta_\tau(t) &= z(t - \tau) - z(t_0) + z(t_0 + \tau), & t_0 - \delta + \tau &\leqq t < t_0 + \tau, \\
\zeta_\tau(t) &= z(t), & t_0 + \tau &\leqq t \leqq t_2.
\end{aligned}$$

(Thus $C_{0,0}$ coincides with C. For τ near 0, it is clear that $C_{\tau,0}$ is in S.)

Given any set of m (vector) functions $\eta_i(t) = (\eta_i{}^0(t), \cdots, \eta_i{}^q(t))$ of class D' on $[t_1, t_2]$ and vanishing at t_1 and t_2, we shall define

(4.9)
$$\begin{aligned}
\Phi(\tau, b) &\equiv \Phi(\tau, b_1, \cdots, b_m) \equiv \mathcal{J}(C_{\tau, b_1, \cdots, b_m}), \\
\Gamma^i(\tau, b) &\equiv \Gamma^i(\tau, b_1, \cdots, b_m) \equiv G^i(C_{\tau, b_1, \cdots, b_m}),
\end{aligned}$$

where $C_{\tau,b}$ is given by the following:

(4.10) $C_{\tau,b} \equiv C_{\tau, b_1, \cdots b_m}$ *is defined by* $z = z(t; \tau, b) \equiv \zeta_\tau(t) + b_\alpha \eta_\alpha(t)$, $(t_1 \leqq t \leqq t_2)$.

For τ and b near 0, this lies in S. As we know,

(4.11) $$\left.\frac{\partial \Gamma^i}{\partial b_\kappa}\right|_{\tau=b=0} = \int_{t_1}^{t_2} \eta_\kappa{}^\alpha(t) \left\{ -\frac{d}{dt} G_\alpha^{\,j}(z, \dot{z}) + G_{z^\alpha}^{\,j}(z, \dot{z}) \right\} dt.$$

We have assumed that C is normal. Hence there exist functions $\eta_1(t), \cdots, \eta_m(t)$ of the type described above for which the jacobian

(4.12) $$\left| \frac{\partial \Gamma^i}{\partial b_\kappa} \right|_{\tau=b=0} \neq 0.$$

The equations

(4.13) $$\Gamma^i(\tau, b) = \gamma_j, \qquad\qquad j = 1, \cdots, m,$$

have the initial solutions $\tau = b = 0$, and at this solution the jacobian with re-

spect to the b_j is not zero by (4.12). So by the implicit functions theorem*
there are functions $b = b_\kappa(\tau)$, $(\kappa = 1, \cdots, m)$, defined and of class C' for all
small non-negative τ, such that

(4.14) $\Gamma^j(\tau, b(\tau)) \equiv \gamma_j$, $j = 1, \cdots, m$.

Let us define

(4.15) $H(\tau, b) = \Phi(\tau, b) + l_\alpha \Gamma^\alpha(\tau, b) \equiv \int_C H(z, \dot{z}) dt$.

Since the first variation of $\int H dt$ vanishes along C, we have

(4.16) $\dfrac{\partial H}{\partial b_k} = 0$ for $\tau = b = 0$, $k = 1, \cdots, m$.

By (3.10) and the mean value theorem, for $b = 0$ we have

(4.17) $\dfrac{\partial H}{\partial \tau} = -\int_{t_0-\delta}^{t_0} \Omega_H(z(t), \dot{z}(t), r) dt = -\delta \Omega_H(z(\bar{t}), \dot{z}(\bar{t}), r)$

for some $\bar{t}$ between $t_0 - \delta$ and t_0. Combining (4.16) and (4.17) we have

(4.18) $\dfrac{d}{d\tau} H(\tau, b(\tau)) = \dfrac{\partial H}{\partial \tau} + b_\alpha'(\tau) \dfrac{\partial H}{\partial b_\alpha} = \dfrac{\partial H}{\partial \tau} < 0$

for $\tau = 0$. But by (4.14) and (4.15),

(4.19) $\dfrac{d}{d\tau} H(\tau, b(\tau)) = \dfrac{d}{d\tau} \Phi(\tau, b(\tau))$.

Now by (4.18) and (4.19) we see that

(4.20) $\mathcal{J}(C_{\tau, b(\tau)}) = \Phi(\tau, b(\tau)) < \Phi(0, b(0)) = \Phi(0, 0) = \mathcal{J}(C)$

for all small positive τ, while by (4.14)

(4.21) $\mathcal{G}^i(C_{\tau, b(\tau)}) = \Gamma^i(\tau, b(\tau)) = \gamma_j$.

This contradicts the minimizing property of C, and establishes the theorem.

   5. **Dresden's corner condition: the non-parametric form.** In order that the
proof of Theorem 1 shall apply also to integrands $F(z, z')$, $G^i(z, z')$ arising
by (1.1), it is only necessary to verify that $z^{0\prime}(t) > 0$ along each of the com-
parison curves $C_{\tau, b}$ used. If C has a class D' representation $z^i = z^i(z^0)$,

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   * The fact that Γ^i is defined only for $\tau \geq 0$ does not prevent our use of the theorem; we could
for example extend the range of definition of Γ^i as a function of τ and then finally disregard the
extension.

$(i = 1, \cdots, q)$, then on identifying the parameter t with z^0 we have $\dot{z}^0(t) \equiv 1$ along C. But along $C_{\tau, b(\tau)}$ we have

$$\frac{d}{dt} z^0(t, \tau, b) = \dot{\zeta}_a{}^0(t) + b_a \dot{\eta}_a{}^0(t) = 1 + b_a \dot{\eta}_a{}^0(t) \geq 1 - |b| \max |\dot{\eta}^0(t)| > \tfrac{1}{2}$$

if $|b|$ be small enough. So the comparison curves used, for which $|b|$ is near 0, are allowable, and the proof applies to the non-parametric case also. Recalling (3.15), we have the following result:

THEOREM 2. *Assume the following conditions satisfied:*

(5.1) *The curve $y = y(x)$, $(x_1 \leq x \leq x_2)$, is interior to S and gives a strong relative minimum to the integral*

$$\mathcal{J}[y] = \int_{x_1}^{x_2} f(x, y, \dot{y}) dx$$

in the class of all D' curves which join $(x_1, y(x_1))$ to $(x_2, y(x_2))$ and give assigned values γ_j to the integrals

$$\mathcal{G}^j[y] \equiv \int_{x_1}^{x_2} g^j(x, y, \dot{y}) dx, \qquad\qquad j = 1, \cdots, m.$$

(5.2) *$y = y(x)$ has a corner at x_0.*

(5.3) *The curve $y = y(x)$ is normal.*

(5.4) *$1, l_1, \cdots, l_m$ are the isoperimetric constants, and*

$$h(x, y, y') \equiv f(x, y, y') + l_a g^a(x, y, y').$$

Then

$$\omega_h(x_0, y(x_0), y'(x_0 - 0), y'(x_0 + 0)) \leq 0.$$

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