SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

II. EXISTENCE THEOREMS FOR ISOPERIMETRIC PROBLEMS IN THE PLANE*

BY

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If we seek to find a curve $y=y(x)$, $(x_1 \leqq x \leqq x_2)$, which minimizes an integral $\mathcal{J}[y]=\int f(x, y, \dot{y})dx$ in the class of curves joining two points (x_1, y_1) and (x_2, y_2), a reasonable beginning is to choose a sequence $y=y_n(x)$ for which $\mathcal{J}[y_n]$ tends to the lower bound μ of values of $\mathcal{J}[y]$, and then (under suitable hypotheses) show that a subsequence of the $y_n(x)$ tends uniformly to a limit function $y_0(x)$. However, the uniform convergence of y_n to y_0 does not ensure that $\mathcal{J}[y_n]$ tends to $\mathcal{J}[y_0]$. One way of overcoming this difficulty is to assume that $\mathcal{J}[y]$ is quasi-regular, from which we find $\mathcal{J}[y_0] \leqq \lim\inf \mathcal{J}[y_n]=\mu$. Since $\mathcal{J}[y_0]$ cannot be less than μ, by the definition of μ, it follows that $\mathcal{J}[y_0]=\mu$. This method of attack was devised by L. Tonelli, and has been applied by him and others, including Graves, Manià, Cinquini, and myself, to a number of different types of variation problems. A second method is to add hypotheses $f(x, y, y')$ which will guarantee that for some sequence $\{y_n(x)\}$ not only does $y_n(x)$ converge uniformly to $y_0(x)$, but also $y_n'(x)$ tends in some manner to $y_0'(x)$, so that $f(x, y_0(x), \dot{y}_0(x))$ is the limit of $f(x, y_n(x), \dot{y}_n(x))$ in some manner which will ensure the convergence of $\mathcal{J}[y_n]$ to $\mathcal{J}[y_0]$. This method does not seem to have been nearly as thoroughly exploited as the first. A very interesting existence theorem, established by this type of reasoning, is to be found in a paper by Hans Lewy.†

Consider now the isoperimetric problem of minimizing $\mathcal{J}[y]$ while keeping $\mathcal{G}[y] \equiv \int g(x, y, y')dx$ constantly equal to a number γ. If we try to use the first of these methods, we find a sequence for which $\mathcal{J}[y] \rightarrow$min. while $\mathcal{G}[y_n]=\gamma$. But now the limit curve $y=y_0(x)$ must satisfy the equation $\mathcal{G}[y_0]=\gamma=\lim \mathcal{G}[y_n]$, and in order that this shall follow from the uniform convergence of y_n to y_0 the integrand $g(x, y, y')$ must be strongly restricted, in fact, it must be linear in y'. And, in fact, for isoperimetric problems in

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† H. Lewy, *Ueber die Methode der Differenzengleichungen* . . . , Mathematische Annalen, vol. 98 (1928), pp. 107–124.

non-parametric form the only existence theorems* known to me make exactly this requirement on $\mathcal{G}[y]$ or on $\mathcal{J}[y]$.

This suggests turning to the second proof-pattern, and putting conditions on $\mathcal{J}$ and $\mathcal{G}$ which will guarantee that $y_n'(x)$ tends to $y_0'(x)$ in some manner strong enough to ensure that $\mathcal{G}[y_n]$ tends to $\mathcal{G}[y_0]$ and $\mathcal{J}[y_n]$ tends to $\mathcal{J}[y_0]$. In this note (and again in the fourth and fifth of the series) we set forth conditions guaranteeing this.

The method here used is based on a formula (3.18) of I. If Π is a polygon, and two consecutive sides of Π have slopes α, β, respectively, and if it be known that

$$\omega_f(x, y, \alpha, \beta) \geqq 0$$

for all (x, y), then the interchange of these sides does not increase $\mathcal{J}[y]$. Furthermore, if the integrand $g(x, y, y')$ happens to be independent of x and y, the interchange leaves $\mathcal{G}(\Pi)$ unaltered. Suppose then that $\omega_f(x, y, p, r) \geqq 0$ if $p \geqq r$. If we choose a minimizing sequence of polygons $\Pi_n : y = y_n(x)$, we may suppose that for each n the slope of the sides of Π_n increases monotonically. For if ever a side is succeeded by one of lesser slope, we may interchange these sides, leaving $\mathcal{G}$ unaltered and not increasing $\mathcal{J}$. Therefore each function $y_n(x)$ is convex, and when we select a subsequence converging to a limit $y_0(x)$ it will follow that $y_n'(x) \to y_0'(x)$ for almost all x.

In the theorem to be proved, we assume somewhat less than the condition that $\omega_f(x, y, p, r) \geqq 0$ if $p \geqq r$, but the essence of the proof is unchanged; our minimizing sequence is made to consist of several convex arcs.

The notation and definitions used in I will be continued in this note. Also, we add the rather obvious abbreviation "a.c." for "absolutely continuous."

1. **Proofs of some lemmas.** Suppose that $F(z, z')$ is a parametric integrand having the continuity properties required in I, §1, and that C is a rectifiable curve. It is easy to show that if $\{\Pi_n\}$ is a sequence of polygons inscribed in C and having the same initial and final points as C, and the length of the longest side of Π_n tends to 0 as $n \to \infty$, then $\mathcal{J}(\Pi_n) \to \mathcal{J}(C)$. For problems in non-parametric form this is no longer true, as may be shown by examples.†

Suppose, however, that $F(z, z')$ is obtained from a non-parametric integrand $f(x, y, y')$ by (1.1) of I and satisfies the following condition:

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* L. Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2, pp. 552, 553.

† M. Lavrentieff, *Sur quelques problèmes du calcul des variations*, Annali di Matematica, (4), vol. 4 (1927), pp. 7–28.

L. Tonelli, *Sur une question du calcul des variations*, Matematicheskii Sbornik, vol. 33 (1936), pp. 87–98.

(1.1) *For every bounded subset S_0 of S there are positive numbers δ and a and a number $b \geqq 0$ such that*

$$|F_{z^i}(z, z')| \leqq a|z'| + bF(z_0, z')$$

whenever z_0 is in S_0 and $|z - z_0| < \delta$.

Then it can be shown* that if $y = y(x)$, $(a \leqq x \leqq b)$, is a.c., then for every $\epsilon > 0$ there exists a function $y_\epsilon(x)$ having a continuous derivative on $a \leqq x \leqq b$, such that $y_\epsilon(a) = y(a)$ and $y_\epsilon(b) = y(b)$, and such also that

$$\left| \int_a^b f(x, y_\epsilon, \dot{y}_\epsilon)dx - \int_a^b f(x, y, \dot{y})dx \right| < \epsilon.$$

Moreover, if several integrands $f^1, \cdots, f^k$ are such that for each of them (1.1), holds an examination of Tonelli's proof shows that $y_\epsilon(x)$ can be so chosen that

$$\left| \int_a^b f^i(x, y_\epsilon, \dot{y}_\epsilon)dx - \int_a^b f^i(x, y, \dot{y})dx \right| < \epsilon, \qquad i = 1, \cdots, k.$$

In particular, if $f(x, y, y')$ is a function of y' alone, (1.1) surely holds; we need only take $a = \delta = 1$, $b = 0$.

We shall proceed to establish a theorem on isoperimetric problems. Several of the somewhat lengthy hypotheses of this theorem occur again as hypotheses of later theorems, and some of the stages of the proof will also recur; so we separate the hypotheses and split the proof into a sequence of lemmas.

The hypotheses for our first theorem are the following:

(1.2) *The functions $f(x, y, y')$ and $g^j(x, y, y')$, $(j = 1, \cdots, m)$, satisfy the continuity conditions of* I, §1, *on a closed set S_1.*

(1.3) *The functions $f(x, y, y')$ and $g^j(x, y, y')$ satisfy* (1.1) , *and on every bounded portion of S the relations*

$$\lim_{|y'| \to \infty} f(x, y, y')/|y'| = \infty,$$

$$\lim_{|y'| \to \infty} g^j(x, y, y')/f(x, y, y') = 0, \qquad j = 1, \cdots, m,$$

hold uniformly in (x, y).

(1.4) *The class K consisting of all a.c. curves $y = y(x)$ joining two fixed points (x_0, y_0) and (X, Y) with $x_0 < X$ and such that the integrals $\mathcal{G}^j[y]$ have given fixed values γ_j, $(j = 1, \cdots, m)$, is not empty.*

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* The proof requires only a minor modification of that given by Tonelli, loc. cit. The condition on F_{z^i}, or f_{y}, here stated is superfluous in this connection.

(1.5) *The points* (x_0, y_0) *and* (X, Y) *and the integrals* $\mathcal{J}$, $\mathcal{G}^i$ *have the property that there exist numbers* $a_0, a_1, \cdots, a_m$ *with* $a_0 \geqq 0$ *such that for every number* H *there is a bounded subset* S_H *of* S *containing all a.c. curves* $y=y(x)$ *lying in* S, *joining* (x_0, y_0) *to* (X, Y), *and having*

$$a_0\mathcal{J}[y] + a_\alpha \mathcal{G}^\alpha[y] < H.$$

(1.6) *The symbol* y *stands for a single variable, and the infinite interval* $-\infty < y' < \infty$ *can be subdivided into a finite number of subintervals* $I_1, I_2, \cdots, I_k$ *(not necessarily in that order of precedence from* $-\infty$ *to* ∞*) such that for all* (x,y) *the following relations hold*:

$$\omega_f(x, y, p, r) \geqq 0 \text{ if } p \,\varepsilon\, I_j \text{ and } r \,\varepsilon\, I_h, j > h;$$
$$\sigma_j\omega_f(x, y, p, r) \geqq 0 \text{ if } p \,\varepsilon\, I_j \text{ and } r \,\varepsilon\, I_j \text{ and } p \geqq r,$$

where $\sigma_j = \pm 1, j=1, \cdots, k$.

(1.7) *The functions* $g^i(x, y, y')$ *are independent of* x *and* y.

(1.8) S *is the entire* (x, y)*-space.*

(Observe that only in (1.6) do we restrict the number q of variables y^i.) With these hypotheses we can state the following theorem:

THEOREM 1. *Under hypotheses* (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), *and* (1.8), *the class* K *contains a curve* $y=y_0(x)$, $(x_0 \leqq x \leqq X)$, *for which* $\mathcal{J}[y]$ *assumes its least value.*

To establish the theorem we first prove six lemmas.

LEMMA 1. *Under hypotheses* (1.2), (1.3), (1.4), *and* (1.5), *the greatest lower bound* μ *of* $\mathcal{J}[y]$ *on the class* K *is finite, and all the curves of* K *for which* $\mathcal{J}[y] < \mu+1$ *lie interior to a sphere* Q.

First, let C^* be a curve of the class K. By hypothesis (1.5), the subclass K_1 of K on which $\mathcal{J}(C) < \mathcal{J}(C^*)+1$ lies in a bounded part S_H of S. Let Q be a sphere (including boundary) large enough so that S_H is interior to Q. Then all curves of K_1, and a fortiori all curves C for which $\mathcal{J}(C) < \mu+1 \leqq \mathcal{J}(C^*)+1$, lie in Q. This establishes the second statement of the lemma. The g.l.b. of $\mathcal{J}(C)$ on K_1 is clearly the same as its g.l.b. on K. By hypothesis (1.3), there is a number c such that $f(x, y, y') > 1$ if (x, y) is in QS and $|y'| > c$. On the bounded closed set $[(x, y) \text{ in } QS, |y'| \leqq c]$ the function $f(x, y, y')$ is continuous, hence bounded. So $f(x, y, y')$ is bounded below for all (x, y) in QS and all y'; say $f(x, y, y') \geqq \nu$. It follows that for all curves C of K_1

$$\mathcal{J}(C) \geqq \int_{x_0}^{X} \nu dx = \nu(X - x_0),$$

and $\mathcal{J}(C)$ is bounded below on K_1. Therefore its g.l.b. μ is finite.

LEMMA 2. *If the integrands f, g all satisfy* (1.1) *and hypothesis* (1.4) *holds, then there exists a sequence* $\{\Pi_n\}$ *of polygons* $\Pi_n: y=y_n(x)$, $(x_0 \leqq x \leqq X)$, *such that* $y_n(x_0)=y_0$ *and* $y_n(X)=Y$ *for all* n, *and*

$$\lim_{n\to\infty} \mathcal{J}(\Pi_n) = \mu, \qquad \lim_{n\to\infty} \mathcal{G}^j(\Pi_n) = \gamma_j, \qquad j = 1, \cdots, m.$$

Let $C_1, C_2, \cdots$ be a sequence of curves of K such that $\mathcal{J}(C_n)\to\mu$. Since all the integrands satisfy (1.1), for each n there exists a curve C_n^*: $y=y_n^*(x)$, $(x_0\leqq x\leqq X)$, joining (x_0, y_0) to (X, Y), having $y_n^{*\prime}$ continuous, such that

$$(1.8) \qquad |\mathcal{J}(C_n^*) - \mathcal{J}(C_n)| < 1/n, \qquad |\mathcal{G}^j(C_n^*) - \mathcal{G}^j(C_n)| < 1/n, \qquad j = 1, \cdots, m.$$

Therefore $\mathcal{J}(C_n^*)\to\mu$ and $\mathcal{G}(C_n^*)\to\gamma_j$.

If $C: y=y(x)$, $(x_0\leqq x\leqq X)$, is such that $y'(x)$ is continuous, we form a sequence of inscribed polygons Π_p: $y=y_p(x)$ whose successive vertices are the points

$$(x_0, y_0), (x_0+\delta_p, y(x_0+\delta_p)), \cdots, (x_0+k\delta_p, y(x_0+k\delta_p)), \cdots, (X, Y),$$

where $\delta_p=(X-x_0)/p$. From the continuity of $y'(x)$ it is easily seen that $y_p(x)$ tends uniformly to $y(x)$ and that (neglecting the vertices of the polygons) $y_p'(x)$ tends uniformly to $y'(x)$. Hence for every $\epsilon>0$ there is a Π_p such that $|\mathcal{J}(\Pi_p)-\mathcal{J}(C)|<\epsilon$ and $|\mathcal{G}^j(\Pi_p)-\mathcal{G}^j(C)|<\epsilon$, $(j=1, \cdots, m)$. Applying this to each C_n^*, we see that for each n there is a polygon Π_n: $y=y_n(x)$, $(x_0\leqq x\leqq X)$, with $y_n(x_0)=y_0$ and $y_n(X)=Y$ for which

$$(1.9) \qquad |\mathcal{J}(\Pi_n) - \mathcal{J}(C_n^*)| < \frac{1}{n}, \qquad |\mathcal{G}^j(\Pi_n) - \mathcal{G}^j(C_n^*)| < \frac{1}{n}\cdot$$

From (1.8) and (1.9) we obtain

$$\mathcal{J}(\Pi_n) \to \mu, \qquad \mathcal{G}^j(\Pi_n) \to \gamma_j.$$

The lemma is therefore established.

LEMMA 3. *Assume that the following conditions hold*:

(a) *Q is a bounded closed set of points (x, y).*

(b) *The functions $f(x, y, y')$ and $g(x, y, y')$ are continuous functions of their arguments for all (x, y) in Q and all y'.*

(c) *f is non-negative for (x, y) in Q and all y'.*

(d) *For every positive number N there is an $M\geqq 0$ such that if (x, y) is in Q and $|g(x, y, y')|\geqq M$, then $f(x, y, y')\geqq N|g(x, y, y')|$.*

(e) *The a.c. functions $z=z_n(t)$, $(a_n\leqq t\leqq b_n)$, represent a sequence of curves $\{C_n\}$ lying in Q and such that the integrals $\mathcal{J}(C_n)$ are bounded, and the functions*

*$z_n^0(t)$ all satisfy the same Lipschitz condition.**

Then the integrals†

$$\int_E G(z_n(t),\ \dot{z}_n(t))dt$$

are equi-absolutely continuous, in the sense that for every $\epsilon > 0$ there is a $\delta > 0$ such that if E is in $[a_n, b_n]$ and $mE < \delta$, then

$$\left| \int_E G(z_n,\ \dot{z}_n)dt \right| < \epsilon.$$

By hypothesis, there are numbers H, L such that $\mathcal{J}(C_n) \leqq H$ and $\dot{z}_n^0(t) \leqq L$. Let ϵ be a positive number, and let $N = 2H/\epsilon$. Hypothesis (d), written in parametric notation, informs us that there is an $M \geqq 0$ such that

$$F(z, \dot{z}) \geqq N\,|\,G(z, \dot{z})\,| \quad \text{if} \quad z \in Q \quad \text{and} \quad |\,G(z, \dot{z})\,| \geqq M\dot{z}^0.$$

Therefore, since $F \geqq 0$, we have for all $z \in Q$ and all $\dot{z}$ with $\dot{z}^0 > 0$,

$$F(z, \dot{z}) + MN\dot{z}^0 \geqq N\,|\,G(z, \dot{z})\,|.$$

In particular, since $L \geqq \dot{z}_n^0(t) > 0$ almost everywhere in $[a_n, b_n]$, the inequality

$$F(z_n, \dot{z}_n) + MNL \geqq F(z_n, \dot{z}_n) + MN\dot{z}_n^0 \geqq N\,|\,G(z_n, \dot{z}_n)\,|$$

holds for almost all t in the interval $[a_n, b_n]$. Now take $\delta = \epsilon/2ML$. If E is in $[a_n, b_n]$ and $mE < \delta$, then

$$\begin{aligned} \left| \int_E G(z_n, \dot{z}_n)dt \right| &\leqq \int_E |\,G(z_n, \dot{z}_n)\,|\,dt \leqq N^{-1} \int_E [F(z_n, \dot{z}_n) + MNL]dt \\ &\leqq N^{-1}\left[\int_{a_n}^{b_n} F(z_n, \dot{z}_n)dt + MNL\cdot mE \right] \\ &\leqq N^{-1}[H + NML\delta] = \epsilon. \end{aligned}$$

The lemma is therefore established.

Remark. Clearly it would be enough to assume in place of (c) that $f(x, y, y')$ is bounded below for $(x, y) \in Q$ and all y', and to assume in place of (d) that for each $N > 0$ there is an $M > 0$ and a k independent of M and N such that if $(x, y) \in Q$ and $|\,g(x, y, y')\,| \geqq M$, then $f(x, y, y') \geqq N\,|\,g(x, y, y')\,| - k$. For if k' is a lower bound for $f(x, y, y')$ for $(x, y) \in Q$, we take K to be the

* This condition on $z_n^0(t)$ certainly holds if $z_n^0(t) = t$; that is, if the C_n are all represented in the form $y = y(x)$, $(a_n \leqq x \leqq b_n)$. As always, we tacitly assume that $z_n^{0\prime}(t) > 0$ for almost all t.

† $F(z, \dot{z})$ and $G(z, \dot{z})$ denote, respectively, the parametric integrands associated with $f(x, y, y')$ and $g(x, y, y')$.

larger of k and $-k'$. Then the functions $f(x, y, y')+K$ and $g(x, y, y')$ satisfy the hypotheses of Lemma 3.

LEMMA 4.* *Assume that the following conditions hold*:

(a) *Q is a bounded closed set of points (x, y).*

(b) *$f(x, y, y')$ is continuous for all (x, y) in Q and all y'.*

(c) $\lim_{|y'|\to\infty} f(x, y, y')/|y'| = \infty$ *uniformly for all (x, y) in Q.*

(d) *$\{C_n\}$ is a sequence of a.c. curves $y=y_n(x)$, $(a_n \leqq x \leqq b_n)$, lying in Q and such that the integrals $\mathcal{J}(C_n)$ are bounded.*

Then the functions $y_n(x)$ are equi-absolutely continuous.

The hypotheses of Lemma 3 are satisfied if we take $g(x, y, y')=|y'|$; so the integrals

$$\int_E |\dot{y}_n|\, dx$$

are equi-absolutely continuous. Let ϵ be a positive number. There is a $\delta>0$ such that the integral above is less than ϵ if E is in $[a_n, b_n]$ and $mE<\delta$. If $(\alpha_1, \beta_1), \cdots, (\alpha_k, \beta_k)$ is a set of non-overlapping subintervals of $[a_n, b_n]$ having length $\sum(\beta_i-\alpha_i)<\delta$, then

$$(1.10)\qquad \sum_{i=1}^{k} |y_n(\beta_i) - y_n(\alpha_i)| = \sum_{i=1}^{k} \left| \int_{\alpha_i}^{\beta_i} \dot{y}_n(x)dx \right| \leqq \sum_{i=1}^{k} \int_{\alpha_i}^{\beta_i} |\dot{y}_n(x)|\, dx < \epsilon.$$

This establishes our lemma.

LEMMA 5. *Let hypotheses* (a), (b), (c), *and* (d) *of Lemma* 3 *be satisfied. Let $\{y_n(x)\}$ be a sequence of a.c. functions all defined on the same interval $[x_0, X]$ and converging everywhere in $[x_0, X]$ to a limit $y_0(x)$. Suppose further that $\dot{y}_n(x)$ tends to $\dot{y}_0(x)$ for almost all x in $[x_0, X]$. Then* $\lim_{n\to\infty} \mathcal{G}[y_n]=\mathcal{G}[y_0]$ *and* $\liminf_{n\to\infty} \mathcal{J}[y_n] \geqq \mathcal{J}[y_0]$. *Moreover, if hypothesis* (c) *of Lemma* 4 *holds, the limit curve $y=y_0(x)$ is a.c.*

For almost all x we have $y_n(x)\to y_0(x)$ and $\dot{y}_n(x)\to\dot{y}_0(x)$; so for all such x the limit of $f(x, y_n, \dot{y}_n)$ is $f(x, y_0, \dot{y}_0)$, and the limit of $g(x, y_n, \dot{y}_n)$ is $g(x, y_0, \dot{y}_0)$. The function $f(x, y, y')$ is non-negative; therefore by the lemma of Fatou† we obtain

$$\liminf_{n\to\infty} \int_{x_0}^{X} f(x, y_n, \dot{y}_n)dx \geqq \int_{x_0}^{X} f(x, y_0, \dot{y}_0)dx.$$

* Lemmas 3 and 4 are closely related to some theorems established by M. Nagumo (*Ueber die gleichmässige Summierbarkeit und ihre Anwendung auf ein Variationsproblem*, Japanese Journal of Mathematics, vol. 6 (1929), pp. 173–182).

† P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Mathematica, vol. 30 (1906), p. 375

The integrals of the $g(x, y_n, \dot{y}_n)$ are equi-absolutely continuous functions of sets, by Lemma 3; so by a known convergence theorem

$$\lim_{n\to\infty}\int_{x_0}^{X} g(x, y_n, \dot{y}_n)dx = \int_{x_0}^{X} g(x, y_0, \dot{y}_0)dx.$$

The final conclusion is obtained at once from (1.10) if we let $n\to\infty$.

LEMMA 6. *If $y=y_n(x)$, $(x_1\leqq x\leqq x_2)$, is a sequence of real-valued functions defined and convex on the interval $x_1\leqq x\leqq x_2$, and the $y_n(x)$ converge uniformly to a limit function $y_0(x)$ on $x_1\leqq x\leqq x_2$, then* $\lim \dot{y}_n(x)=\dot{y}_0(x)$ *for almost all x.*

It is easy to see that $y_0(x)$ is also a convex function. Hence each derivative $y_j'(x)$, $(j=0, 1, 2, \cdots)$, is defined almost everywhere on $[x_1, x_2]$ and is monotone increasing. Let E be the set of measure x_2-x_1 on which all these derivatives are defined. Let x_0 be a point of E. If $h>0$ is such that $x+h\leqq x_2$, then

$$\lim_{n\to\infty}\frac{y_n(x_0+h)-y_n(x_0)}{h}=\frac{y_0(x_0+h)-y_0(x_0)}{h}. \tag{1.11}$$

But since $y_n(x)$ is convex, we know that $y_n'(x_0)\leqq[y_n(x_0+h)-y_n(x_0)]/h$. Hence by this and (1.11) we obtain

$$\limsup_{n\to\infty} y_n'(x_0)\leqq\frac{y_0(x_0+h)-y_0(x_0)}{h},\qquad h>0. \tag{1.12}$$

Now let $h\to 0$; this yields

$$\limsup_{n\to\infty} y_n'(x_0)\leqq y_0'(x_0). \tag{1.13}$$

Repeating the argument with $h<0$, we find

$$\liminf_{n\to\infty} y_n'(x_0)\geqq y_0'(x_0). \tag{1.14}$$

Inequalities (1.13) and (1.14) show that $y_n'(x_0)\to y_0'(x_0)$. Since x_0 is any point of E, this establishes the lemma.

2. **Proof of the theorem; examples.** The preliminaries being disposed of, we take up the proof of the theorem. Let μ be the greatest lower bound of $\mathcal{J}[y]$ on the class K; by Lemma 1 this is finite. By Lemma 2, we can select a sequence of polygons Π_n^*: $y=y_n^*(x)$, $(x_0\leqq x\leqq X)$, joining (x_0, y_0) and (X, Y) and such that

$$\mathcal{J}(\Pi_n^*)\to\mu,\qquad \mathcal{G}^j(\Pi_n^*)=\gamma_{j,n}\to\gamma_j,\qquad j=1,\cdots,m.$$

Suppose that AB and BC are consecutive sides of one of the polygons Π_n^*, having respective slopes α and β. Let D be the fourth vertex of the parallelogram having AB and BC as sides. By "interchanging" AB and BC in the polygon Π_n^* we shall mean (as in I) the operation of forming the polygon Π_n' which has all the sides of Π_n^* except AB and BC, and has sides AD and DC to replace them. From (3.13) or (3.18) of I we know that if $\omega_f(x, y, \alpha, \beta) \geqq 0$ for all (x, y) in the parallelogram $ABCD$, then

$$\mathcal{J}(ADC) - \mathcal{J}(ABC) \leqq 0;$$

whence $\mathcal{J}(\Pi_n') \leqq \mathcal{J}(\Pi_n^*)$. Since by hypothesis (1.7) the integrands $g^j(x, y, y')$ are independent of x and y, it is obvious that $\mathcal{G}^j(\Pi_n') = \mathcal{G}^j(\Pi_n^*)$, $(j=1, \cdots, n)$.

In particular, if the slope α of AB belongs to an interval I_j and β belongs to an interval I_h with $h<j$, then $\omega_f(x, y, \alpha, \beta) \geqq 0$ for all (x, y), by hypothesis (1.6). So on Π_n^* we search for the first side (that is, side with the least x) whose slope belongs to the interval I_1. If this is not already the first side of Π_n^*, we can interchange it successively with all preceding sides so as to bring it to first place. These interchanges do not increase the integral $\mathcal{J}$ and leave the integrals $\mathcal{G}^j$ unchanged. Next we locate the second one of these sides whose slope belongs to the interval I_1. We can interchange this with preceding sides (if any) so as to bring it to second place; the integral $\mathcal{J}$ is not increased, and the integrals $\mathcal{G}^j$ are unchanged. Proceeding thus, we finally bring all those sides of Π_n^* whose slopes are in I_1 into first, second, $\cdots$, places in unbroken succession. Next we locate the sides whose slopes are in I_2, and bring them together after the sides whose slopes are in I_1; and continue so until all intervals I_h have been considered. We thus have a polygon Π_n' joining (x_0, y_0) to (X, Y), having $\mathcal{J}(\Pi_n')$ not greater than $\mathcal{J}(\Pi_n^*)$, and such that $\mathcal{G}^j(\Pi_n') = \mathcal{G}^j(\Pi_n^*)$, $(j=1, \cdots, m)$. Moreover, if AB and CD are sides of Π_n' such that the slope of AB is in I_j and the slope of CD is in I_h with $h>j$, then AB precedes CD on the polygon.

Now for any particular number h of the set $1, \cdots, p$ we consider the aggregate of sides of Π_n' whose slopes belong to I_h. Suppose, to be specific, that the number σ_h of (2.5) is $+1$. If AB and BC are consecutive sides of the aggregate having respective slopes α and β, and $\beta<\alpha$, then by hypothesis (1.6) we have $\omega_f(x, y, \alpha, \beta) \geqq 0$ for all (x, y). So, by (3.13) of I, if we interchange AB and BC, the value of $\mathcal{J}$ is not increased. Therefore in this aggregate of sides we seek one having least slope. It can be interchanged successively with all preceding sides so as to bring it to first place in the aggregate; in the interchange the value of $\mathcal{J}$ is not increased and the values of the $\mathcal{G}^j$ are unchanged. Of the remaining sides of the aggregate we seek the one with least slope; this can likewise be brought to second place in the aggregate. Proceeding thus, we find that we can rearrange the sides of the aggregate so as to have their slopes monotonically increasing; the rearrangement leaves the $\mathcal{G}^j$ unchanged and does not increase the $\mathcal{J}$. If the value of σ_h had been -1, we could have carried out a rearranging process so as to have the slopes of the sides steadily decreasing instead of increasing.

This process having been carried out for each value of h, $(h=1, \cdots, p)$, we arrive finally at a polygon Π_n: $y=y_n(x)$, $(x_0 \leqq x \leqq X)$, joining (x_0, y_0) and (X, Y), having $\mathcal{J}(\Pi_n) \leqq \mathcal{J}(\Pi_n^*)$ and $\mathcal{G}^j(\Pi_n) = \mathcal{G}^j(\Pi_n^*)$, $(j=1, \cdots, m)$. The polygon Π_n consists of at most k polygonal arcs on each of which the slope is either monotonic increasing or monotonic decreasing; that is, Π_n is composed of at most k convex or concave polygonal arcs.

Let us suppose that this has been done for each Π_n^*. For all the polygons Π_n thus obtained the sum

$$a_0\mathcal{J}(\Pi_n) + a_\alpha \mathcal{G}^\alpha(\Pi_n) \leqq a_0\mathcal{J}(\Pi_n^*) + a_\alpha \mathcal{G}^\alpha(\Pi_n^*)$$

is bounded above, since the sequences $\mathcal{J}(\Pi_n^*)$ and $\mathcal{G}^j(\Pi_n^*)$ converge. So by hypothesis (1.5) all the polygons Π_n lie in a bounded set. By Lemma 4, the functions $y_n(x)$ are equi-absolutely continuous, hence equi-continuous; hence by Ascoli's theorem it is possible to select a subsequence which converges uniformly to a limit function $y_0(x)$, $(x_0 \leqq x \leqq X)$. We suppose that this subsequence is the whole sequence $\{y_n(x)\}$.

For each n, we can choose points $x_{0,n}=x_0, x_{1,n}, \cdots, x_{k,n}=X$ such that for $x_{i-1,n} \leqq x \leqq x_{i,n}$ the derivative $y_n'(x)$ is in I_i whenever it is defined. (If y_n' is never in I_i, then $x_{i,n}=x_{i-1,n}$.) It is possible to choose a subsequence of y_n (we suppose it to be the whole sequence) for which $x_{i,n}$ tends to a limit $x_{i,0}$ as $n \to \infty$. On each interval interior to $x_{i-1,0} \leqq x < x_{i,0}$ the convex (or concave) functions $y_n(x)$ tend uniformly to $y_0(x)$; hence $y_0(x)$ is convex (or concave) on the interval $x_{i-1,0} \leqq x < x_{i,0}$. By Lemma 6, on every interval (α, β) interior to $[x_{i-1,0}, x_{i,0}]$ the derivative $\dot{y}_n(x)$ tends almost everywhere to $\dot{y}_0(x)$. Hence the derivative of $y_n(x)$ tends to $\dot{y}_0(x)$ for almost all x in $[x_0, X]$; and by Lemma 5 $y_0(x)$ is a.c. and

$$\mathcal{J}[y_0] \leqq \liminf_{n\to\infty} \mathcal{J}[y_n] \leqq \mu, \quad \mathcal{G}^j[y_0] = \lim_{n\to\infty} \mathcal{G}^j[y_n] = \gamma_j, \quad j = 1, \cdots, m.$$

But $y_0(x)$ is in the class K; so $\mathcal{J}[y_0] \geqq \mu$ by the definition of μ. Hence $\mathcal{J}[y_0]=\mu$, and the theorem is proved.

Examples. (1) An example of a function $f(x, y, y')$ such that $\omega_f(x, y, p, q) > 0$ if $p > q$ is

$$f(x, y, y') = y'^2 + y(1 + y'^2)^{1/2}.$$

Here $f_x=0$, $f_y=(1+y'^2)^{1/2}$; so (1.1) is satisfied with $a=\delta=1$, $b=0$. Also,

$$\omega_f(x, y, p, q) = p(1 + q^2)^{1/2} - q(1 + p^2)^{1/2}$$
$$= [(1 + p^2)(1 + q^2)]^{1/2}\left\{\frac{p}{(1 + p^2)^{1/2}} - \frac{q}{(1 + q^2)^{1/2}}\right\},$$

which is positive if $p > q$. Thus the whole interval $-\infty < y' < \infty$ can be taken as one interval I_1, with $\sigma_1 = +1$, in (1.6). We take, for example, $g^1 = (1+y'^2)^{1/2}$, $g^2 = |y'|^{3/2}$. Hypotheses (1.2) and (1.3) are obviously satisfied. Hypothesis (1.5) holds if we take $a_0 = a_2 = 0$, $a_1 = 1$, for $g^1[y]$ is the length of the curve $y = y(x)$. Let (x_0, y_0) and (X, Y) be any two points with $x_0 < X$, and let γ_1, γ_2 be any two numbers. Then if the class K of a.c. curves joining (x_0, y_0) to (X, Y) and giving $g^1[y]$ and $g^2[y]$ the respective values γ_1, γ_2 is not empty, it contains a minimizing curve for $\mathcal{J}[y]$.

(2) For another example we use the same integral

$$\mathcal{J}[y] = \int_0^1 [\dot{y}^2 + y(1 + \dot{y}^2)^{1/2}]dx,$$

but now we impose no side conditions and require that (x_0, y_0) and (X, Y) be (0, 0) and (0, 1), respectively. Hypotheses (1.2) and (1.3) again hold, as does (1.6). Hypothesis (1.4) is satisfied vacuously. To show that (1.5) also holds (with $a_0 = 1$; there are no other a_i) we use Schwarz' inequality. For $0 \leqq x \leqq 1$ we have

$$(y(x))^2 = \left(\int_0^x \dot{y}(x)dx\right)^2 \leqq x\int_0^x \dot{y}^2dx,$$

$$(y(x))^2 = \left(\int_x^1 \dot{y}(x)dx\right)^2 \leqq (1 - x)\int_0^x \dot{y}^2dx,$$

$$(y(x))^2 \leqq \min(x, 1 - x)\int_0^x \dot{y}^2dx \leqq \frac{1}{2}\int_0^1 \dot{y}^2dx, \qquad \int_0^1 y^2dx \leqq \frac{1}{2}\int_0^1 \dot{y}^2dx.$$

Hence, again using Schwarz' inequality, we have

$$\mathcal{J}[y] = \int_0^1 [(1 + \dot{y}^2) + y(1 + \dot{y}^2)^{1/2}]dx - 1$$
$$\geqq \int_0^1 (1 + \dot{y}^2)dx - \left\{\int_0^1 y^2dx\int_0^1 (1 + \dot{y}^2)dx\right\}^{1/2} - 1$$
$$\geqq (1 - 2^{-1/2})\int_0^1 (1 + \dot{y}^2)dx - 1$$
$$\geqq \frac{1}{4}\int_0^1 [\dot{y}^2 - 4]dx.$$

The last integral satisfies (1.5) (with $a_0=1$); so $\mathcal{J}[y]$ does also. (Note, however, that for pairs of end points other than (0, 0) and (1, 0) the condition (1.5) may fail.)

By Theorem 1 we see that in the class of all a.c. curves joining (0, 0) and (0, 1) there is a minimizing curve for $\mathcal{J}[y]$. This result is not quite trivial; for the integral $\mathcal{J}[y]$ is not even quasi-regular, since

$$f_{y'y'}(x, y, y') = 2 + y(1 + y'^2)^{-3/2},$$

which is not invariant in sign. It is clear that a large class of non-regular problems comes under our theorem, for ω_f involves only the partial derivatives f_x and f_y, and if f satisfies (1.6) so does $f(x, y, y')+\phi(y')$ for every continuously differentiable function $\phi(y')$.

(3) For a third example we take $f(x, y, y')=e^y y'^2$. Here

$$\omega_f(x, y, p, r) = e^y\{r^2p - p^2r\} = e^y pr(r - p).$$

So if we take I_1 to be the interval $-\infty<y'<0$ and I_2 to be $0\leqq y'<\infty$, we have

$$\omega_f(x, y, p, r) > 0, \qquad p \in I_2,\ r \in I_1,$$

while

$$-\omega_f(x, y, p, r) > 0, \quad p, r \in I_1 \text{ or } p, r \in I_2,\ p > r.$$

Hence (1.6) holds with $\sigma_1=\sigma_2=-1$. If, for example, we take $\mathcal{G}^1$ and $\mathcal{G}^2$ as in the first example, the class K contains a minimizing curve for $\mathcal{J}[y]$. This curve will consist either of two concave arcs, on the first of which $y'\leqq 0$ and on the second of which $y'\geqq 0$, or else it will consist of just one concave arc on which y' does not change sign.

3. **Integrals in parametric form.** Related to Theorem 1 there is a theorem on integrals in parametric form; but it applies only to a very restricted class of integrands. The hypotheses on these integrands will be the following:

(3.1) *The function $F(y, x', y')$ is independent of x and of the sign of x':*

$$F(y, -x', y') = F(y, x', y').$$

(3.2) *The functions $G^j(x', y')$, $(j=1, \cdots, m)$, depend only on $|x'|$ and y':*

$$G^j(-x', y') = G^j(x', y').$$

(3.3) *There is a set of constants $a_0, a_1, \cdots, a_m$ with $a_0\geqq 0$ such that for every number H all the rectifiable curves C beginning at a fixed point and having $a_0\mathcal{J}(C)+a_\alpha\mathcal{G}^\alpha(C)\leqq H$ have lengths less than a constant A_H.*

(3.4) *The interval $-\pi/2\leqq\theta\leqq\pi/2$ can be subdivided into a finite number of subintervals $I_1, \cdots, I_k$ such that:*

$\Omega_F(x, y, \cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2) \geqq 0$ *if* $\theta_1 \epsilon I_j$ *and* $\theta_2 \epsilon I_h$ *with* $j > h$,

$\sigma_j\Omega_F(x, y, \cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2) \geqq 0$ *if* $\theta_1 \epsilon I_j$ *and* $\theta_2 \epsilon I_j$ *and* $\theta_1 \geqq \theta_2$,

where $\sigma_j = \pm 1$, $(j=1, \cdots, k)$.

(3.5) *The class* K *of all rectifiable curves* C *joining two fixed points* (x_0, y_0) *and* (X, Y) *and having* $\mathcal{G}^j(C) = \gamma_j$, *where the* γ_j *are fixed numbers, is a non-empty class.*

THEOREM 2. *Under hypotheses* (3.1), (3.2), (3.3), (3.4), *and* (3.5), *the class* K *contains a curve* C_0 *for which* $\mathcal{J}(C)$ *is least.*

Let $\{C_n\}$ be a sequence of curves of K for which $\mathcal{J}(C_n)$ tends to the greatest lower bound μ of $\mathcal{J}$ on the class K. Then the numbers $a_0\mathcal{J}(C_n)+a_\alpha\mathcal{G}^\alpha(C_n)$ are bounded; so by (3.3) the lengths $\bar{L}_n$ of the C_n are bounded. Consequently, all the C_n lie in a circle Q. For (x, y) in Q the function $F(y, \cos\theta, \sin\theta)$ is bounded below, say by $-\nu$; hence if C_n is represented with arc length as parameter by functions $x = \bar{x}_n(s)$, $y = \bar{y}_n(s)$, $(0 \leqq s \leqq \bar{L}_n)$, we have

$$\mathcal{J}(C_n) = \int_0^{\bar{L}_n} F(\bar{y}_n, \dot{\bar{x}}_n, \dot{\bar{y}}_n)ds \geqq \int_0^{\bar{L}_n} (-\nu)ds = -\nu\bar{L}_n,$$

and $\mathcal{J}(C_n)$ is bounded below. That is, the number μ is finite.

As we saw in §1, it is possible for each n to construct a sequence of polygons Π_m joining (x_0, y_0) to (X, Y) and tending to C_n such that $\mathcal{J}(\Pi_m) \to \mathcal{J}(C_n)$ and $\mathcal{G}^j(\Pi_m) \to \mathcal{G}^j(C_n)$. We choose for each n one of these polygons (which we rename Π_n) such that

$$|\mathcal{J}(\Pi_n) - \mathcal{J}(C_n)| < 1/n, \qquad |\mathcal{G}^j(\Pi_n) - \mathcal{G}^j(C_n)| < 1/n.$$

Then $\mathcal{J}(\Pi_n) \to \mu$ and $\mathcal{G}^j(\Pi_n) \to \gamma_j$, $(j=1, \cdots, m)$. We may assume, if we wish, that no side of Π_n is parallel to the y-axis, since we can bring this about by an arbitrarily small change, causing an arbitrarily small change in $\mathcal{J}(\Pi_n)$ and $\mathcal{G}^j(\Pi_n)$.

Suppose that Π_n is defined in terms of arc-length by functions $x = \xi_n(s)$, $y = \eta_n(s)$, $(0 \leqq s \leqq L_n)$. We define a new set of functions x_n, y_n by the relations

$$y_n(s) = \eta_n(s), \qquad x_n(s) = x_0 + \int_0^s |\dot{\xi}_n(s)| ds.$$

Then $y_n(L_n) = Y$ and $x_n(L_n) - x_0 \geqq |X - x_0|$, and

$$\int_0^{L_n} F(y_n, \dot{x}_n, \dot{y}_n)ds = \mathcal{J}(\Pi_n), \qquad \int_0^{L_n} G^j(\dot{x}_n, \dot{y}_n)ds = \mathcal{G}^j(\Pi_n).$$

Exactly as in the preceding proof, we find that the curve $x = x_n(s)$, $y = y_n(s)$

can be considered to be composed of at most k arcs, each one of which is either concave or convex; the jth arc consists of segments along which $(x_n', y_n')=(\cos\theta, \sin\theta)$ with θ in the interval I_j. Along such an arc y_n' is either monotonic increasing or monotonic decreasing. Likewise x_n' is monotonic unless y_n' changes sign; so each of the (at most k) convex or concave arcs can be split into at most two arcs on each of which both x_n' and y_n' are monotonic.

Let us change parameter from s to $t=s/L_n$. The polygon $x=x_n(s)$, $y=y_n(s)$, $(0\leqq s\leqq L_n)$, is then represented in the form $x=X_n(t)$, $y=Y_n(t)$, $(0\leqq t\leqq 1)$; and the interval $[0, 1]$ can be split into at most $2k$ subintervals on each of which $X'(t)$ and $Y'(t)$ are monotonic. That is, $X_n(t)$ is concave or convex as a function of t on each separate subinterval, and likewise $Y_n(t)$. The functions $x_n(s)$ and $y_n(s)$ satisfy a Lipschitz condition of constant 1; so $X_n(t)$ and $Y_n(t)$ satisfy a Lipschitz condition of constant L_n, which is bounded. Hence we can select a subsequence (we suppose it the whole sequence) for which $X_n(t)$ and $Y_n(t)$ converge uniformly to limit functions $X_0(t)$ and $Y_0(t)$, respectively. As in the preceding proof, the interval $0\leqq t\leqq 1$ can be split into at most $2k$ subintervals on each of which $X_0(t)$ is concave or convex, and likewise $Y_0(t)$; and

$$\dot{X}_n(t)\rightarrow \dot{X}_0(t), \qquad \dot{Y}_n(t)\rightarrow \dot{Y}_0(t)$$

for almost all t.

Since the derivatives of the X_0 and Y_0 are bounded, we have at once

$$\int_0^1 G^j(\dot{X}_0, \dot{Y}_0)dt = \lim_{n\to\infty}\int_0^1 G^j(\dot{X}_n, \dot{Y}_n)dt = \lim_{n\to\infty}\mathcal{G}^j(\Pi_n) = \gamma_j, \qquad j=1,\cdots,m,$$

$$\int_0^1 F(Y_0, \dot{X}_0, \dot{Y}_0)dt = \lim_{n\to\infty}\int_0^1 F(Y_n, \dot{X}_n, \dot{Y}_n)dt = \lim_{n\to\infty}\mathcal{J}(\Pi_n) = \mu.$$

The curve $x=X_0(t)$, $y=Y_0(t)$ still is not a solution of our problem; for although we have $Y_n(1)=y_n(L_n)=Y$ for all n, so that $Y_0(1)=Y$, still we have only $X_n(1)-x_0=x_n(L_n)-x_0\geqq|X-x_0|$, so that $X_0(1)-x_0\geqq|X-x_0|$. However, let us define t_0 to be that value of t for which

$$x_0+\int_0^{t_0}\dot{X}_0(t)dt-\int_{t_0}^1\dot{X}_0(t)dt = X.$$

Such a t_0 exists, for the function

$$x_0+\int_0^t\dot{X}_0dt-\int_t^1\dot{X}_0dt$$

is a continuous function of t, and as t goes from 0 to 1, it goes from $x_0 - |X_0(1) - x_0|$ to $x_0 + |X_0(1) - x_0|$, and the number X is not less than the first of these and not greater than the second. If we now define

$$y_0(t) = Y_0(t),$$

$$x_0(t) = x_0 + \int_0^t \sigma(t)\dot{X}_0(t)dt, \qquad 0 \leqq t \leqq 1,$$

where $\sigma(t) = 1$ for $0 \leqq t \leqq t_0$ and $\sigma(t) = -1$ for $t_0 < t \leqq 1$, then $x_0(0) = x_0$, $x_0(1) = X$, $y_0(0) = y_0$, $y_0(1) = Y$; and since $\dot{x}_0(t) = \pm \dot{X}_0(t)$,

$$\int_0^1 F(y_0, \dot{x}_0, \dot{y}_0)dt = \int_0^1 F(Y_0, \dot{X}_0, \dot{Y}_0)dt = \mu,$$

$$\int_0^1 G^j(\dot{x}_0, \dot{y}_0)dt = \int_0^1 G^j(\dot{X}_0, \dot{Y}_0)dt = \gamma_j, \quad j = 1, \cdots, m.$$

Thus the curve $x = x_0(t)$, $y = y_0(t)$ is in the class K and minimizes the integral $\mathcal{J}$.

For an example let us take

$$F = \phi(y)(x'^2 + y'^2)^{1/2}, \quad G^1 = (ax'^2 + by'^2)^{1/2}, \quad G^2 = (x'^4 + y'^4)^{1/4},$$

where a and b are positive and $\phi'(y) > 0$ for all y. Conditions (3.1), (3.2), and (3.3) are easily seen to be satisfied. We readily calculate

$$\Omega_F(x, y, \cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2) = \phi'(y)(\sin\theta_1 - \sin\theta_2),$$

which is positive if $-\pi/2 \leqq \theta_2 < \theta_1 \leqq \pi/2$. Hence (3.4) holds with $k = 1$ and $\sigma_1 = +1$. By Theorem 2, if (x_0, y_0) and (X, Y) are any two points and γ_1, γ_2 any two numbers such that there are curves C joining (x_0, y_0) to (X, Y) and having $\mathcal{G}^j(C) = \gamma_j$, $(j = 1, 2)$, then there is a curve of that class for which $\mathcal{J}(C)$ is least.

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