SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

V. THE ISOPERIMETRIC PROBLEM IN PARAMETRIC FORM*

BY

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1. First existence theorem. We continue the notation of preceding papers with the trifling change that \( z \) will denote \((z^1, \ldots, z^n)\) instead of \((z^0, \ldots, z^n)\) as heretofore, and a similar change for \( z' \). The class of all rectifiable curves joining two fixed points \( z_1, z_2 \) (not necessarily distinct) will be denoted by \( K \). The functions \( F(z, z') \) and \( G(z, z') \) are defined for all points \( z = (z^1, \ldots, z^n) \) in a closed point set \( S \) and all \( z' \). They are positively homogeneous of degree 1 in \( z' \), are continuous for all \( z \) in \( S \) and all \( z' \), and possess partial derivatives of first and second orders continuous except at \( z' = 0 \). We write

\[
\mathcal{J}(C) = \int_C F(z, z') \, dt, \quad \mathcal{G}(C) = \int_C G(z, z') \, dt.
\]

A hypothesis which we shall henceforth impose on our integrals is the following:

\[(1.1) \text{ To each pair of numbers } l, m \text{ there corresponds a number } L \text{ such that if } C \text{ is in } K \text{ and } \left| \mathcal{G}(C) \right| \leq l, \mathcal{J}(C) \leq m, \text{ then } \mathcal{L}(C) \leq L.\]

There are well known conditions which ensure this. For instance, if there are numbers \( a \geq 0 \) and \( b \) such that \( aF + bG \) is positive definite and the set \( S \) is bounded, then (1.1) is satisfied. Or, \( S \) being unbounded, if there exist numbers \( a \geq 0 \) and \( b \) for which

\[
aF(z, z') + bG(z, z') \geq k \left| z' \right| (1 + \left| z \right| )^{-1},
\]

\( k > 0 \), then (1.1) holds.

The proofs in note IV give us, with hardly any modification, a proof of the following theorem:

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Theorem 1. Let the set $S$ consist of the entire $z$-space, and let (1.1) hold. Let $G(C)$ be quasi-regular normal. Suppose further that for each $z$, each approach set $A$ at $z$ contains only a finite number of unit vectors $p_1, \ldots, p_k$ which can be so ordered that $Q_H(z, p_i, p_j) < 0$ if $i < j$; where $H(z, r) = H(z, r; \lambda(z, A))$. Then for every number $l$, either the class $K_{[G=1]}$ is empty or it contains a curve which minimizes $J(C)$ on the class $K_{[G=1]}$.

The proofs in IV leading up to equations (8.2) and (8.3) of IV were so designed as to apply to problems in parametric form as well as to those in non-parametric form. Hence these equations, which are $\phi'(t) = \phi_0'(t)$ and $\gamma'(t) = \gamma_0'(t)$ for all $t$ in $E$ (that is, for almost all $t$ in $[0, 1]$) are here valid. But for problems in parametric form the functions $\phi, \phi_0, \gamma, \gamma_0$ are Lipschitzian; so this implies that $\phi(1) = \phi_0(1)$ and $\gamma(1) = \gamma_0(1)$. As remarked at the end of §4 of IV, these equations imply the conclusion of the theorem.

2. Statement of the principal existence theorem. Before stating the next problem we introduce several definitions:

(2.1) The point $z_0$ of $S$ is an ordinary point if the following conditions hold:
(a) It is interior to $S$.
(b) $G(z, r)$ is quasi-regular normal (either positive or negative) at $z_0$.
(c) Each approach set $A$ at $z_0$ contains only a finite number of unit vectors $p_1, \ldots, p_k$, and these can be so ordered that $Q_H(z_0, p_i, p_j) < 0$ if $i < j$; here $H(z, r) = F(z, r) - \lambda(z_0, A)G(z, r)$.

Theorem 1 required that $S$ be the whole space and that every point $z_0$ be ordinary. Subject to further hypotheses, our next theorem will permit $S$ to be a closed subset of the space and will allow $S$ to contain singular points (that is, points which are not ordinary points).

One of our hypotheses will be the following:

(2.2) For each $z_0$ in $S$ there is a number $\theta$ such that $F(z, r) - \theta G(z, r)$ is p.q.r.† at $z_0$.

We denote by $m(z_0)$, $M(z_0)$, respectively, the greatest lower bound and the least upper bound of all numbers $\theta$ for which $F(z, r) - \theta G(z, r)$ is p.q.r. at $z_0$. It is easy to see that $m(z) = -\infty$ if and only if $G(z, r)$ is p.q.r. at $z_0$. For if $G(z, r)$ is p.q.r. at $z_0$ and $\theta_0$ is any number such that $F - \theta_0 G$ is p.q.r. at $z_0$, then $F(z, r) - \theta G(z, r) = [F(z, r) - \theta_0 G(z, r)] + (\theta_0 - \theta)G(z, r)$ is also p.q.r. for all $\theta < \theta_0$; so $m(z) = -\infty$. However, if $G$ is not p.q.r. at $z_0$ there are orthogonal unit vectors $p, u$ for which $u^\alpha G_{\alpha\beta}(z, p)u^\beta < 0$. Then $u^\alpha[F_{\alpha\beta}(z, p) - \theta G_{\alpha\beta}(z, p)]u^\beta$ is negative if $\theta$ is negative and numerically large; so $m(z_0)$ is finite. Likewise $M(z_0) = +\infty$ if and only if $G(z, r)$ is n.q.r. at $z_0$.

† Defined in (6.2) of III.
For each curve $C: z = z(t), \ (t_1 \leq t \leq t_2)$, lying in $S$ we define two sets of points $t$ as follows:

1. $T_+(C)$ is the set of all points $t$ such that $z(t)$ is a singular point and $G(z, r)$ is not n.q.r. at $z(t)$.

2. $T_-(C)$ is the set of all points $t$ such that $z(t)$ is a singular point and $G(z, r)$ is not p.q.r. at $z(t)$.

These sets may overlap: if $z(t)$ is a singular point and $G(z, r)$ is neither p.q.r. nor n.q.r. at $z(t)$, then $t$ belongs to both $T_+(C)$ and $T_-(C)$. Also they may vanish simultaneously; this happens whenever $G(z, r)$ is linear in the variables $r^i$ at each singular point $z(t)$.

Our next definition is as follows:

3. If $C: z = z(t)$ is a curve lying in $S$, then if $T_+(C)$ is not empty, we define $M(C)$ to be the greatest lower bound of $M(z(t))$ for all $t$ in $T_+(C)$; if $T_-(C)$ is not empty, we define $m(C)$ to be the least upper bound of $m(z(t))$ for all $t$ in $T_-(C)$.

If $M(C)$ is defined, it is not $+ \infty$. For if $M(C)$ is defined, the set $T_+(C)$ is not empty. Let $t_0$ be a point in it. At $z(t_0)$ the function $G(z, r)$ is not n.q.r.; so $M(z(t_0))$ is not $+ \infty$. Hence $M(C) \leq M(z(t_0)) < \infty$. Likewise, if $m(C)$ is defined, it is not $- \infty$. It is interesting to observe that if $G(z(t), r)$ is positive regular for all $t$ and if $C$ contains singular points $z(t)$, then $M(C)$ is defined and finite, while $m(C)$ is undefined. Likewise, if $G(z(t), r)$ is negative regular for all $t$ and if $C$ contains singular points, then $m(C)$ is defined and finite, while $M(C)$ is undefined. We prove the first statement; the proof of the second is similar. The set $T_+(C)$ here consists of all singular points, and by hypothesis is not empty; so $M(C)$ is defined. As always, $M(C) < \infty$. The quadratic form $u^*G_{ab}(z(t), \ p)u^b$ is positive for all $t$ and all pairs of orthogonal unit vectors $u, p$. Let $\nu$ be its greatest lower bound; then $\nu > 0$.

The quadratic form $u^*F_{ab}(z(t), \ p)u^b$ is bounded, say by $N$, in absolute value, for the same arguments. Then $F(z(t), r) + (N/\nu)G(z(t), r)$ is p.q.r. for all $t$; so $M(z(t)) \geq -N/\nu$ for each $t$, and $M(C) \geq -N/\nu > -\infty$. Therefore $M(C)$ is finite.

We now state our principal theorem.

**Theorem 2.** Let the following hypotheses be satisfied:

(a) $S$ is closed.

(b) Hypotheses (1.1) and (2.2) hold.

(c) For every curve $C$ of $K$, either $T_+(C)$ is empty or there exists† a curve $\Gamma^* : x = \xi(\tau), \ (0 \leq \tau \leq e)$, with the properties:

† This implies that $M(C)$ is finite. For $M(C)$ is never $+ \infty$, and if $M(C)$ were $- \infty$, inequalities (2.6) and (2.7) could not hold. Likewise, from (d) we conclude that $m(C)$ is undefined or is finite.
(i) The length of \( T^* \) is not zero.
(ii) \( \xi(0) \) is on \( C \).
(iii) For almost all \( \tau \) in \( [0, \epsilon] \) the inequalities

\[
G(\xi(\tau), \xi'(\tau)) + G(\xi(\tau), -\xi'(\tau)) > 0, \tag{2.6}
\]

\[
F(\xi(\tau), \xi'(\tau)) + F(\xi(\tau), -\xi'(\tau)) \leq M(C) [G(\xi(\tau), \xi'(\tau)) + G(\xi(\tau), -\xi'(\tau))]
\]

hold.

(d) For every curve \( C \) of \( K \), either \( T_-(C) \) is empty or there exists a curve \( T^*: z = \xi(\tau), (0 \leq \tau \leq \epsilon) \), with the properties:

(i) The length of \( T^* \) is not zero.
(ii) \( \xi(0) \) is on \( C \).
(iii) For almost all \( \tau \) in \( [0, \epsilon] \) the inequalities

\[
G(\xi'(\tau), \xi(\tau)) + G(\xi'(\tau), -\xi(\tau)) < 0, \tag{2.8}
\]

\[
F(\xi'(\tau), \xi(\tau)) + F(\xi'(\tau), -\xi(\tau)) \leq m(C) [G(\xi'(\tau), \xi(\tau)) + G(\xi'(\tau), -\xi(\tau))]
\]

hold.

Then for every \( l \) the class \( K[\mathcal{G}=l] \) either is empty or contains a curve \( C \) for which \( J(C) \) assumes its least value on \( K[\mathcal{G}=l] \).

Sections 3 to 5 will be devoted to the proof of this theorem, and throughout these sections the hypotheses of Theorem 2 will be assumed to hold.

3. Construction of a minimizing sequence. If the class \( K[\mathcal{G}=l] \) is not empty, we denote by \( \mu \) the greatest lower bound of \( J(C) \) on the class \( K[\mathcal{G}=l] \). Also we define \( \mu_0 \) to be the greatest lower bound of numbers \( m \) for which there exists a sequence \( \{C_n\} \) of curves of \( K \) for which there exists a sequence \( \{C_n\} \) of curves of \( K \) having

\[
\lim_{n \to \infty} J(C_n) = m, \quad \lim_{n \to \infty} G(C_n) = l.
\]

Since \( \mu \) is such a number \( m \), we evidently have \( \mu_0 \leq \mu \). If \( \{h_n\} \) is a sequence of numbers greater than \( \mu_0 \) and tending to \( \mu_0 \), for each \( n \) there is a \( C_n^* \) such that

\[
J(C_n^*) < h_n, \quad |G(C_n^*) - l| < 1/n.
\]

Hence

\[
(3.1) \quad \lim_{n \to \infty} J(C_n^*) = \mu_0, \quad \lim_{n \to \infty} G(C_n^*) = l.
\]

By (1.1), the \( C_n^* \) have bounded lengths, which ensures the finiteness of \( \mu_0 \).

Suppose \( C_n^* \) has the Lipschitzian representation \( z = z_n^*(t), (0 \leq t \leq 1) \). The set of values of \( t \) such that \( z_n^*(t) \) has a distance greater than \( 1/2n \) from the boundary of \( S \) is open relative to \( [0, 1] \) and so falls into a finite or denumerable aggregate of subintervals. For only a finite number of these, say
\[ [\alpha_1, \beta_1], \ldots, [\alpha_h, \beta_h], \] can the corresponding arc of \( C_n^* \) have length as great as \( 1/2n \); the others we disregard. Thus if \( t \) is not in any of the intervals \([\alpha_i, \beta_i]\), the point \( z_n^*(t) \) has distance less than \( n^{-1} \) from the boundary of \( S \).

Consider now one of the intervals \([\alpha_i, \beta_i]\). In the corresponding arc of \( C_n^* \) we inscribe a polygon \( \Pi_{i,n}^* \) with the successive vertices

\[ z_n^*(\alpha_i), z_n^*(t_1), \ldots, z_n^*(t_m), z_n^*(\beta_i), \quad \alpha_i < t_1 < \cdots < t_m < \beta_i. \]

If the sides of this polygon are short enough, we will have

\[
(3.2) \quad \left| J(\Pi_{i,n}^*) - \int_{\alpha_i}^{\beta_i} F(z_n^*, z_n^*) \, dt \right| < 1/\hbar n,
\]

and

\[
(3.3) \quad \left| G(\Pi_{i,n}^*) - \int_{\alpha_i}^{\beta_i} G(z_n^*, z_n^*) \, dt \right| < 1/\hbar n.
\]

Now let \( \Pi_{i,n}^* \) be the polygon having the same number of vertices as \( \Pi_{i,n}^* \), joining \( z_n^*(\alpha_i) \) to \( z_n^*(\beta_i) \), having \( \mathcal{C}(\Pi_{i,n}^*) = \mathcal{C}(\Pi_{i,n}^*) \), and minimizing \( J(C) \) in that class of polygons. Such a polygon exists, by Lemma 3 of IV. For this polygon we have

\[
(3.4) \quad J(\Pi_{i,n}) < \int_{\alpha_i}^{\beta_i} F(z_n^*, z_n^*) \, dt + 1/\hbar n,
\]

\[
(3.5) \quad G(\Pi_{i,n}) - \int_{\alpha_i}^{\beta_i} G(z_n^*, z_n^*) \, dt < 1/\hbar n.
\]

Now we define the curve \( C_n \) to be the curve \( C_n^* \) with the arcs corresponding to the intervals \([\alpha_i, \beta_i]\) replaced by the respective polygons \( \Pi_{i,n} \). Since we can consider the functions \( z_n^* \) unaltered except on the intervals \([\alpha_i, \beta_i]\), we obtain from (3.4) and (3.5) the relations

\[
(3.6) \quad J(C_n) < J(C_n^*) + 1/n,
\]

\[
(3.7) \quad \left| G(C_n) - G(C_n^*) \right| < 1/n.
\]

Hence

\[
(3.8) \quad \limsup_{n \to \infty} J(C_n) \leq \lim_{n \to \infty} J(C_n^*) + 0 = \mu_0,
\]

\[
(3.9) \quad \lim_{n \to \infty} G(C_n) = l.
\]

But by the definition of \( \mu_0 \) we cannot have \( \liminf J(C_n) < \mu_0 \), in the presence of (3.9). So from this and (3.8) we conclude that
(3.10) \[ \lim_{n \to \infty} J(C_n) = \mu_0 \leq \mu. \]

By (1.1), the curves \( C_n \) are of uniformly bounded lengths; so we can select a subsequence for which \( \mathcal{L}(C_n) \) converges to a finite limit \( L \):

(3.11) \[ \lim_{n \to \infty} \mathcal{L}(C_n) = L < \infty. \]

Clearly we may suppose (since we can discard a finite number of the \( C_n \)) that we have

(3.12) \[ \mathcal{L}(C_n) \leq L + 1. \]

For this subsequence (3.9) and (3.10) still hold.

On each curve \( C_n \) we introduce as parameter \( t = s / \mathcal{L}(C_n) \), where \( s \) is arc length on \( C_n \). Then \( C_n \) has the representation

\[ C_n: z = z_n(t), \quad 0 \leq t \leq 1. \]

These functions \( z_n(t) \) satisfy a Lipschitz condition of constant \( \mathcal{L}(C_n) \), which is less than or equal to \( L+1 \) by (3.12). Hence

(3.13) \[ |\dot{z}_n(t)| \leq L + 1, \quad 0 \leq t \leq 1. \]

By Ascoli's theorem, we can select a subsequence of the \( C_n \) (we suppose it the whole sequence) such that the functions \( z_n(t) \) converge uniformly to a limit function \( z_0(t) \):

(3.14) \[ \lim_{n \to \infty} z_n(t) = z_0(t) \text{ uniformly for } 0 \leq t \leq 1. \]

For this subsequence (3.9), (3.10), and (3.11) remain valid. The curve \( z = z_0(t), \) \((0 \leq t \leq 1)\), we denote by \( C_0 \).

Next we define

(3.15) \[ \phi_n(t) = \int_0^t F(z_n(s), \dot{z}_n(s))ds, \quad n = 0, 1, \ldots, \]

(3.16) \[ \gamma_n(t) = \int_0^t G(z_n(s), \dot{z}_n(s))ds, \quad n = 0, 1, \ldots. \]

These functions all satisfy the same Lipschitz condition, since the integrands are bounded. So we may select a subsequence (we again denote it by \( \{C_n\} \)) such that \( \phi_n(t) \) and \( \gamma_n(t) \) tend uniformly to limit functions \( \phi(t) \) and \( \gamma(t) \), respectively:

(3.17) \[ \lim_{n \to \infty} \phi_n(t) = \phi(t), \quad \lim_{n \to \infty} \gamma_n(t) = \gamma(t). \]
uniformly for \(0 \leq t \leq 1\). For this subsequence (3.9), (3.10), (3.11), and (3.14) remain valid.

Equations (3.9), (3.10), (3.15), (3.16), and (3.17) imply

\[
\phi(1) = \mu_0, \quad \gamma(1) = l.
\]

From the manner of constructing the curves \(C_n\), we notice that for every \(\epsilon > 0\) all the curves \(C_n\) with large \(n\) are polygons except for their arcs which are within the \(\epsilon\)-neighborhood of the boundary of \(S\). So if \(z_0(t_0)\) is a point of \(C_0\) interior to \(S\), there is a neighborhood \((t_0 - \delta, t_0 + \delta)\) of \(t_0\) such that the functions \(z_n(t), (t_0 - \delta < t \leq t_0 + \delta)\), represent polygonal arcs whenever \(n\) is greater than a certain \(n_0\). (Modifications if \(t_0 = 0\) or \(1\) are obvious.)

4. Some lemmas. Let us first dispose of the trivial case for which we have \(L = \lim \mathcal{L}(C_n) = 0\). This is only possible if \(z_1 = z_2\), and it implies that \(C_0\) has length 0 and consists of the one point \(z_1\). By (3.9) and (3.10) we have \(l = 0 = \mu_0 \leq \mu\). By trivial computation we obtain \(\mathcal{G}(C_0) = 0 = l\), \(\mathcal{J}(C_0) = 0 \leq \mu\). Since \(C_0\) is thus in \(K[\mathcal{G} = l]\), \(\mathcal{J}(C_0) \geq \mu\); whence \(\mathcal{J}(C_0) = \mu = \mu_0 = 0\) and \(C_0\) is the curve sought. This leaves for consideration the principal case, in which

\[
L = \lim \mathcal{L}(C_n) > 0.
\]

Since \(\phi, \phi_0, \gamma, \gamma_0,\) and \(z_0\) are all Lipschitzian functions of \(t\), the interval \([0, 1]\) contains a set \(E\) of measure 1 such that for all \(t\) in \(E\) all the functions mentioned have derivatives and

\[
(4.2) \quad \phi'_0(t) = F(z_0(t), z'_0(t)), \quad \gamma'_0(t) = G(z_0(t), z'_0(t)).
\]

It will be supposed (without loss of generality) that neither 0 nor 1 is in \(E\). We now begin to prove a sequence of lemmas.

**Lemma 1.** If \(t_0\) is in \(E\), and \(a\) and \(b\) are numbers such that \(a F(z, r) + b G(z, r)\) is p.q.r. at \(z_0(t_0)\), then

\[
a[\phi'(t_0) - \phi'_0(t_0)] + b[\gamma'(t_0) - \gamma'_0(t_0)] \geq 0.
\]

Except for notation, this is merely a restatement of Lemma 5 of III.

**Lemma 2.** If \(t\) is in \(E\) and \(z(t)\) is an ordinary point, then

\[
(4.3) \quad \phi'(t_0) = \phi'_0(t_0), \quad \gamma'(t_0) = \gamma'_0(t_0).
\]

By (2.2), (a), \(z_0(t_0)\) is interior to \(S\). Therefore, as remarked at the end of §3, for all large \(n\) the arcs of \(C_n\) lying in a neighborhood of \(z_0(t_0)\) are polygonal. All the hypotheses leading up to equations (8.2) and (8.3) of IV (which together are (4.3) above) are here satisfied, except that in IV the set \(S\) was the whole space. However, the proof of (8.2) and (8.3) was purely local in nature; the only reason for taking the whole space for \(S\) was to be sure that each point of
Lemma 3. If \( z_0(t_0) \) is a singular point, and \( t_0 \) belongs to \( E \) but does not belong either to \( T_+(C_0) \) or to \( T_-(C_0) \), then \( \gamma'(t_0) = \gamma_0'(t_0) \) and \( \phi'(t_0) \geq \phi_0'(t_0) \).

If \( G(z, r) \) were not p.q.r. at \( z_0(t_0) \), the point \( t_0 \) would be in \( T_+(C_0) \). If it were not n.q.r. at \( z_0(t_0) \), then \( t_0 \) would be in \( T_-(C_0) \). So \( G(z, r) \) must be both p.q.r. and n.q.r. at \( z_0(t_0) \). That is, \( G(z_0(t_0), r) \) is linear in the variables \( r \). By hypothesis (2.3), \( F(z, r) - \theta G(z, r) \) is p.q.r. at \( z_0(t_0) \) for some \( \theta \). But since \( G(z, r) \) is merely linear in the \( r \), this implies that \( F(z, r) \) itself is p.q.r. at \( z_0(t_0) \). The application of Lemma 1 with \( a = 1, b = 0 \) yields \( \phi'(t_0) - \phi_0'(t_0) \geq 0 \).

Since \( G(z, r) \) is both p.q.r. and n.q.r. at \( z_0(t_0) \), we apply Lemma 1 with \( a = 0, b = 1 \) and with \( a = 0, b = -1 \). This yields two inequalities which together imply \( \gamma'(t_0) = \gamma_0'(t_0) \), completing the proof.

Lemma 4. If \( t_0 \) is in \( ET_+(C_0) \), then

\[ \phi'(t_0) - \phi_0'(t_0) \geq M(z_0(t_0)) [\gamma'(t_0) - \gamma_0'(t_0)] .\]

By definition of \( M(z) \), there is a sequence \( \{ \theta_n \} \) of numbers tending to \( M(z_0(t_0)) \) such that for each \( n \), \( F(z, r) - \theta_n G(z, r) \) is p.q.r. at \( z_0(t_0) \). By lemma 1,

\[ \phi'(t_0) - \phi_0'(t_0) \geq \theta_n [\gamma'(t_0) - \gamma_0'(t_0)] .\]

Letting \( n \rightarrow \infty \) establishes the desired inequality.

Lemma 5. If \( t_0 \) is in \( ET_-(C_0) \), then

\[ \phi'(t_0) - \phi_0'(t_0) \geq m(z_0(t_0)) [\gamma'(t_0) - \gamma_0'(t_0)] .\]

Choose a sequence \( \{ \theta_n \} \) such that \( \theta_n \rightarrow m(z_0(t_0)) \) and \( F(z, r) - \theta_n G(z, r) \) is p.q.r. at \( z_0(t_0) \) for each \( n \). The rest of the proof is a repetition of that of Lemma 4.

Lemma 6. If \( t_0 \) is in \( E \) and \( \gamma'(t_0) = \gamma_0'(t_0) \), then \( \phi'(t_0) \geq \phi_0'(t_0) \).

If \( z_0(t_0) \) is ordinary, this follows from Lemma 2. If \( z_0(t_0) \) is singular, either it is in neither \( T_+(C_0) \) nor \( T_-(C_0) \), in which case \( \phi'(t_0) \geq \phi_0'(t_0) \) by Lemma 3, or it is in one (or both) of the sets \( T_+(C_0) \) and \( T_-(C_0) \), in which case \( \phi'(t_0) - \phi_0'(t_0) \geq 0 \) by Lemma 4 or Lemma 5.

Lemma 7. If \( t_0 \) is in \( E \) and \( \gamma'(t_0) > \gamma_0'(t_0) \), then \( t_0 \) is in \( T_+(C_0) \); if \( t_0 \) is in \( E \) and \( \gamma'(t_0) < \gamma_0'(t_0) \), then \( t_0 \) is in \( T_-(C_0) \).

The point \( z_0(t_0) \) must be singular by Lemma 2. If \( t_0 \) is not in \( T_+(C_0) \), then \( G(z, r) \) is n.q.r. at \( z_0(t_0) \). Applying Lemma 1 with \( a = 0, b = -1 \), we obtain \( \gamma'(t_0) - \gamma_0'(t_0) \leq 0 \). Hence if \( \gamma'(t_0) - \gamma_0'(t_0) > 0 \), then \( t_0 \) is in \( T_+(C_0) \). The proof of the other statement is similar.
5. **Proof of the theorem.** We now subdivide the set \( E \) into three subsets. The set \( T_0 \) will be the subset of \( E \) on which \( \gamma'(t) = \gamma'_0(t) \); the set \( T_1 \) will be the subset of \( E \) on which \( \gamma'(t) > \gamma'_0(t) \); and the set \( T_2 \) will be the subset of \( E \) on which \( \gamma'(t) < \gamma'_0(t) \). These sets are clearly measurable, since \( \gamma'(t) \) and \( \gamma'_0(t) \) are measurable functions. By Lemma 7, \( T_1 \) is contained in \( T_+(C_0) \) and \( T_2 \) is contained in \( T_-(C_0) \).

First we shall construct a curve \( \Gamma_1: z = \xi_1(\tau), \; (0 \leq \tau \leq \tau_1) \), beginning and ending at a point \( z_0(\xi_1) \) on \( C_0 \), and such that

\[
(5.1) \quad \mathcal{G}(\Gamma_1) = \int_{\tau_1} \left[ \gamma'(t) - \gamma'_0(t) \right] dt,
\]

\[
(5.2) \quad \mathcal{J}(\Gamma_1) \leq \int_{\tau_1} \left[ \phi'(t) - \phi'_0(t) \right] dt.
\]

Whenever \( T_1 \) is empty we can take \( \Gamma_1 \) to be a degenerate curve consisting of a single point on \( C_0 \). Then (5.1) and (5.2) obviously hold. If \( T_1 \) is not empty, then \( T_+(C_0) \) is also not empty. By hypothesis (c) there is a curve \( \Gamma^* \) corresponding to \( C_0 \) and having the properties there specified. For \( 0 \leq \tau \leq \epsilon \) we define \( \Gamma(\tau) \) to be the curve obtained by traversing \( \Gamma^* \) from \( \xi(0) \) to \( \xi(\tau) \) and then returning to \( \xi(0) \). Thus \( \Gamma(\tau) \) is defined by the equations

\[
z = \xi(t), \; 0 \leq t \leq \tau; \quad z = \xi(2\tau - t), \; \tau < t \leq 2\tau.
\]

This is a rectifiable continuous curve beginning and ending at \( \xi(0) \), which, by hypothesis (c), is a point \( z_0(\xi_1) \) on \( C_0 \). We calculate

\[
(5.3) \quad \mathcal{G}(\Gamma(\tau)) = \int_0^\tau G(\xi(t), \xi(t)) dt + \int_\tau^{2\tau} G(\xi(2\tau - t), -\xi(2\tau - t)) dt
\]

Likewise

\[
(5.4) \quad \mathcal{J}(\Gamma(\tau)) = \int_0^\tau \left[ F(\xi(t), \xi(t)) + F(\xi(t), -\xi(t)) \right] dt.
\]

By (2.6) the integrand in (5.3) is almost everywhere positive; hence \( \mathcal{G}(\Gamma(\epsilon)) > 0 \). Let \( m \) be an integer for which

\[
m \mathcal{G}(\Gamma(\epsilon)) > \int_{\tau_1} \left[ \gamma'(t) - \gamma'_0(t) \right] dt.
\]

Since \( \mathcal{G}(\Gamma(\tau)) \) is a continuous function of \( \tau \), there is a \( \tau_0 \) such that
\[ G(\Gamma(t_0)) = m^{-1} \int_{T_1} [\gamma'(t) - \gamma_0'(t)] dt. \]

We now define \( \Gamma_1 \) to be the curve obtained by traversing \( \Gamma(t_0) \) a total of \( m \) times. Then (5.1) holds.

Recalling that \( M(C_0) \) is the greatest lower bound of \( M(z_0(t)) \) for all \( t \) in \( T_+(C_0) \), by Lemma 4 we find that whenever \( t_0 \) is in \( T_1 \) the inequality

\[ \phi'(t_0) - \phi_0'(t_0) \geq M(C_0) [\gamma'(t_0) - \gamma_0'(t_0)] \]

holds. Hence

\[ (5.5) \quad \int_{T_1} [\phi'(t) - \phi_0'(t)] dt \geq M(C_0) \int_{T_1} [\gamma'(t) - \gamma_0'(t)] dt. \]

On the other hand, by (2.7), (5.3), and (5.4) we find

\[ \mathcal{J}(\Gamma_1) = m \int_0^{\tau_0} [F(\xi(t), \xi(t)) + F(\xi(t), -\xi(t))] dt \]

\[ (5.6) \quad \leq mM(C_0) \int_0^{\tau_0} [G(\xi(t), \xi(t)) + G(\xi(t), -\xi(t))] dt \]

\[ = M(C_0) G(\Gamma_1). \]

From (5.1), (5.5), and (5.6) we obtain (5.2).

Next we prove that there is a curve \( \Gamma_2 \) beginning and ending at a point \( z_0(t_2) \) on \( C_0 \) and such that

\[ (5.7) \quad G(\Gamma_2) = \int_{T_2} [\gamma'(t) - \gamma_0'(t)] dt, \]

\[ (5.8) \quad \mathcal{J}(\Gamma_2) \leq \int_{T_2} [\phi'(t) - \phi_0'(t)] dt. \]

We could prove this as we did (5.1) and (5.2). But it is much simpler to observe that if we replace \( G(z, r) \) by \(-G(z, r)\), then \( \gamma(t) \) is replaced by \(-\gamma(t)\) and \( M(C_0) \) by \(-m(C_0)\), while hypotheses (c) and (d) are interchanged. Then (5.7) and (5.8) are merely (5.1) and (5.2) as rewritten for \( F \) and \(-G\) in place of \( F \) and \( G \).

We can now define the minimizing curve \( \overline{C} \). Suppose to be specific that \( t_1 \leq t_2 \). We obtain \( \overline{C} \) by traversing \( C_0 \) from \( t=0 \) to \( t=t_1 \), traversing \( \Gamma_1 \), continuing along \( C_0 \) from \( t=t_1 \) to \( t=t_2 \), traversing \( \Gamma_2 \), then proceeding along \( C_0 \) from \( t=t_2 \) to \( t=1 \). We therefore have, by (3.16), (5.1), (5.7), the definition of \( T_0 \), and (3.18),
\[ \mathcal{G}(\overline{C}) = \mathcal{G}(C_0) + \mathcal{G}(\Gamma_1) + \mathcal{G}(\Gamma_2) \]

\[ = \int_{T_0} + \int_{T_1} + \int_{T_2} \gamma'_i(t) dt + \int_{T_1} [\gamma'(t) - \gamma'_0(t)] dt \]

\[ + \int_{T_2} [\gamma'(t) - \gamma'_0(t)] dt \]

\[ = \int_{\mathbb{R}} \gamma'(t) dt = \int_0^1 \gamma(t) dt = \gamma(1) = 1. \]

Similarly, using (3.15), (5.2), (5.8), the definition of \( T_0 \), Lemma 6, and (3.18), we obtain

\[ \mathcal{J}(\overline{C}) = \mathcal{J}(C_0) + \mathcal{J}(\Gamma_1) + \mathcal{J}(\Gamma_2) \]

\[ \leq \int_{T_0} + \int_{T_1} + \int_{T_2} \phi'_i(t) dt + \int_{T_1} [\phi'(t) - \phi'_0(t)] dt \]

\[ + \int_{T_2} [\phi'(t) - \phi'_0(t)] dt \]

\[ \leq \int_{\mathbb{R}} \phi'(t) dt = \int_0^1 \phi(t) dt = \phi(1) = \mu_0. \]

But by (5.9) the curve \( \overline{C} \) is in \( K[\mathcal{G}=1] \); so \( \mathcal{J}(\overline{C}) \geq \mu \geq \mu_0 \). This, with (5.10), implies

\[ (5.11) \quad \mathcal{J}(\overline{C}) = \mu_0 = \mu, \]

and the proof of the theorem is complete.

Incidentally we have proved that under the hypotheses of Theorem 2 the equation \( \mu_0 = \mu \) holds. It follows with little difficulty that the value of \( \mu \), considered as a function of \( l \), is lower semicontinuous.

6. Corollaries and examples. Let us define \( T^*_+(C) \) and \( T^*_-(C) \) by deleting the words "\( z(t) \) is a singular point and" in (2.3) and (2.4), and let \( M^*(C) \) and \( m^*(C) \) be the numbers defined by replacing \( T_+(C), T_-(C) \) by \( T^*_+(C), T^*_-(C) \), respectively, in (2.5). Then if \( m(C) \) is defined, so is \( m^*(C) \); and \( m^*(C) \geq m(C) \), for \( T^*_-(C) \) contains \( T_-(C) \). Likewise, if \( M(C) \) is defined, so is \( M^*(C) \); and \( M^*(C) \leq M(C) \). The following corollary is then immediately evident:

**Corollary 1.** If the hypotheses of Theorem 2 hold with \( m^*(C), M^*(C) \) in place of \( m(C), M(C) \), respectively, then the class \( K[\mathcal{G}=1] \) either is empty or contains a curve for which \( \mathcal{J}(C) \) assumes its least value on \( K[\mathcal{G}=1] \).

For if the hypotheses of Corollary 1 are satisfied, so are the hypotheses of Theorem 2.
In my dissertation I established an existence theorem which overlaps considerably with Corollary 1 but neither contains it nor is contained in it. Nor does Corollary 1 cover the five existence theorems for isoperimetric problems given by Tonelli; for Tonelli allows his class \( K \) to be a “complete class of total ramification,” where our class \( K \) consists of the family of curves in \( S \) joining two fixed points. § In all other respects, however, Corollary 1 contains Tonelli’s theorems. Take for example Tonelli’s Theorem 3 (p. 473) whose generalization to \( q \) dimensions is as follows:

Let \( S \) be bounded and closed, and let \( K \) be a complete class of curves of total ramification lying in \( S \). Let \( F(z, r) \) be \( p.q.r. \) on \( S \), and let \( G(z, r) = g(z)G(z, r) + a_4(z)r^a \), where \( g(z) \) is nonnegative \( \) on \( S \), and through each point \( z_1 \) of \( S \) there passes an arc \( \Gamma^* \) on which \( g(z) > g(z_1) \), provided that any continuous curve at all passes through \( z_1 \). Then \( K[G=1] \) either is empty or contains a curve for which \( F(C) \) assumes its least value on \( K[G=1] \).

We disregard the statements in brackets; they interchange with the unbracketed statements if \( G \) is replaced by \(-G\), which replacement does not affect the hypotheses of Theorem 2 or Corollary 1. The set \( T^*_+(C) \) consists of all \( t \) at which \( g(z(t)) > 0 \) and \( F(z, r) \) is not linear in the \( r^i \). The set \( T^*_-(C) \) is empty; so hypothesis (d) is satisfied. For each \( t \) in \( T^*_+(C) \) the function

\[
F(z, r) - \theta G(z, r) = F(z, r)[1 - \theta g(z)] - \theta a_4 r^a
\]

is \( p.q.r. \) for all \( \theta \leq 1/g(z) \). Hence \( M(z) = 1/g(z) \). If \( T^*_+(C) \) is not empty, then the greatest lower bound of \( M(z(t)) \) for \( t \) in \( T^*_+(C) \) is at least \( g.l.b. \) \([1/g(z(t))]\). That is, \( M^*(C) \geq 1/\max g(z(t)) \). Let \( z_1 = z(t_1) \) be a point at which \( g(z(t)) \) assumes its maximum (greater than zero), and let \( \Gamma^* \) be the curve along which \( g(z) \geq g(z_1) \). Then along \( \Gamma^* \) the conditions (2.6) and (2.7) are satisfied. Hypotheses (1.1) and (2.2) obviously hold. So except for the added generality of the class \( K \), this theorem is contained in Corollary 1.

As an example covered by Theorem 2 but not by Corollary 1 or any other theorems cited, let us consider

\[
F(x, y, x', y') = -(x - y)^2(x'^2 + 8y'^2)^{1/2},
G(x, y, x', y') = (x'^2 + y'^2)^{1/2}.
\]
where the range $S$ of $(x, y)$ is the whole plane. Hypothesis (d) of Corollary 1 is not satisfied; inequality (2.7) cannot be satisfied unless $C$ lies entirely along the line $y = x$. However, every point $(x, y)$ not on the line $y = x$ is an ordinary point. For, first, $G$ is regular. Second, the matrix $\Delta(x, y; p_1, p_2; q_1, q_2)$ has the form

$$\begin{pmatrix}
- (y - x) \{ p_1(p_1^2 + 8p_2^2)^{-1/2} & - q_1(q_1^2 + 8q_2^2)^{-1/2} \\
q_1(q_1^2 + 8q_2^2)^{-1/2} & - (y - x) \{ p_2(p_2^2 + 8p_1^2)^{-1/2}
\end{pmatrix}$$

if we assume (as we may) that $p_1^2 + p_2^2 = q_1^2 + q_2^2 = 1$. If a vector $(p_1, p_2)$ is given, the vector $(q_1, q_2) = (p_1, -p_2)$ is an approach set containing $(p_1, p_2)$, and $(p_1, -p_2)$ is in another approach set containing $(p_1, p_2)$. It is possible (though not very easy) to show that no approach set contains any other unit vectors than these. Computing $\Omega_H$ we see that it is not zero for any of the sets except in the trivial case in which the two formally different unit vectors of the approach set coincide ($p_1 = 0$ or $p_2 = 0$). Hence every point of $S$ with $y \neq x$ is an ordinary point.

The singular points of $S$ are thus the points $(x, x)$. For these the function $F(x, y, x', y') - \theta G(x, y, x', y')$ reduces to $- \theta (x'^2 + y'^2)^{1/2}$, which is positive quasi-regular if and only if $\theta \leq 0$. So $m(x, x) = - \infty$ and $M(x, x) = 0$. The set $T_-(C)$ is always empty, since $G(z, r)$ is positive regular. If $T_+(C)$ is not empty, then for every $t$ in $T_+(C)$ we have $x(t) = y(t)$ and $M(x(t), y(t)) = 0$. Therefore $M(C)$ is 0 whenever it is defined; that is, whenever $C$ intersects the line $y = x$. Thus if $C$ does not intersect the line, then $T_+(C)$ is empty; and if $C$ intersects the line at a point $(x_0, x_0)$, we can take $\Gamma^*$ to be a segment of $y = x$ beginning at $(x_0, x_0)$. Hypothesis (d) therefore is satisfied.

If we use the same $G$, but take

$$F = e^{(x^n + x^2)(x'^2 + 8y'^2)^{1/2}}$$

and let $S$ be the whole $(x, y)$-plane, we find similarly that there are no singular points.

7. A generalization. There are several ways of strengthening Theorem 2 without great difficulty. An obvious one is as follows: If $z_1$, $z_2$, and $l$ are given, under hypothesis (1.1) there is only a bounded subset of $S$ which can contain points of curves $C$ of $K \{ G = l \}$ with $\gamma(C) \leq \mu + \epsilon$ for any given $\epsilon > 0$. Let $S_*$ be this subset. We need then assume only that hypotheses (c) and (d) of Theorem 2 hold on the closure of $S_*$. A less trivial generalization is obtained by redefining $M(z)$ and $m(z)$ at singular points $z$ which are interior to $S$. Let $z$ be such a singular point, and
let \( A \) be an approach set at \( z \). Consider the aggregate of numbers \( \theta \) for which
\[
(7.1) \quad a_\theta E_{r-\theta}^0(z, a_\alpha p_a, p_a) \geq 0
\]
for all finite collections \( p_1, \ldots, p_n \) of vectors of \( A \) and all sets \( a_1, \ldots, a_n \) of nonnegative numbers such that \( |a_\alpha p_a| \neq 0 \). We make the following definitions:

(7.2) \( M_1(z, A) \) is the least upper bound of all numbers \( \theta \) such that (7.1) holds, and \( m_1(z, A) \) is their greatest lower bound.

(7.3) \( M_1(z) \) is the greatest lower bound of \( M_1(z, A) \) for all approach sets \( A \) at \( z \), and \( m_1(z) \) is the least upper bound of \( m_1(z, A) \) for all approach sets \( A \) at \( z \).

Under hypothesis (2.2) such numbers \( \theta \) exist. For if \( \theta \) serves in (2.2), then
\[
\varepsilon_{r-\theta_0}(z, p, r) \geq 0
\]
for all \( \theta \neq 0 \) and all \( r \), and (7.1) follows at once. This argument shows moreover that every \( \theta \) which serves in (2.2) serves in (7.1), no matter which approach set \( A \) we use. Hence if \( A \) is any approach set at \( z \), \( M_1(z, A) \geq M(z) \) and \( m_1(z, A) \leq m(z) \); so by (7.3)
\[
(7.4) \quad M_1(z) \geq M(z), \quad m_1(z) \leq m(z).
\]

Our theorem is given as follows:

**Theorem 3.** At all singular points \( z \) interior to \( S \) let \( M(z), m(z) \) be redefined to mean \( M_1(z), m_1(z) \), respectively. Then with this new meaning of \( m(z) \) and \( M(z) \) Theorem 2 remains valid.

The numbers \( M(z), m(z) \) entered the proof of Theorem 2 by way of Lemmas 4 and 5. Therefore we need only establish Lemmas 4 and 5 with \( M_1, m_1 \) in place of \( M, m \), respectively. Suppose then that \( z_0(\iota) \) is a singular point interior to \( S \). By Lemma 5 of IV there is an approach set \( A \) and a subsequence \( \{z_\alpha(t)\} \) with the properties there specified. (We disregard case (i) of that Lemma, for then the proof that \( \phi'(\iota) = \phi'(\iota_0) \) and \( \gamma'(\iota) = \gamma'(\iota_0) \) goes through as before.) By the definition of the \( E \)-function, inequality (7.1) can be written
\[
(7.5) \quad a_\alpha[F(z_\alpha, p_a) - \theta G(z_\alpha, p_a)] \geq F(z_\alpha, a_\alpha p_a) - \theta G(z_\alpha, a_\alpha p_a),
\]
where we have written \( z_\alpha \) for \( z_\alpha(\iota) \). Let \( R \) be the (convex) set consisting of all nonzero vectors \( r \) which can be written in the form \( a_1 p_1 + \cdots + a_n p_n \), where each \( p_i \) is in \( A \) and \( a_i \geq 0 \). Each \( r \) in \( R \) can be written in one or many ways as a sum \( a_\alpha p_a \). The lower bound of the left member of (7.5), for all such ways of writing \( r \), is known to be a convex function \( H(z_\alpha, r) \) on \( R \). By (7.5),
\[
H(z_\alpha, r) \geq F(z_\alpha, r) - \theta G(z_\alpha, r), \quad r \text{ in } R.
\]
From this (and the differentiability of $F$ and $G$) there is, for each $r_0$ in $R$, a linear function $l_\alpha r^\alpha$ such that

$$F(z_0, r_0) - \theta G(z_0, r_0) \leq l_\alpha r^\alpha,$$

$$H(z_0, r) \geq l_\alpha r^\alpha \text{ for all } r \text{ in } R.$$  

In particular, $F(z_0, \rho) - \theta G(z_0, \rho) \leq H(z_0, \rho)$ for all $\rho$ in $A$, as we find by taking $n=1, \rho_1 = \rho, a_1=1$, in (7.5). So by (7.7)

$$F(z_0, \rho) - \theta G(z_0, \rho) \geq l_\alpha \rho^\alpha, \quad \rho \text{ in } A.$$  

Let us denote the closed $\gamma$-neighborhoods of $z_0, A, R$ by $(z_0)_\gamma, (A)_\gamma, (R)_\gamma$, respectively. For each $\gamma > 0$, if $m$ is large and $\delta$ small, the point $z_m(t)$ is $(z_0)_\gamma$ and $z_m'(t)$ is in $(A)_\gamma$ for $t_0 - \delta \leq t \leq t_0 + \delta$. A fortiori, $z_m'(t)$ is in the closed convex set $(R)_\gamma$. By Jensen’s inequality, if $t_0 - \delta \leq t < t_0 + h \leq t_0 + \delta$, then

$$\frac{1}{h} \int_0^{t_0 + h} \dot{z}_m(t) dt \equiv [z_m(t_0 + h) - z_m(t_0)]/h$$

is in $(R)_\gamma$. Let $m \to \infty$; the vector $[z_0(t_0 + h) - z_0(t_0)]/h$ is in $(R)_\gamma$. Let $h \to 0$; the vector $z'_0(t)$, if defined, is in $(R)_\gamma$. By use of fairly obvious estimates, we find by (7.6) and (7.8) that for all sufficiently small positive numbers $\gamma$

$$F(z_0(t), z_0'(t)) - \theta G(z_0(t), z_0'(t)) \leq l_\alpha z_0(t_0 + h) - z_0(t_0) + \epsilon,$$

$$F(z_m(t), z_m'(t)) - \theta G(z_m(t), z_m'(t)) \geq l_\alpha z_m(t_0 + h) - z_m(t_0) - \epsilon,$$

Integrating from $t_0$ to $t_0 + h$ yields

$$\phi_0(t_0 + h) - \phi_0(t_0) = \theta [\gamma_0(t_0 + h) - \gamma_0(t_0)] \leq l_\alpha [z_0(t_0 + h) - z_0(t_0)] + \epsilon h,$$

$$\phi_m(t_0 + h) - \phi_m(t_0) = \theta [\gamma_m(t_0 + h) - \gamma_m(t_0)] \geq l_\alpha [z_m(t_0 + h) - z_m(t_0)] - \epsilon h.$$  

If we let $m \to \infty$, divide by $h$, and let $h \to \infty$, we get

$$\phi'_0(t_0) - \theta \gamma'_0(t_0) \leq l_\alpha z_0(t_0 + h) + \epsilon, \quad \phi'_m(t_0) - \theta \gamma'_m(t_0) \geq l_\alpha z_m(t_0 + h) - \epsilon.$$  

Since $\epsilon$ is arbitrary,

$$\phi'(t_0) - \phi'_0(t_0) \geq \theta [\gamma'_0(t_0) - \gamma'_0(t_0)].$$

If we let $\theta$ run through a sequence of values approaching $M_1(z_0(t_0), A)$, we find

$$\phi'(t_0) - \phi'_0(t_0) \geq M_1(z_0(t_0), A) [\gamma'_0(t_0) - \gamma'_0(t_0)].$$

This holds for all $M_1(z_0(t_0), A)$; so it holds for their greatest lower bound $M_1(z_0(t_0))$. The generalization of Lemma 4 is therefore established. Lemma 5 can be discussed similarly; or we can obtain the result from the proof above by replacing $G$ by $-G$.  

From the definitions it is evident that any alteration in the definitions of \(M_i(z, A), m_i(z, A), M_i(z),\) and \(m_i(z)\) which enables us to discard vectors \(p\) from approach sets \(A\) at \(z\) or enables us to disregard entire approach sets \(A\) either leaves these numbers unchanged or improves them; that is, if \(M_i(z, A)\) and \(M_i(z)\) are altered by the change of definition they are increased, and if \(m_i(z, A)\) and \(m_i(z)\) are altered they are decreased. In §8 we shall establish a criterion which will permit us to ignore certain types of approach sets. Here we establish two simpler criteria. Suppose that \(z\) is a singular point interior to \(S\), at which \(G(z, r)\) is quasi-regular normal. If \(A\) is an approach set containing only a finite number of unit vectors \(p_1, \ldots, p_k\), and if these can be so ordered that \(\Omega(z, p_i, p_j) < 0\) if \(i < j\), then \(A\) can be disregarded in defining \(m_i(z)\) and \(M_i(z)\). For if, in Lemma 5 of IV, the set \(A\) can be so chosen as to have these properties, all the proofs leading up to equations (8.2) and (8.3) of IV remains valid without change, and we obtain \(\gamma'(t_0) = \gamma'_0(t)\) and \(\phi'(t_0) = \phi'_0(t)\).

Retaining the assumption concerning \(G(z, r)\), let us suppose that \(A\) is an approach set containing a finite number of unit vectors. If there is a unit vector \(p_1\) in \(A\) such that \(\Omega(z, p_1, p) < 0\) for all unit vectors \(p \neq p_1\) in \(A\), we say that \(p_1\) is the first vector in \(A\). If there is also a unit vector \(p_2\) in \(A\) such that \(\Omega(z, p_2, p) < 0\) for all unit vectors \(p\) in \(A\) except \(p_1\) and \(p_2\), then \(p_2\) is the second vector in \(A\); and so on. If there is a \(p_1\) in \(A\) such that \(\Omega(z, p_1, p) < 0\) for all unit vectors \(p = p_1\) in \(A\), we say that \(p_1\) is the last vector in \(A\); and so on for \(p_{1-1}, p_{1-2}, \ldots\). Unless \(z\) is an ordinary point, there may remain some unit vectors not thus classified. These and their multiples we call the non-ordered nucleus of \(A\). In defining \(M_i(z, A)\) and \(m_i(z, A)\) we can discard from \(A\) all vectors not belonging to the non-ordered nucleus. The details of proof I shall omit.

8. \(\mathcal{E}\)-Admissibility. In §8 of III we introduced a concept called \(\mathcal{E}\)-admissibility, and showed that we could restrict our attention to those approach sets which were \(\mathcal{E}\)-admissible. The set \(A\) was \(\mathcal{E}\)-admissible if \(\mathcal{E}(z, p_0, p) \geq 0\) for all \(p_0\) in \(A\) and all \(p\). If we wish to define an analogous notion for isoperimetric problems, we must be guided by the way in which the Weierstrass condition is stated for those problems. The Weierstrass condition is to the effect that \(\mathcal{E}(z_0(t), z'_0(t), p) \geq 0\) if \(z = z_0(t)\) is the minimizing curve and \(H = F - \lambda G\). This suggests the following definition:

\[
(8.1) \text{Let } A \text{ be an approach set at } z, \text{ and let } H(z, r) \text{ be the function } F(z, r) - \lambda(z, A)G(z, r). \text{ Then the set } A \text{ is } \mathcal{E}\text{-admissible if}
\]

\[
\mathcal{E}(z, p_0, p) \geq 0
\]

for all \(p_0\) in \(A\) and all \(p\).
In complete analogy with III, we can prove the following theorem:

**Theorem 4.** If in the definition of ordinary point we replace the words (in (2.1), (c)) "every approach set" by "every $\mathcal{E}$-admissible approach set," Theorems 2 and 3 remain valid.

I have been unable to find any proof of this theorem which is not extremely long and involved. Therefore I shall here content myself with a sketch of a proof; the reader will probably be able to furnish the omitted details if he is interested.

Let $\mu_0$ be (as before) the least number which is the limit of $\mathcal{J}(\Pi_n)$ for a sequence of polygons $\Pi_n$ of $\Lambda$ such that $G(\Pi_n) \to l$. We may suppose that $\Pi_n$ has the usual minimizing property with respect to curves having not more vertices than $\Pi_n$ has. We thus come to Lemma 2 and must establish that lemma. Suppose the contrary, that one of the equations (4.3) fails. In particular, we suppose that the second one fails. In the proof of Lemma 1 an approach set $A$ entered, via Lemma 5 of IV. If this approach set is $\mathcal{E}$-admissible, the whole argument leading to equations (4.3) is valid without alteration, and (4.3) holds. This is a contradiction. It remains to consider the possibility that $A$ is not $\mathcal{E}$-admissible and show that this leads to a contradiction.

If $A$ is not $\mathcal{E}$-admissible, there is a vector $p_1$ such that $\mathcal{E}(\Pi, p_0, p_1) < 0$ for some (hence for all) $p_0$ in $A$. Choose a small interval $[t_0 - \delta, t_0 + \delta]$. We treat the two subintervals $[t_0 - \delta, t_0]$ and $[t_0, t_0 + \delta]$ differently.

If, as usual, we write $H(z, r) = F(z, r) - \lambda(z_0(t_0), A)G(z, r)$, we may assume $H_t(z_0(t_0), p) = 0$ for all $p$ in $A$; for this may be brought about by adding the linear function $-H_\alpha(z_0(t_0), p_0)t^\alpha$ to $H(z, r)$, where $p_0$ is in $A$. Let $\xi_m(t, 0)$ be the linear function for which $\xi_m(t_0, 0) = z_m(t_0)$ and $\xi_m(t_0 + \delta, 0) = z_m(t_0 + \delta)$, and define

$$\xi(t, \sigma) = \xi(t, 0) + \sigma[z_m(t) - \xi(t, 0)], \quad t_0 \leq t \leq t_0 + \delta.$$
\[ \gamma_m(t_0 + \delta) - \gamma_m(t_0) - [\gamma_0(t_0 + \delta) - \gamma_0(t_0)] > 2\kappa \delta. \]

Since \( G(t) \) is independent of \( z \) and is positive quasi-regular, the line segment \( z = \xi_m(t, 0) \) furnishes an absolute minimum for \( G(C) \) in the class of curves joining its ends. Hence

\[
\lim_{m \to \infty} \int_{t_0}^{t_0 + \delta} G(\xi_m) dt \leq \int_{t_0}^{t_0 + \delta} G(\xi_0) dt.
\]

With the preceding inequality, this shows that for large \( m \) and small \( \delta \)

\[ I_m(1) = \gamma_m(t_0 + \delta) - \gamma_m(t_0) \geq \int_{t_0}^{t_0 + \delta} G(\xi_m) dt + \kappa \delta = I_m(0) + \kappa \delta. \]

From the convexity of \( I_m(\sigma) \) we find that

\[ I_m(1 - \sigma) - I_m(1) < - \sigma \kappa \delta \quad \text{for} \quad 0 \leq \sigma \leq 1. \]

Now we consider the interval \( (t_0 - \delta, t_0) \). On this we use the construction of \( \xi \) of \( \Sigma \). We thereby replace the arc \( z = \xi_m(t), \quad (t_0 - \delta \leq t \leq t_0) \), of \( \Pi_m \) by an arc \( z = \xi_m(t, \varepsilon) \) with the same ends and having

\[ \int_{t_0 - \delta}^{t_0} H(\xi_m(t, \varepsilon)) dt - \int_{t_0 - \delta}^{t_0} H(\xi_m(t)) dt < - 2 \gamma \delta \varepsilon, \quad \gamma > 0, \]

for all \( m \). Because of the convexity of \( G(t) \) the integral of \( G \) is increased, but it is easy to estimate that the increase is less than \( K \delta \varepsilon, \quad K \) a constant.

Choose \( \varepsilon \) small enough so that \( K \delta \varepsilon < \kappa \delta \). Then by (8.3) there is a \( \sigma \) such that \( 0 \leq \sigma \leq 1 \) and

\[ I_m(1 - \sigma) - I_m(1) = \int_{t_0 - \delta}^{t_0} G(\xi_m'(t)) dt - \int_{t_0 - \delta}^{t_0} G(\xi_m'(t, \varepsilon)) dt. \]

That is, if we let \( \xi_m(t) \) be \( \xi_m(t, \varepsilon) \) on \( [t_0 - \delta, t_0) \) and \( \xi_m(t, \sigma) \) on \( [t_0, t_0 + \delta] \), then

\[ \int_{t_0 - \delta}^{t_0 + \delta} G(\xi_m'(t)) dt = \int_{t_0 - \delta}^{t_0 + \delta} G(\xi_m'(t, \varepsilon)) dt. \]

The right member of equation (8.5) has a value between \(- K \delta \varepsilon \) and zero; so by inequality (8.3) we conclude that

\[ 0 < \sigma < K \varepsilon / \kappa. \]

By reducing \( \varepsilon \) if necessary, we can ensure that \( \sigma \) is less than \( 1/12 \).

Since \( A \) is an approach set on which \( H \) vanishes identically, all first-order partial derivatives of \( H \) also vanish on \( A \). Therefore there is a positive number \( \lambda \) such that

\[ H_n(r) H_n(r) < \left[ \gamma \kappa / 3KL \right]^2 \]
if \( L/2 \leq |r| \leq 3L/2 \) and \( r \) is in the \( 2\lambda \)-neighborhood of \( A \). If \( m \) is large, the inequality
\[
3L/4 < \left| z_m'(t) \right| < 5L/4
\]
holds; and moreover \( z_m'(t) \) is in the \( \lambda \)-neighborhood of \( A \) if \( |t-t_0| < \delta \), provided that \( \delta \) is small enough. By definition of \( \zeta_m \) we find that
\[
\left| \zeta_m'(t, 1-\sigma) - z_m'(t) \right| = \sigma \left| z_m'(t) - \zeta_m'(t, 0) \right|
\]
\[
< \sigma 3L < L/4.
\]
By using the theorem of the mean, together with (8.7) and (8.8), whose use is permitted by (8.9) and (8.10), we obtain
\[
|H(\zeta_m'(t, 1-\sigma)) - H(z_m'(t))| < \sigma \gamma \varepsilon /K < \gamma \varepsilon.
\]
Recalling the definition of \( \bar{z}_m(t) \); inequalities (8.4) and (8.11) (integrated from \( t_0 \) to \( t_0+\delta \)), we obtain
\[
\int_{t_0-\delta}^{t_0+\delta} F(z_m') dt - \int_{t_0-\delta}^{t_0+\delta} F(z_m') dt = \int_{t_0-\delta}^{t_0+\delta} H(\bar{z}_m') dt - \int_{t_0-\delta}^{t_0+\delta} H(z_m') dt < -\gamma \delta \varepsilon.
\]
Extending \( \bar{z}_m(t) \) by setting it equal to \( z_m(t) \) for \( 0 \leq t < t_0 - \delta \) and \( t_0 + \delta < t \leq 1 \), we obtain a polygon \( \Pi_m \) such that
\[
G(\Pi_m) = G(\Pi_m), \quad \mathcal{J}(\Pi_m) < \mathcal{J}(\Pi_m) - \gamma \delta \varepsilon.
\]
But then \( \lim sup \mathcal{J}(\Pi_m) \leq \mu_0 - \gamma \delta \varepsilon \), contrary to the definition of \( \mu_0 \), and the desired contradiction has been reached.

We thus see that \( \gamma'(t_0) = \gamma_0'(t_0) \) for almost all \( t_0 \) such that \( z_0(t_0) \) is interior to \( S \). If \( t_1 \leq t \leq t_2 \) defines an interior arc of \( z = z_0(t) \), this proves that
\[
\lim_{n \to \infty} \int_{t_1}^{t_2} G(z_n, \hat{z}_n) dt = \int_{t_1}^{t_2} G(z_0, \hat{z}_0) dt.
\]
But \( G \) is quasi-regular normal; so by a known theorem this implies
\[
\lim_{n \to \infty} \int_{t_1}^{t_2} F(z_m, \hat{z}_m) dt = \int_{t_1}^{t_2} F(z_0, \hat{z}_0) dt;
\]
that is, \( \phi(t_2) - \phi(t_1) = \phi_0(t_2) - \phi_0(t_1) \). Hence \( \phi'(t_0) = \phi_0'(t_0) \) for all points \( z_0(t_0) \) interior to \( S \), and equations (4.3) are established.

Besides the added generality, Theorem 4 offers another advantage. The search for \( \mathcal{E} \)-admissible approach sets may be easier than the determination of all approach sets. For one thing, if \( A \) is an approach set at \( z \), and there is a \( z' \) such that
(8.13) \[ H(z, z'; \lambda(z, A)) + H(z, -z'; \lambda(z, A)) < 0, \]
then \( A \) is not \( \mathcal{E} \)-admissible, as we see if we rewrite (8.13) in the form
\[ \mathcal{E}_H(z, \rho, z') + \mathcal{E}_H(z, \rho, -z') < 0. \]

If \( F \) and \( G \) belong to the important special class of integrands such that
\[ F(z, -z') = F(z, z'), \quad G(z, -z') = G(z, z'), \]
then in order that the approach set \( A \) at \( z \) be \( \mathcal{E} \)-admissible it is necessary that \( H(z, z'; \lambda(z, A)) \) be nonnegative. The example of §6 is of this type. More generally, let
\[ F = \phi(x, y)(x'^2 + a^2y'^2)^{1/2}, \quad G = \psi(x, y)(x'^2 + y'^2)^{1/2}, \]
where \( a > 1 \) and \( \psi > 0 \). Here
\[ H = \phi(x'^2 + a^2y'^2)^{1/2} - \lambda\psi(x'^2 + y'^2)^{1/2}, \]
and in order that this be nonnegative we must have
(8.14) \[ \lambda \leq a\phi/\psi. \]

Suppose to be specific that \( \phi \leq 0 \). If the equality holds in (8.14), then \( H(x, y, 0, y'; \lambda) = 0 \) for all \( y' \), and \( (0, 1) \) and \( (0, -1) \) are in an \( \mathcal{E} \)-admissible approach set. No other unit vectors are in this set unless \( \phi = 0 \). If \( \lambda < a\phi/\psi \) then \( H \) is positive. The graph of \( H = 1 \) is either convex (if \( -\lambda \) is large) or dumb-bell shaped, with its narrowest section along the \( x' \)-axis. It is then geometrically evident that the only \( \mathcal{E} \)-admissible approach sets are those containing only two unit vectors, \( (\rho, q) \) and \( (\rho, -q) \). This applies, in particular, to the example of §6.

Again, let \( F = -e^y(x'^2 + 4y'^2)^{1/2}, \quad G = (x'^2 + y'^2)^{1/2} \). As we have just seen, the only \( \mathcal{E} \)-admissible approach sets contain at most two unit vectors, \( (\rho, q) \) and \( (\rho, -q) \). Suppose \( q > 0 \); then \( \Omega_H(x, y, \rho, q, \rho, -q) = -e^{2q(\rho^2 + 4q^2)^{1/2}} \). So (2.1), (c) holds for the \( \mathcal{E} \)-admissible approach sets \( A \), and Theorem 4 applies. But Theorems 1 and 2 do not apply. For if \( (\rho, q) \) is a unit vector, then \( (-\rho, q) \) is in an approach set with \( (\rho, q) \), and \( \Omega_H(x, y, \rho, q, -\rho, q) = 0 \). Hence (2.1) is not satisfied for all approach sets \( A \), and no point is ordinary as defined in (2.1).

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