CONDITIONS ON $u(x, y)$ AND $v(x, y)$ NECESSARY AND SUFFICIENT FOR THE REGULARITY OF $u+iv$*

BY
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Introduction. On what subset of an open set must a continuous function of a complex variable be assumed to have a derivative in order that the regularity of the function be implied? A series of researches on this problem culminates in the following theorem due to Besicovitch: † A complex function f, continuous on an open set G, is regular in G if it is derivable at almost all the points of G and if further,

$$\limsup_{h\to 0} \left| [f(z+h) - f(z)]/h \right| < +\infty$$

at each point z of G except at most those of a set which is the sum of a sequence of sets of finite length.‡

At the same time the problem of reducing the conditions on $u(x, y)$ and $v(x, y)$, where $u+iv=f(z)$, necessary and sufficient for the regularity of $f(z)$, has also received much attention,§ the most general result being the theorem of Looman and Menchoff: || If the functions $u(x, y)$ and $v(x, y)$, continuous in an open set G, are derivable with respect to x and with respect to y at each point of G except at most at the points of an enumerable set, and if $u_x(x, y)=v_y(x, y)$ and $u_y(x, y)=-v_x(x, y)$ at almost all the points (x, y) of G, then the function $u+iv$ is regular in G.

In the first part of this paper we investigate a question raised by Saks,¶ as to the existence of a more general theorem including these two results. The answer obtained is affirmative in the case when the sequence of sets of finite length mentioned in the theorem of Besicovitch is an F_σ with respect to the open set considered.

In the second part we further extend the set on which the partial deriva-

* Presented to the Society, April 16, 1938; received by the editors March 14, 1938.

† Proceedings of the London Mathematical Society, vol. 32, pp. 1–9. The version given here is due to S. Saks, *The Theory of the Integral*, New York, 1937, p. 197.

‡ For a definition of the length of a planar set, see C. Carathéodory, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, 1914, pp. 404–426.

§ For a review of recent results, see D. Menchoff, *Les Conditions de Monogenéité*, Actualités Scientifiques et Industrielles, no. 329, 1936.

|| Quoted from Saks, p. 199.

¶ Saks, p. 201.

tives of u and v may fail to exist by placing certain restrictions on u and v in addition to continuity. In particular, we state a condition on u and v according to which if the Dini partial derivates of u and v are finite except at most on an F_σ of measure zero and if the partials, where they exist, satisfy the Cauchy-Riemann equations ($u_x = v_y$, $u_y = -v_x$) a.e. (almost everywhere), then $u+iv$ is regular.

Finally, in the third part we obtain conditions necessary and sufficient in order that a function of two real variables be harmonic.

We agree to the following conventions: G will be used to denote an open set in the complex plane; R an open rectangle with sides $x=a_1$, $x=a_2$, $y=b_1$, $y=b_2$, $(a_1<a_2,\ b_1<b_2)$; (R) will denote the boundary of R; R will be said to be *in* G if $R+(R)$ is contained in G.

1. **Extension of the Looman-Menchoff theorem.** We prove the following theorem:

THEOREM 1. *Let $\mathfrak{J}$ be any class of continuous functions defined in G. Let E_n, $(n=1, 2, \cdots)$, be subsets of G, closed with respect to G, and such that regularity of any function of $\mathfrak{J}$ on $G-E_n$ implies its regularity throughout G.*

If $f(z)=u(x, y)+iv(x, y)$, belonging to $\mathfrak{J}$, has its Dini derivates infinite on $\sum_{n=1}^{\infty}E_n$ at most, and if the partial derivatives, where they exist, satisfy the Cauchy-Riemann equations a.e., $f(z)$ is regular in G.

Proof.* Let F be the points of G where $f(z)$ is not regular. F is evidently closed in G, and it has to be proved that F is empty.

Suppose therefore, if possible, that $F\neq 0$, and let F_n denote, for each positive integer n, the set of points of G such that whenever $|h|\leqq 1/n$, none of the four differences

$$u(x+h, y)-u(x, y), \qquad v(x+h, y)-v(x, y),$$
$$u(x, y+h)-u(x, y), \qquad v(x, y+h)-v(x, y)$$

exceeds $|nh|$ in absolute value. By continuity of u and v each set F_n is closed in G. Every point of G, except for the set $\sum_{n=1}^{\infty}E_n$, has all derivates finite and so falls in some F_n. The subset F of G can therefore be expressed as

$$F=\sum_{n=1}^{\infty}F_n\cdot F+\sum_{n=1}^{\infty}E_n\cdot F.$$

Since F is closed with respect to G, we have by Baire's theorem† that some term in the right-hand side is everywhere dense with respect to a portion of

* The first part of this proof follows closely the first part of the proof which Saks gives of the Looman-Menchoff theorem, p. 199.

† Saks, p. 54.

F, say $S\cdot F$ in the square S. We have two cases according as this is true of a term in the first or the second sum.

If $F_n\cdot F$ is supposed everywhere dense with respect to $S\cdot F$, then, since $F_n\cdot F$ is closed in G and so with respect to S, $F_n\cdot F$ contains $S\cdot F$. Hence each point of $G\cdot S$ has finite derivates; and by an extension* of the Looman-Menchoff theorem, $S\cdot F=0$.

In the second case E_n is everywhere dense with respect to $S\cdot F$. Since E_n is closed with respect to G, it contains $S\cdot F$. We have therefore the case of $f(z)$ regular in $G\cdot S-E_n$, and by our assumption about the sets E_n it follows that $f(z)$ is regular throughout $G\cdot S$, contrary to the assumption that $S\cdot F\neq 0$. This completes the proof of Theorem 1.

By the theorem of Besicovitch it follows that a continuous function is regular in G if it is regular except for a set of finite length closed with respect to G. As an immediate consequence of Theorem 1 we obtain the following generalization of the Looman-Menchoff theorem:

THEOREM 2. *If $f(z)=u+iv$ is continuous in G, and if the partial derivates of u and v are infinite at most on the sum of a sequence of sets of finite length closed in G, and if the Cauchy-Riemann equations hold a.e. where the partials exist, $f(z)$ is regular in G.*

It might be supposed that if the partial derivates of u and v are assumed finite except for a closed set of measure zero, then the continuous function $u+iv$ is regular in G. The following example shows that this is not the case.

Let $u(0, y)$, $(0\leqq y\leqq 1)$, be the function defined by Hille and Tamarkin,† monotone and continuous but not absolutely continuous, and let $u(x, y) =u(0, y)$, $(0<x\leqq 1; 0\leqq y\leqq 1)$. Then $u(x, y)$ is continuous (even of bounded variation, a property to be defined in §2) with $u_x(x, y)=0$ and $u_y(x, y)=0$ a.e. The function $u+iu$ is regular a.e. but not regular throughout the unit square.

THEOREM 3. *There exists a function $f(z)$ defined in the unit square, continuous and of bounded variation there, and regular everywhere except on a closed set of measure zero.*

The function $f(z)$ just defined is not regular on a set of parallel lines which divide the region into an infinite number of separated sets. The following question then arises: If $f(z)$ is continuous in R and regular a.e. with the points of regularity connected, is $f(z)$ regular in R? For example, take a Cantor set

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* Saks, p. 200: "instead of assuming partial derivability of the functions u and v, it is sufficient to suppose that at each point of G (except at most those of an enumerable set) these functions have with respect to each variable, x and y, their partial Dini derivates finite."

† American Mathematical Monthly, vol. 36 (1929), pp. 255–264.

on two adjacent sides of R and pass parallels to the sides through these sets, and take the intersection points of these lines as the set E. If $f(z)$ is continuous in R and regular in $R-E$, is $f(z)$ regular on E as well? In the following section we shall show that this is the case under the added assumption that u and v are of bounded variation.

2. **Functions with summable partials.** In this section we assume explicitly or otherwise, that the partial derivatives u_x, u_y, v_x, v_y exist a.e. and are summable as functions of two variables. We begin by considering a function $f(z)$ defined only on a certain subset of G and state conditions under which $f(z)$ is equal there to a function regular throughout G.

We need the following definitions: The intersection of an open set G and any set of almost all lines parallel to the x (y) axis will be called a G_x (G_y). We define R_x and R_y similarly. The sum of a G_x and a G_y will be denoted by G_x+G_y.

To avoid the phrase "continuity of $u(x, y)$ as a function of x for almost all values of y," we shall say "$u(x, y)$ is continuous in x for almost all y," or more simply, "$u(x, y)$ is continuous in a G_x."

THEOREM 4. *Let $f(z)$ be defined in a G_x+G_y with $f(z)=u(x, y)+iv(x, y)$. Suppose that in every R in G the following conditions hold:*

(a) *$u(x, y)$ and $v(x, y)$ are absolutely continuous in x for almost all y, and in y for almost all x.*

(b) *The partials u_x, u_y, v_x, v_y, are Lebesgue-summable in (x, y).*

(c) *The Cauchy-Riemann equations hold a.e.*

Then $f(z)$ is equal in G_x+G_y to a function regular in G.

Proof. Let P be any point of G, and let R be a rectangle in G containing P, with the vertex (a_1, b_1) chosen so that $u(x, b_1)$ and $v(x, b_1)$ are continuous in x for $a_1 \leqq x \leqq a_2$, and $u(a_1, y)$ and $v(a_1, y)$ are continuous in y for $b_1 \leqq y \leqq b_2$. By Fubini's theorem and the Cauchy-Riemann relation we have

$$\int_{a_1}^{x}\int_{b_1}^{y} u_y(\xi, \eta)d\eta d\xi = \int_{b_1}^{y}\int_{a_1}^{x} u_y(\xi, \eta)d\xi d\eta = -\int_{b_1}^{y}\int_{a_1}^{x} v_x(\xi, \eta)d\xi d\eta, \qquad (x, y) \text{ in } R.$$

By the absolute continuity of $u(x, y)$ in R_y we have

$$\int_{b_1}^{y} u_y(\xi, \eta)d\eta = u(\xi, y) - u(\xi, b_1) \text{ in } R_y.$$

Similarly

$$\int_{a_1}^{x} v_x(\xi, \eta)d\xi = v(x, \eta) - v(a_1, \eta) \text{ in some } R_x.$$

Substituting these in the extreme members of the first equation, we have the relation

$$\int_{a_1}^{x}[u(\xi, y) - u(\xi, b_1)]d\xi = -\int_{b_1}^{y}[v(x, \eta) - v(a_1, \eta)]d\eta, \qquad (x, y) \text{ in } R.$$

Let

$$\begin{aligned} w_1(x, y) &= \int_{a_1}^{x} u(\xi, y)d\xi - \int_{b_1}^{y} v(a_1, \eta)d\eta \\ &= -\int_{b_1}^{y} v(x, \eta)d\eta + \int_{a_1}^{x} u(\xi, b_1)d\xi. \end{aligned}$$

By the continuity of u in R_x and the first equality, $w_{1x}(x, y) = u(x, y)$ in R_x. By the continuity of v in R_y and the second equality, $w_{1y}(x, y) = -v(x, y)$ in R_y.

Again,

$$\int_{b_1}^{y}\int_{a_1}^{x} u_x(\xi, \eta)d\xi d\eta = \int_{a_1}^{x}\int_{b_1}^{y} u_x(\xi, \eta)d\eta d\xi = \int_{a_1}^{x}\int_{b_1}^{y} v_y(\xi, \eta)d\eta d\xi,$$

and as before

$$\int_{b_1}^{y}[u(x, \eta) - u(a_1, \eta)]d\eta = \int_{a_1}^{x}[v(\xi, y) - v(\xi, b_1)]d\xi, \quad (x, y) \text{ in } R.$$

Let

$$\begin{aligned} w_2(x, y) &= \int_{b_1}^{y} u(x, \eta)d\eta + \int_{a_1}^{x} v(\xi, b_1)d\xi \\ &= \int_{a_1}^{x} v(\xi, y)d\xi + \int_{b_1}^{y} u(a_1, \eta)d\eta. \end{aligned}$$

By the continuity of u in R_y and the first equality, $w_{2y}(x, y) = u(x, y)$ in R_y. By the continuity of v in R_x and the second equality, $w_{2x}(x, y) = v(x, y)$ in R_x.

By a theorem due to H. Rademacher,* a function $\phi(x, y) + i\psi(x, y)$ is regular in R when the following conditions are satisfied there: (1) $\phi(x, y)$ and $\psi(x, y)$ are absolutely continuous in x and y separately for all values of x and y; (2) $\phi(x, y)$ and $\psi(x, y)$ are summable in (x, y); (3) ϕ_x, ϕ_y, ψ_x, and ψ_y, which necessarily exist a.e., are summable in (x, y); (4) the Cauchy-Riemann equations hold a.e. This result is applicable to $w_1 + iw_2$. For (1) is true since by definition w_1 and w_2 are integrals; (4) has already been verified; (3) is true since the summability of u and v is a consequence of the summability of the partials and the continuity of $u(a_1, y)$, as the following inequality shows:

* Mathematische Zeitschrift, vol. 4 (1919), p. 184, Theorem II.

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} | u | \, dydx = \int_{a_1}^{a_2}\int_{b_1}^{b_2} \left| \int_{a_1}^{x} u_x(\xi, y) d\xi \right| dydx$$
$$+ \int_{a_1}^{a_2}\int_{b_1}^{b_2} | u(a_1, y) | \, dydx$$
$$\leqq \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} | u_x(\xi, y) | \, d\xi dydx$$
$$+ \int_{a_1}^{a_2}\int_{b_1}^{b_2} | u(a_1, y) | \, dydx.$$

By repeating this argument for w_1 and w_2, the partials of which are summable by (3), we obtain (2). Hence w_1+iw_2 is regular in R.

If R_1 and R_2 are any two separated rectangles lying in a domain of G, the regular function to which $u+iv$ is equal in R_1 is readily shown to be the analytic continuation of the regular function to which $u+iv$ is equal in R_2; hence the proof is complete.

Corollary. *If, in addition to satisfying the conditions of Theorem* 4, $u(x, y)$ *is continuous in* x *or in* y, *and* $v(x, y)$ *is continuous in* x *or in* y, *throughout* G, *then* $u+iv$ *is regular in* G.

For, in G_x+G_y, $u+iv$ is equal to a regular function. On any line parallel to the x axis, for example, the points of G_y are everywhere dense. Hence the continuity of $u(x, y)$ with respect to x implies that $u(x, y)$ is equal everywhere in G to the real part of a regular function, and by a similar argument applied to $v(x, y)$, the proof is complete.

A function $f(x)$ satisfies the (N) condition in the set A if the measure of the image of E ($E \subset A$) with respect to $f(x)$ is zero when the measure of E is zero. A function $f(x, y)$ will be said to *satisfy the* (N) *condition linearly in* A, a plane set, if for almost all x_0 and y_0 in the interval $(-\infty, +\infty)$, $f(x_0, y)$ and $f(x, y_0)$ satisfy the (N) condition in A.

Theorem 5. *Theorem* 4 *holds when condition* (a) *is replaced by either of the following*:

(i) u_x *and* v_x *exist and are finite in a* G_x; u_y *and* v_y *exist and are finite in a* G_y.

(ii) $u(x, y)$ *and* $v(x, y)$ *are continuous in* x *in a* G_x, *in* y *in a* G_y, *and satisfy the* (N) *condition in a* G_x *and a* G_y.

Proof. It is readily seen from the proof of Theorem 4, and the similarity of the conditions on u and v in x and in y, that it will suffice to show that (i) and (ii) imply the absolute continuity of u with respect to x in an R_x for all R of G.

If (i) is given, from the summability in (x, y) of u_x it follows that u_x is summable in x for almost all y. Hence for almost all y,

$$\int_{a_1}^{x} u_x(\xi, y)d\xi = u(x, y) - u(a_1, y)$$

by a well known theorem;* so $u(x, y)$ is absolutely continuous in x in a G_x.

To prove that (ii) implies the absolute continuity of $u(x, y)$ in x in R_x, let E_y denote the linear set in R with ordinate y at which $u(x, y)$ has a finite nonnegative partial with respect to x, and let E be all such points in R; then since $u_x(x, y)$ is summable on E, $\int_{E_y} u_x(\xi, y)d\xi < +\infty$ for almost all y, $(b_1 \leqq y \leqq b_2)$, and by a theorem due to N. Bary† this is sufficient for the absolute continuity of $u(x, y)$ in x in G_x, assumed continuous and satisfying the (N) condition for almost all y.

Given $f(x, y)$ continuous in $\overline{R}$, denote by $T_1(f; x; b_1, b_2)$, for any x such that $a_1 \leqq x \leqq a_2$, the total variation of $f(x, y)$ with respect to y from b_1 to b_2, and by $T_2(f; y; a_1, a_2)$, for any y such that $b_1 \leqq y \leqq b_2$, the total variation of $f(x, y)$ with respect to x from a_1 to a_2. When T_1 and T_2 are finite for almost all x and y, respectively, and the Lebesgue integrals $\int_{a_1}^{a_2} T_1(f; x; b_1, b_2)dx$ and $\int_{b_1}^{b_2} T_2(f; y; a_1, a_2)dy$ exist (finite), $f(x, y)$ is said to be *of bounded variation* (in the sense of Tonelli).‡

LEMMA. *If the continuous function* $f(x, y)$ (a) *is absolutely continuous in* x *in* R_x *and in* y *in* R_y, *or* (b) *has* f_x *existing and finite in* R_x *and* f_y *existing and finite in* R_y, *a necessary and sufficient condition that* $f(x, y)$ *be of bounded variation in* R *is that* f_x *and* f_y *be Lebesgue-summable there.*

Proof. To prove the necessity, assume that both $\int_{a_1}^{a_2} T_1(f; x; b_1, b_2)dx$ and $\int_{b_1}^{b_2} T_2(f; y; a_1, a_2)dy$ exist and are finite. Since by either (a) or (b) $f(x, y)$ is absolutely continuous in x in some R_x and in y in some R_y,

$$T_1(f; x; b_1, b_2) = \int_{b_1}^{b_2} | f_y | dy, \qquad T_2(f; y; a_1, a_2) = \int_{a_1}^{a_2} | f_x | dx$$

for almost all x and y, respectively.§ Substituting these values for T_1 and T_2, we get $\int_{a_1}^{a_2}\int_{b_1}^{b_2} |f_y| dydx$ and $\int_{b_1}^{b_2}\int_{a_1}^{a_2} |f_x| dydx$ finite; hence the double integrals exist.‖

The converse follows from the fact that the Lebesgue integral is absolutely

* E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. 1, 3d edition, Cambridge, 1927, p. 601.

† Saks, p. 285.

‡ Saks, p. 169.

§ Hobson, p. 605.

‖ Hobson, p. 631.

convergent. For example, the summability of f_y implies the finite existence of

$$\int_{a_1}^{a_2} T_1(f;\ x;\ b_1,\ b_2)dx = \int_{a_1}^{a_2}\int_{b_1}^{b_2} |f_y|\ dydx.$$

As a consequence of this lemma and Theorem 4, it follows that if u and v are of bounded variation in every R of G and absolutely continuous in x in G_x and in y in G_y, and if the Cauchy-Riemann equations are satisfied a.e. in $G, f(z)$ is regular in G. Using the following definition, we can state Theorem immediately.

The function $f(x, y)$ is *absolutely continuous* in R if it is of bounded variation in (x, y) and absolutely continuous in x in R_x and in y in R_y.*

THEOREM 6. *The function $u+iv$ is regular in G if and only if u and v are absolutely continuous in every R of G and if the Cauchy-Riemann equations hold a.e. in G.*

THEOREM 7. *If u and v are of bounded variation in every R of G, and if their partial derivates are infinite at most on an F_σ with respect to G, of measure zero, on which u and v satisfy the* (N) *condition linearly, and if the Cauchy-Riemann equations hold a.e. where the partials exist, $u+iv$ is regular in G.*

Proof. We first show that if u and v are of bounded variation in R, and $u+iv$ is regular except possibly for a set E of measure zero, necessarily closed with respect to R, on which u and v satisfy the (N) condition, then $u+iv$ is regular in R. By Theorem 6, it will be sufficient to show that u and v are absolutely continuous in R; we need therefore to prove that u and v are absolutely continuous in x in an R_x and in y in an R_y. By the symmetry of the conditions of u and v with respect to x and y, it will suffice to prove that u is absolutely continuous in x in an R_x. Any line $L \equiv [y = y_0]$ in R is composed of $\sum_{n=1}^{\infty} I_n + E\cdot L$, where I_n, $(n = 1, 2, \cdots)$, is an interval free of E and like $E\cdot L$ may be null. Since $u(x, y_0)$ satisfies the (N) condition on each of these subsets, it clearly satisfies the condition on their sum, that is, on L. Hence, because u is of bounded variation and satisfies the (N) condition in an R_x, u is absolutely continuous in x in an R_x.†

Now use Theorem 1 and the proof is complete.

In the example given in the first section, u and v are of bounded variation, and their partials exist a.e. satisfying the Cauchy-Riemann equations; hence the assumption of the (N) condition or its equivalent must necessarily be made in this theorem. However there is a less general class of sets of measure zero for which this (N) condition obviously need not be assumed.

* Saks, p. 169.

† Saks, p. 227.

COROLLARY. *If $u(x, y)$ and $v(x, y)$ are of bounded variaton in every R of G, and if their partial derivates are infinite at most on an F_σ with respect to G such that the lines parallel to the axes which intersect F_σ in a nondenumerable number of points are of measure zero, and if the Cauchy-Riemann equations hold a.e. where the partials exist, then $u+iv$ is regular in G.*

COROLLARY. *Let E be a non-empty subset of G such that almost every line parallel to an axis intersects E in at most a denumerable set. The function $f(z)$, regular in $G-E$, is regular throughout G if and only if u and v are of bounded variation in every R of G.*

This corollary should be compared with a similar result of V. Fedoroff.* He requires that $G-E$ be connected; otherwise E is any set of measure zero. On the other hand, his condition (D) is more restrictive than the present condition that u and v be of bounded variation.

3. **Morera's theorem and its application to harmonic functions.** Rademacher† has shown that if for every R in G, u and v are summable in (x, y) and, as functions of x and y separately, are summable for each value of x and y, and if $\int_{(R)}(u+iv)dz=0$, then $u+iv$ is regular in G except for removable discontinuities of measure zero.

In extending this result we introduce the following definition: Let E be a set of measure zero in G, and let R be any rectangle in G with sides $x=a_1$, $x=a_2$, $y=b_1$, $y=b_2$, $(a_1<a_2,\ b_1<b_2)$, with (a_1, b_1) and (a_2, b_2) in $G-E$; the set of all such rectangles will be called *almost all R in G*.

THEOREM 8. *If for almost all R in G the functions u and v are summable in (x, y) and $\int_{(R)}(u+iv)dz=0$, $u+iv$ is regular in G except possibly for removable discontinuities of measure zero.*

Proof. Let

$$
\begin{aligned}
w_1(x, y) = & -\int_{a_1}^{x}\int_{b_1}^{y} v(\xi, \eta)d\eta d\xi + \int_{a_1}^{x}\int_{a_1}^{\xi} u(\xi_1, b_1)d\xi_1 d\xi \\
& -\int_{b_1}^{y}\int_{b_1}^{\eta} u(a_1, \eta_1)d\eta_1 d\eta,
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
w_2(x, y) = & \int_{a_1}^{x}\int_{b_1}^{y} u(\xi, \eta)d\eta d\xi + \int_{a_1}^{x}\int_{a_1}^{\xi} v(\xi_1, b_1)d\xi_1 d\xi \\
& -\int_{b_1}^{y}\int_{b_1}^{\eta} u(a_1, \eta_1)d\eta_1 d\eta.
\end{aligned} \tag{2}
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* Recueil Mathématique de la Société de Moscou, vol. 41 (1934), p. 97.

† Rademacher, loc. cit., p. 183, Theorem I.

The condition $\int_{(R)}(u+iv)dz=0$ yields

$$(3)\quad \int_{a_1}^{x} u(\xi, b_1)d\xi - \int_{a_1}^{x} u(\xi, y)d\xi - \int_{b_1}^{y} v(x, \eta)d\eta + \int_{b_1}^{y} v(a_1, \eta)d\eta = 0,$$

$$(4)\quad \int_{a_1}^{x} v(\xi, b_1)d\xi - \int_{a_1}^{x} v(\xi, y)d\xi + \int_{b_1}^{y} u(x, \eta)d\eta - \int_{b_1}^{y} u(a_1, \eta)d\eta = 0.$$

In the first term of $w_1(x, y)$ replace $\int_{b_1}^{y} v(\xi, \eta)d\eta$ by the function

$$\int_{a_1}^{x} u(\xi, b_1)d\xi - \int_{a_1}^{x} u(\xi, y)d\xi + \int_{b_1}^{y} v(a_1, \eta)d\eta$$

to which by (3) it is equal a.e. Since the value of $w_1(x, y)$ has not altered and the new integrand is continuous in x, the latter equals the partial, $w_{1x}(x, y)$ in an R_x.

Hence by (3),

$$(5)\qquad w_{1x}(x, y) = -\int_{b_1}^{y} v(x, \eta)d\eta + \int_{a_1}^{x} u(\xi, b_1)d\xi \quad \text{a.e.}$$

Replacing $\int_{a_1}^{x} v(\xi, \eta)d\xi$ in the first term of $w_1(x, y)$ by

$$\int_{a_1}^{x} v(\xi, b_1)d\xi + \int_{b_1}^{y} u(x, \eta)d\eta - \int_{b_1}^{y} u(a_1, \eta)d\eta$$

and using a similar argument, we have

$$(6)\qquad w_{1y}(x, y) = -\int_{a_1}^{x} v(\xi, y)d\xi - \int_{b_1}^{y} u(a_1, \eta)d\eta \quad \text{a.e.}$$

Applying a similar proof to $w_2(x, y)$, we obtain

$$(7)\qquad w_{2x}(x, y) = \int_{b_1}^{y} u(x, \eta)d\eta + \int_{a_1}^{x} v(\xi, b_1)d\xi \quad \text{a.e.},$$

$$(8)\qquad w_{2y}(x, y) = \int_{a_1}^{x} u(\xi, y)d\xi - \int_{b_1}^{y} v(a_1, \eta)d\eta \quad \text{a.e.}$$

By (3), (5), and (8), we verify $w_{1x}=w_{2y}$ a.e.; and by (4), (6), and (7), $w_{1y}=-w_{2x}$ a.e. Theorem 4 implies that w_1+iw_2 is regular in R, therefore, and because $u+iv$ is equal a.e. to the second derivative of w_1+iw_2, the theorem is proved.

As an application of Theorem 8, consider a function $f(x, y)$ of two real variables, with partials f_x and f_y. If we take these to be the functions u and v, formally the conditions $\int_{(R)}f_xdx+f_ydy=0$ and $\int_{(R)}(df/dn)ds=0$ are the real and imaginary parts, respectively, of $\int_{(R)}(u+iv)dz=0$. Consequently, $u+iv$ is regular and $f(x, y)$, harmonic, except for removable discontinuities. A precise statement of the conditions for this conclusion is the following:

THEOREM 9. *A necessary and sufficient condition that $f(x, y)$ be equal in a G_x+G_y to a function harmonic in G is that for almost all R in G the following conditions hold:*

(i) *$f(x, y)$ is absolutely continuous in x in an R_x, and in y in an R_y.*

(ii) *$f_x(x, y)$ and $f_y(x, y)$ are summable in R.*

(iii) $\int_{(R)}(df/dn)ds=0$.

As in the case of Theorem 4, the requirement of absolute continuity in x in R_x and in y in R_y can be replaced by conditions (i) or (ii) of Theorem 5.

The further assumption that $f(x, y)$ is continuous in x or in y separately throughout G would imply that f is harmonic everywhere in G.

Finally, in view of the lemma, we state the analogue of Theorem 6:

THEOREM 10. *The function $f(x, y)$ is harmonic in G if and only if for almost all R in G, $f(x, y)$ is absolutely continuous in R and $\int_{(R)}(df/dn)ds=0$.*

Theorems 9 and 10 are similar to results obtained by G. C. Evans* in connection with the theory of potential functions.

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* G. C. Evans, *Fundamental points of potential theory*, The Rice Institute Pamphlet, vol. 7 no. 4, 1920, p. 286.

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