IRREDUCIBLE SYSTEMS OF ALGEBRAIC DIFFERENTIAL EQUATIONS*

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INTRODUCTION

Let $\mathcal{F}$ be a domain of rationality, and let $y_1, \cdots, y_n$ be a set of indeterminates. Then the set of prime ideals in the ring of polynomials $\mathcal{F}[y_1, \cdots, y_n]$ satisfies a divisor-chain condition for decreasing sequences as well as for increasing sequences. That is, a sequence of prime ideals $\Sigma_1, \Sigma_2, \cdots$ in $\mathcal{F}[y_1, \cdots, y_n]$ must be of finite length not only if Σ_{i+1} properly includes Σ_i for every i, but also if Σ_{i+1} is properly included in Σ_i for every i.†

However, if the domain of rationality $\mathcal{F}$ is a set of functions meromorphic in an open region $\mathfrak{A}$, and if $\mathcal{F}$ is closed to differentiation (in other words, if $\mathcal{F}$ is a *field*, in the terminology of algebraic differential equations‡), and if $\mathcal{F}\{y_1, \cdots, y_n\}$ is the differential-ring consisting of all forms in $y_1, \cdots, y_n$ with coefficients in $\mathcal{F}$, then the set of prime differential-ideals in $\mathcal{F}\{y_1, \cdots, y_n\}$ satisfies a divisor-chain condition for increasing sequences,§ but does not satisfy such a condition for decreasing sequences. That is, we can have an infinite sequence $\Sigma_1, \Sigma_2, \cdots$ of prime differential-ideals such that Σ_{i+1} is properly included in Σ_i. In the set-theoretic sense the sequence $\Sigma_1, \Sigma_2, \cdots$ converges to a limiting set Σ which is the intersection of the Σ_i. If $\mathcal{M}_i$ is the manifold of Σ_i, then the sequence $\mathcal{M}_1, \mathcal{M}_2, \cdots$ is a monotonically increasing sequence converging in the set-theoretic sense to a set $\mathcal{N}$ which is the union of the $\mathcal{M}_i$. However, while the limiting set Σ is a prime differential-ideal, the limiting set $\mathcal{N}$ not only is not the manifold of Σ, but is not a manifold at all. We are concerned in this paper with the relation between $\mathcal{N}$ and the manifold $\mathcal{M}$ of Σ.

In the terminology of ADE, what we are considering is an infinite sequence $\Sigma_1, \Sigma_2, \cdots$ of closed irreducible systems in $y_1, \cdots, y_n$ such that $\mathcal{M}_i$, the manifold of Σ_i, is a proper part of the manifold of Σ_{i+1}, $(i=1, 2, \cdots)$.

* Presented to the Society, February 26, 1938, under the title *Sequences of systems of algebraic differential equations*; received by the editors April 12, 1938.

† Cf. Van der Waerden, *Moderne Algebra*, vol. 2, pp. 25, 63. The set of *all* ideals in $\mathcal{F}[y_1, \cdots, y_n]$ satisfies a divisor-chain condition for increasing sequences, but not for decreasing sequences.

‡ See, for example, J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, American Mathematical Society Colloquium Publications, vol. 14, New York, 1932. We shall refer to this book by the letters ADE, and we shall use the terminology of ADE without further reference.

§ As a consequence of Ritt's theorem on the completeness of infinite systems, ADE, §7. The set of *all* differential-ideals in $\mathcal{F}\{y_1, \cdots, y_n\}$ does *not* satisfy a divisor-chain condition for increasing sequences (by ADE, §11).

The set $\mathfrak{N}$ is the union $\mathfrak{M}_1+\mathfrak{M}_2+\cdots$, and $\mathfrak{M}$ is the manifold of the system Σ consisting of all forms F such that F is in every Σ_i.* Not only does $\mathfrak{M}$ contain solutions which do not appear in $\mathfrak{N}$, but there is even a sense in which we may say that $\mathfrak{M}$ is of *higher dimensionality* than $\mathfrak{N}$. This is expressed in the statement that Σ has more arbitrary unknowns than Σ_i, $(i=1, 2, \cdots)$ (Theorem 3). On the other hand, we shall see that $\mathfrak{M}$ may be described as the set of all ordered sets of n analytic functions which can be approximated in a certain manner by solutions in $\mathfrak{N}$ (Theorem 4).

Approximability, as we shall define it, will not imply the familiar uniform approximability in a region. Indeed, for certain sequences $\Sigma_1, \Sigma_2, \cdots$, every solution of Σ which is not in $\mathfrak{N}$ possesses no region of analyticity in which it may be uniformly approximated by solutions of the Σ_i. On the other hand, there exist sequences $\Sigma_1, \Sigma_2, \cdots$ such that every solution of Σ has a region of analyticity in which it can be approximated uniformly by solutions of the Σ_i.

As a converse to Theorem 3, we have the theorem that for every closed irreducible system Σ with a non-empty set of arbitrary unknowns there is a sequence $\Sigma_1, \Sigma_2, \cdots$ of closed irreducible systems such that Σ_{i+1} holds Σ_i, Σ is the set of forms common to the Σ_i, and Σ_i has fewer arbitrary unknowns than Σ, $(i=1, 2, \cdots)$. In fact, Σ_i may be taken to have no arbitrary unknowns.

In studying the sequences $\Sigma_1, \Sigma_2, \cdots$ we use several preliminary theorems which are demonstrated in Part I of this paper. These theorems are extensions of results obtained by Ritt. Theorem 1 deals with the possibility of approximating a solution of a prime algebraic system by solutions which do not annul a specified simple form. Theorem 2 has to do with an analogous question for differential equations.

Lemmas 1 and 2 of Part II are devoted to the study of the degree of freedom which one enjoys in assigning initial conditions to a solution of a prime algebraic system.

PART I. APPROXIMATION THEOREMS

1. The following theorem is due to Ritt: *Let Σ be an indecomposable system of simple forms in $y_1, \cdots, y_n$. Let B be any simple form which does not hold Σ. Given any solution of Σ, analytic in an open region $\mathfrak{A}_1$, there is an open region $\mathfrak{A}'$, contained in $\mathfrak{A}_1$, in which the given solution can be approximated uniformly, with arbitrary closeness, by solutions of Σ for which B is distinct from* 0 *throughout $\mathfrak{A}'$.*†

* We shall see that Σ is closed and irreducible (Theorem 3, below).

† ADE, §64.

We shall use the following modification of Ritt's result:

THEOREM 1. *Let Σ be an indecomposable system of simple forms in $y_1, \cdots, y_n$. Let B be any simple form which does not hold Σ. Given any solution of Σ, analytic in an open region $\mathfrak{A}_1$, there is an open region $\mathfrak{M}$ in $\mathfrak{A}_1$, such that $\mathfrak{A}_1-\mathfrak{M}$ is isolated in $\mathfrak{A}_1$,* and such that for every bounded simply-connected open region $\mathfrak{C}$ which lies with its boundary in $\mathfrak{M}$, there exists a sequence of solutions of Σ, analytic in $\mathfrak{C}$, for each of which B is distinct from zero throughout $\mathfrak{C}$, the sequence converging to the given solution in $\mathfrak{C}$, uniformly in every closed subset of $\mathfrak{C}$.*

Following the procedure in ADE, §64, we introduce the system Σ_1 in $z_1, \cdots, z_n$ with which are associated the simple forms R, G, D, E_{ij}, $(i=q+1, \cdots, n; j=0, 1, \cdots, g-1)$, as in §§59–61. We denote by C_i some simple form of Σ_1 which is of degree m in the z_j, $(j=1, 2, \cdots, q, q+i)$, and of degree m in z_{q+i}, the coefficient of $(z_{q+i})^m$ being unity $(i=1, \cdots, p)$. With Σ_1 and B are associated the simple forms C, N, X, and Y of §64. We let $H=XR+YN$, and we may and do assume that X and Y are so chosen that H is divisible by DG. By $\xi_1, \cdots, \xi_n$ we understand that solution of Σ_1 which corresponds, under the transformation of §57, to the given solution of Σ. Proceeding from this definition of H and the ξ_i as in §63, we introduce constants $b_1, \cdots, b_q$ such that H, under the substitution $z_i=\xi_i+b_i$, $(i=1, \cdots, q)$, becomes a function of x not identically zero. Then H, under the substitution $z_i=\xi_i+b_ih$, $(i=1, \cdots, q)$, becomes a polynomial

$$\alpha_r h^r + \alpha_{r+1}h^{r+1} + \cdots + \alpha_s h^s,$$

where $r\geqq 1$, the α_j are functions of x, meromorphic in $\mathfrak{A}_1$, and $\alpha_r(x)\not\equiv 0$.

Let Π be the set of simple forms C, N, X, Y, R, G, D, E_{ij}, $C_1, \cdots, C_p$. Let $\mathfrak{M}$ be the set of points of $\mathfrak{A}_1$ at which the coefficients in Π are analytic and at which the function α_r is different from zero. Evidently $\mathfrak{A}_1-\mathfrak{M}$ is isolated in $\mathfrak{A}_1$. The functions α_j are analytic in $\mathfrak{M}$.

Let $\mathfrak{C}$ be a bounded simply-connected open region which lies with its boundary in $\mathfrak{M}$. Since $\mathfrak{C}$ is at a positive distance from the boundary of $\mathfrak{M}$, the function α_r is bounded away from zero in $\mathfrak{C}$, and the functions $\alpha_{r+1}, \cdots, \alpha_s$ are bounded in $\mathfrak{C}$. This implies that for every sufficiently small nonzero constant h the polynomial $\alpha_r h^r+ \cdots +\alpha_s h^s$ vanishes nowhere in $\mathfrak{C}$. Therefore in the considerations of ADE, §63, we may take $\mathfrak{A}_2=\mathfrak{C}$. Moreover, we may take $\mathfrak{A}_3=\mathfrak{C}$, since the functions $\xi_1, \cdots, \xi_q$ and the coefficients in $C_1, \cdots, C_p$ are bounded in $\mathfrak{C}$.

* We shall say that a subset $\mathfrak{S}$ of an open region $\mathfrak{R}$ is *isolated in* $\mathfrak{R}$ if $\mathfrak{S}$ is empty, or if $\mathfrak{S}$ is a nonempty set which has no limit points in $\mathfrak{R}$.

Let $\mathfrak{D}$ be a region which lies with its boundary in $\mathfrak{C}$. Taking $\mathfrak{A}'=\mathfrak{D}$, and following the procedure of ADE, §63, we determine a sequence of solutions

$$\zeta_{1,i}, \cdots, \zeta_{n,i}, \qquad i = 1, 2, \cdots,$$

of Σ_1 for each of which H is distinct from zero throughout $\mathfrak{C}$, the sequence converging to $\xi_1, \cdots, \xi_n$ uniformly in $\mathfrak{D}$. Moreover, the sequence is so constructed that there is a positive number d' for which the inequalities

$$|\zeta_{ji}| < d', \quad j = 1, \cdots, n; i = 1, 2, \cdots,$$

are valid in $\mathfrak{C}$. Hence by Vitali's theorem* the sequence $\zeta_{1,i}, \cdots, \zeta_{n,i}$ converges to $\xi_1, \cdots, \xi_n$ in $\mathfrak{C}$, uniformly in every closed subset of $\mathfrak{C}$. Corresponding to this sequence is a sequence of solutions of Σ for each of which B is distinct from zero throughout $\mathfrak{C}$, the sequence converging to the given solution in $\mathfrak{C}$, uniformly in every closed subset of $\mathfrak{C}$.

2. We use Theorem 1 to prove the following lemma:

LEMMA. *Let Σ be a non-trivial closed irreducible system in $y_1, \cdots, y_n$, and let B be a form which does not hold Σ. Given any positive integer m, and any solution $\bar{y}_1(x), \cdots, \bar{y}_n(x)$ of Σ, analytic in an open region $\mathfrak{B}$, let $\mathfrak{B}_m$ be the set of all points x_0 in $\mathfrak{B}$ such that for every $\epsilon>0$ there is a solution $y_1(x), \cdots, y_n(x)$ of Σ, analytic at x_0, for which $\mathfrak{B}$ is different from zero at x_0, and*

$$| y_{ij}(x_0) - \bar{y}_{ij}(x_0) | < \epsilon, \qquad i = 1, \cdots, n; j = 0, 1, \cdots, m.†$$

Then $\mathfrak{B}-\mathfrak{B}_m$ is isolated in $\mathfrak{B}$.

Let $u_1, \cdots, u_q$ be a set of arbitrary unknowns for Σ, let $y_1, \cdots, y_p$ be the remaining unknowns in Σ,‡ and let

$$(1) \qquad A_1, \cdots, A_p$$

be a basic set for Σ with the unknowns ordered $u_1, \cdots, u_q; y_1, \cdots, y_p$.

Let

$$(2) \qquad \bar{u}_1, \cdots, \bar{u}_q; \bar{y}_1, \cdots, \bar{y}_p$$

be a solution of Σ, analytic in an open region $\mathfrak{B}$.

Following the procedure in ADE, §73, without change, we determine the prime algebraic system Ω. Corresponding to (2) is a solution $\bar{u}_{ik}$, $\bar{y}_{jk}$ of Ω, analytic in $\mathfrak{B}$. In accordance with Theorem 1 of this paper there is an open region $\mathfrak{M}$ in $\mathfrak{B}$, whose complement in $\mathfrak{B}$ is isolated in $\mathfrak{B}$, such that, in every open region which with its boundary is included in a bounded simply-con-

* Montel, *Les Familles Normales de Fonctions Analytiques*, p. 30.

† The second subscript is an index of differentiation.

‡ We renumber the unknowns if necessary.

nected subregion $\mathfrak{C}$ of $\mathfrak{M}$, the solution $\bar{u}_{ik}$, $\bar{y}_{jk}$ of Ω can be approximated uniformly by solutions of Ω for which $BS_1 \cdots S_p$ is distinct from zero throughout $\mathfrak{C}$. In particular, if x_0 is a point of $\mathfrak{M}$, then for every $\epsilon>0$ there is a solution u_{ik}, y_{jk} of Ω, analytic at x_0, for which $BS_1 \cdots S_p$ is different from zero at x_0 and

$$(3) \qquad \left| u_{ik}(x_0) - \bar{u}_{ik}(x_0) \right| < \epsilon, \quad \left| y_{jk}(x_0) - \bar{y}_{jk}(x_0) \right| < \epsilon, \qquad i = 1, \cdots, q;\ j = 1, \cdots, p;\ k = 0, 1, \cdots, m.$$

Now the $u_{ik}(x_0)$, $y_{jk}(x_0)$ in (3) furnish initial conditions for a normal solution of the set of differential forms (1). Hence for every point x_0 in $\mathfrak{B}$ and every $\epsilon>0$ there is a solution u_i, y_j of Σ, analytic at x_0, which satisfies (3), and which gives $BS_1 \cdots S_p$ a nonzero value at x_0. Thus $\mathfrak{M}$ is included in $\mathfrak{B}_m$, and therefore $\mathfrak{B}-\mathfrak{B}_m$ is isolated in $\mathfrak{B}$.

3. We use this lemma to prove the following theorem:

THEOREM 2. *Let Σ be a non-trivial closed irreducible system in $y_1, \cdots, y_n$, and let B be a form which does not hold Σ. Then the open region* $\mathfrak{A}$ contains a subset $\mathfrak{P}$ whose complement in $\mathfrak{A}$ is at most denumerably infinite, such that for every point x_0 in $\mathfrak{P}$, every solution $\bar{y}_1, \cdots, \bar{y}_n$ of Σ, analytic at x_0, every positive integer m, and every positive number ϵ there is a solution $y_1, \cdots, y_n$ of Σ, analytic at x_0, for which B is different from zero at x_0 and*

$$(4) \qquad \left| y_{ij}(x_0) - \bar{y}_{ij}(x_0) \right| < \epsilon, \qquad i = 1, 2, \cdots, n;\ j = 0, 1, \cdots, m.$$

We shall use the following notation: If $\bar{y}_1, \cdots, \bar{y}_n$ is a solution of Σ, analytic in an open region $\mathfrak{B}$, then by $\mathfrak{P}(\bar{y}_1, \cdots, \bar{y}_n; \mathfrak{B})$ we shall mean the set of points x_0 in $\mathfrak{B}$ such that for every positive integer m and every $\epsilon>0$ there is a solution $y_1, \cdots, y_n$ of Σ, analytic at x_0, for which B is different from zero at x_0 and (4) holds. Now for every choice of $\bar{y}_1, \cdots, \bar{y}_n$ and $\mathfrak{B}$ the set $\mathfrak{P}(\bar{y}_1, \cdots, \bar{y}_n; \mathfrak{B})$ has a complement in $\mathfrak{B}$ which is at most denumerably infinite. For let $\bar{y}_1, \cdots, \bar{y}_n$ be a solution of Σ, analytic in $\mathfrak{B}$. Then for every positive integer m let $\mathfrak{B}_m$ be the set of points x_0 in $\mathfrak{B}$ such that for every $\epsilon>0$ there is a solution $y_1, \cdots, y_n$ of Σ, analytic at x_0, for which $\mathfrak{B}$ is different from zero at x_0 and (4) holds. It is easy to see that $\mathfrak{P}(\bar{y}_1, \cdots, \bar{y}_n; \mathfrak{B})$ is identical with the intersection of the $\mathfrak{B}_i$, $(i=1, 2, \cdots)$. By the lemma just proved, $\mathfrak{B}-\mathfrak{B}_i$, $(i=1, 2, \cdots)$, is at most denumerably infinite. Consequently $\mathfrak{B}-\mathfrak{P}(\bar{y}_1, \cdots, \bar{y}_n; \mathfrak{B})$ is at most denumerably infinite.†

* We recall that $\mathfrak{A}$ is the open region in which are defined the functions belonging to $\mathcal{J}$, the underlying field of coefficients.

† This statement is an extension of a theorem of Ritt, according to which, for every solution $\bar{y}_1, \cdots, \bar{y}_n$ of Σ, analytic in $\mathfrak{B}$, the set $\mathfrak{P}(\bar{y}_1, \cdots, \bar{y}_n; \mathfrak{B})$ is *dense* in $\mathfrak{B}$. (ADE, §74, and Ritt, *On the singular solutions of algebraic differential equations*, Annals of Mathematics, vol. 37 (1936), note 18.)

Let $\mathfrak{K}_1, \mathfrak{K}_2, \cdots$ be the set of those circles contained in $\mathfrak{A}$ whose centers have rational coordinates and whose radii are rational. Let $\mathfrak{S}_i$, $(i=1, 2, \cdots)$, be the set of all solutions of Σ which are analytic in the closed envelope of $\mathfrak{K}_i$. We consider $\mathfrak{S}_i$ to be a metric space, the distance between two solutions $y_1, \cdots, y_n$ and $z_1, \cdots, z_n$ in $\mathfrak{S}_i$ being given by

$$
\begin{aligned}
(5)\qquad &\delta_i((y_1, \cdots, y_n), (z_1, \cdots, z_n)) \\
&\qquad = \max (| y_1 - z_1| + | y_2 - z_2| + \cdots + | y_n - z_n|),
\end{aligned}
$$

where the maximum is take over the closed envelope of $\mathfrak{K}_i$. Then $\mathfrak{S}_i$ is a separable space.* For $\mathfrak{S}_i$ is a subset of the separable space $\mathcal{A}_i$ consisting of all ordered sets of n functions $y_1, \cdots, y_n$ analytic in the closed envelope of $\mathfrak{K}_i$, the distance between two elements $y_1, \cdots, y_n$ and $z_1, \cdots, z_n$ of $\mathcal{A}_i$ being given by (5); $\mathcal{A}_i$ is separable because the subset of $\mathcal{A}_i$ consisting of all ordered sets of n polynomials with rational complex numbers for coefficients is dense in $\mathcal{A}_i$ and denumerable.

Let $\mathfrak{T}_i$ be a denumerable dense subset of $\mathfrak{S}_i$, $(i=1, 2, \cdots)$. Let $\mathfrak{Q}$ be the set-theoretic sum

$$\sum (\mathfrak{K}_i - \mathfrak{P}(\bar{y}_1, \cdots, \bar{y}_n; \mathfrak{K}_i)),$$

where i ranges over the positive integers, and for each i the solution $\bar{y}_1, \cdots, \bar{y}_n$ ranges over $\mathfrak{T}_i$. Then $\mathfrak{Q}$ is at most denumerably infinite. We define $\mathfrak{P}$ as the complement of $\mathfrak{Q}$ in $\mathfrak{A}$. Let x_0 be a point of $\mathfrak{P}$, $\bar{y}_1, \cdots, \bar{y}_n$ a solution of Σ, analytic at x_0, m a positive integer, and ϵ a positive number. Then there is a $\mathfrak{K}_i$ containing x_0 such that $\bar{y}_1, \cdots, \bar{y}_n$ is analytic in the closed envelope of $\mathfrak{K}_i$. There exists a solution $\bar{y}_1, \cdots, \bar{y}_n$ belonging to $\mathfrak{T}_i$ such that

$$(6)\qquad | \bar{y}_{ij}(x_0) - \bar{y}_{ij}(x_0) | < \epsilon/2, \qquad i = 1, \cdots, n; j = 0, 1, \cdots, m,$$

since there is a sequence of solutions in $\mathfrak{T}_i$ convergent to $\bar{y}_1, \cdots, \bar{y}_n$ uniformly in $\mathfrak{K}_i$. Since x_0 is in $\mathfrak{P}(\bar{y}_1, \cdots, \bar{y}_n; \mathfrak{K}_i)$, there is a solution $y_1, \cdots, y_n$ of Σ, analytic at x_0, for which $\mathfrak{B}$ is different from zero at x_0 and

$$(7)\qquad | y_{ij}(x_0) - \bar{y}_{ij}(x_0) | < \epsilon/2, \qquad i = 1, \cdots, n; j = 0, 1, \cdots, m.$$

By (6) and (7) we have (4). Since $\mathfrak{A} - \mathfrak{P}$ is at most denumerably infinite, we have our theorem.

Part II. Sequences of irreducible systems

4. We state first the following lemma:

Lemma 1. *Let Σ be a prime system in the unknowns $u_1, \cdots, u_q; y_1, \cdots, y_p$, where $u_1, \cdots, u_q$ is a set of unconditioned unknowns for Σ, and let*

* Of course $\mathfrak{S}_i$ may be an empty or finite set.

$$(8) \qquad A_1, \cdots, A_p$$

be a basic set for Σ, *with* A_i *introducing* y_i, $(i=1, \cdots, p)$. *Let* F *be a simple form which does not hold* Σ. *Then there exists a nonzero simple form* G *in* $u_1, \cdots, u_q$ *which is a linear combination of the simple forms* $A_1, \cdots, A_p, F$.

Let J be the product of the initials in (8). Let K be a nonzero simple form in $u_1, \cdots, u_q$ such that every solution of (8) which annuls J is a solution of K.* Let Φ be the system of simple forms $A_1, \cdots, A_p, F$, and let Φ be equivalent to the prime systems $\Pi_1, \cdots, \Pi_r; \Lambda_1, \cdots, \Lambda_s$ where every Π_i is held by K and no Λ_i is held by K. It is easy to see that each Λ_i is held by $\Sigma+F$ and is therefore of lower dimensionality than Σ; that is, has a nonzero simple form G_i in $u_1, \cdots, u_q$.

Set $H=G_1 \cdots G_s K$. Then H is a nonzero simple form in $u_1, \cdots, u_q$ which holds Φ. Consequently, there is a positive integer σ such that H^σ is a linear combination of the simple forms of Φ. We evidently may take H^σ for the simple form G whose existence is to be demonstrated.

5. We can now prove the following lemma:

LEMMA 2. *Let* Λ *be a prime system in the unknowns* $v_1, \cdots, v_t; z_1, \cdots, z_s$, *where* $v_1, \cdots, v_t$ *is a set of unconditioned unknowns for* Λ, *and let* $C_1, \cdots, C_s$ *be a basic set for* Λ, *with* C_i *introducing* z_i, $(i=1, \cdots, s)$. *Let* T_i *be the separant of* C_i, *and let* F *be any form which does not hold* Λ. *Then there is a set* $\mathfrak{N}$ *in* $\mathfrak{A}$ *with the following properties*:

(i) $\mathfrak{A}-\mathfrak{N}$ *is isolated in* $\mathfrak{A}$.

(ii) *If* σ *is any integer with* $1 \leqq \sigma \leqq s$, *if* x_0 *is any point of* $\mathfrak{N}$, *and if* $a_1, \cdots, a_t; b_1, \cdots, b_\sigma$ *is a set of complex numbers such that* $C_i=0$ *and* $T_i \neq 0$, $(i=1, \cdots, \sigma)$, *when* $x=x_0$, $v_j=a_j$, and $z_k=b_k$, $(j=1, \cdots, t; k=1, \cdots, \sigma)$, *then for every* $\delta>0$ *there is a solution* $v_1(x), \cdots, v_t(x); z_1(x), \cdots, z_s(x)$ *of* Λ, *analytic at* x_0, *satisfying the inequalities*

$$|v_j(x_0)-a_j|<\delta, \quad |z_k(x_0)-b_k|<\delta, \qquad j=1, \cdots, t; k=1, \cdots, \sigma,$$

and giving F *a nonzero value at* x_0.†

Let I_i be the initial of C_i, $(i=1, \cdots, s)$, and let k be any integer with $1 \leqq k \leqq s-1$. Now $C_1, \cdots, C_k$ is a basic set for a prime system Λ_k which has $v_1, \cdots, v_t$ for a set of unconditioned unknowns.‡ I_{k+1} does not hold Λ_k, since it does not hold Λ. By Lemma 1 there is an identity

$$(9) \qquad G_k = E_{k,1}C_1 + E_{k,2}C_2 + \cdots + E_{k,k}C_k + E_{k,k+1}I_{k+1},$$

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* Ritt, *Systems of algebraic differential equations*, Annals of Mathematics, (2), vol. 36 (1935), §8.

† We shall apply this lemma only for the case $\sigma=1$.

‡ Cf. ADE, §45.

where G_k is a nonzero simple form in $v_1, \cdots, v_t$. Likewise, since $T_1 \cdots T_sF$ does not hold Λ, there is an identity

$$(10) \qquad G = F_1C_1 + F_2C_2 + \cdots + F_sC_s + F_{s+1}T_1 \cdots T_sF,$$

where G is a nonzero simple form in $v_1, \cdots, v_t$. Set $L = I_1GG_1 \cdots G_{s-1}$. Then L is a nonzero simple form in $v_1, \cdots, v_t$. We now present the set $\mathfrak{N}$. Let $\mathfrak{N}$ consist of all points in $\mathfrak{A}$ at which the coefficients of the simple forms appearing in the identities (10) and (9), $(k=1, \cdots, s-1)$, are analytic, and at which L has one or several nonzero coefficients. Evidently $\mathfrak{A}-\mathfrak{N}$ is isolated in $\mathfrak{A}$. Now let σ be any positive integer with $1 \leqq \sigma \leqq s$, let x_0 be any point of $\mathfrak{N}$, assume $\delta > 0$, and let $a_1, \cdots, a_t; b_1, \cdots, b_\sigma$ be complex numbers such that $C_i = 0$ and $T_i \neq 0$, $(i=1, \cdots, \sigma)$, when $x = x_0$, $v_j = a_j$, $z_k = b_k$, $(j=1, \cdots, t; k=1, \cdots, \sigma)$. Since $T_1 \cdots T_\sigma$ is equal to the Jacobian $\partial(C_1, \cdots, C_\sigma)/\partial(z_1, \cdots, z_\sigma)$, there is a unique set of functions $f_1(x, v_1, \cdots, v_t), \cdots, f_\sigma(x, v_1, \cdots, v_t)$, analytic near $(x_0, a_1, \cdots, a_t)$ such that $b_i = f_i(x_0, a_1, \cdots, a_t)$, $(i=1, \cdots, \sigma)$, and such that the substitution of $f_i(x, v_1, \cdots, v_t)$ for z_i in $C_1, \cdots, C_\sigma$ yields σ functions of $x, v_1, \cdots, v_t$ each of which is identically zero.

Let $\mathfrak{R}$ be a neighborhood of $(x_0, a_1, \cdots, a_t)$ in which every $f_i(x, v_1, \cdots, v_t)$ is analytic $(i=1, \cdots, \sigma)$ such that for every point $(x_1, c_1, \cdots, c_t)$ in $\mathfrak{R}$ the relations

$$|c_j - a_j| < \delta, \quad |f_i(x_0, c_1, \cdots, c_t) - b_i| < \delta, \qquad i = 1, \cdots, \sigma,$$

are valid. Let $c_1, \cdots, c_t$ be chosen so that $(x_0, c_1, \cdots, c_t)$ is a point of $\mathfrak{R}$ at which L is not zero. Such a point exists because L has one or several coefficients different from zero at x_0. Let $d_i = f_i(x_0, c_1, \cdots, c_t)$, $(i=1, \cdots, \sigma)$. Then the substitution

$$(11) \qquad x = x_0,\ v_j = c_j,\ z_i = d_i, \qquad i = 1, \cdots, \sigma; j = 1, \cdots, t,$$

annuls $C_1, \cdots, C_\sigma$ but not G_σ, and therefore does not annul $I_{\sigma+1}$ (by (9), with $k=\sigma$). Therefore the polynomial in $z_{\sigma+1}$ obtained from $C_{\sigma+1}$ by the substitution (11) has at least one root $d_{\sigma+1}$. The substitution

$$(12) \qquad x = x_0,\ v_j = c_j,\ z_i = d_i, \qquad i = 1, \cdots, \sigma+1; j = 1, \cdots, t,$$

annuls $C_1, \cdots, C_{\sigma+1}$ but not $G_{\sigma+1}$, and therefore does not annul $I_{\sigma+2}$ (by (9), with $k=\sigma+1$). Hence the polynomial in $z_{\sigma+2}$ obtained from $C_{\sigma+2}$ by the substitution (12) has at least one root $d_{\sigma+2}$.

Continuing in this manner, we obtain a set of values $c_1, \cdots, c_t; d_1, \cdots, d_s$ such that the substitution

$$(13) \qquad x = x_0,\ v_j = c_j,\ z_i = d_i, \qquad i = 1, \cdots, s; j = 1, \cdots, t,$$

annuls $C_1, \cdots, C_s$ but not G, and therefore does not annul $T_1 \cdots T_sF$ (by

(10)). Since $T_1 \cdots T_s = \partial(C_1, \cdots, C_s)/\partial(z_1, \cdots, z_s)$, there is a unique set of functions $\zeta_1(x, v_1, \cdots, v_t), \cdots, \zeta_s(x, v_1, \cdots, v_t)$, analytic near $(x_0, c_1, \cdots, c_t)$, such that $d_i = \zeta_i(x_0, c_1, \cdots, c_t)$, $(i=1, \cdots, s)$, and such that the substitution $z_i = \zeta_i(x, v_1, \cdots, v_t)$, $(i=1, \cdots, s)$, transforms $C_1, \cdots, C_s$ into s functions of $x, v_1, \cdots, v_t$, each of which is identically zero.

Let $v_j(x) = c_j$, $z_i(x) = \zeta_i(x, c_1, \cdots, c_t)$, $(j=1, \cdots, t;\ i=1, \cdots, s)$. Then $v_j(x), z_i(x)$ is evidently a solution of Λ, analytic at x_0, for which F is different from zero at x_0, and

$$|v_j(x_0) - a_j| < \delta, \quad |z_k(x_0) - b_k| < \delta, \qquad j = 1, \cdots, t;\ k = 1, \cdots, \sigma.$$

6. We now prove the following theorem:

THEOREM 3. *Let*

$$\Sigma_1, \Sigma_2, \cdots \tag{14}$$

be a sequence of closed irreducible systems in the unknowns $y_1, \cdots, y_n$ *such that the manifold of* Σ_i *is a proper part of the manifold of* Σ_{i+1}, $(i=1, 2, \cdots)$. *Let* Σ *be the set of all forms* F *such that* F *is in every system* Σ_i, $(i=1, 2, \cdots)$. *Then* Σ *is a closed irreducible system having more arbitrary unknowns than* Σ_i, $(i=1, 2, \cdots)$.*

Σ is obviously closed. Σ is irreducible because if GH holds Σ, then either G holds an infinite set of the Σ_i, hence all the Σ_i, or H does; so either G is in Σ or H is.

Now let Σ_i be any system in (14). Evidently, if there is a set of unknowns in which Σ_i has no nonzero form, then Σ has no nonzero form in the unknowns of that set. Hence Σ has at least as many arbitrary unknowns as Σ_i. Now suppose that there is an m such that Σ_m has the same number of arbitrary unknowns as Σ. Then there is a set of unknowns $y_{i_1}, y_{i_2}, \cdots, y_{i_q}$ which is a set of arbitrary unknowns for Σ, and which is also a set of arbitrary unknowns for Σ_m. Now Σ_j, $(j \geqq m)$, has no nonzero form in $y_{i_1}, \cdots, y_{i_q}$ because Σ_m has no such form. But Σ_j has not more than q arbitrary unknowns, because Σ has q arbitrary unknowns. Hence $y_{i_1}, \cdots, y_{i_q}$ is a set of arbitrary unknowns for every Σ_j, $(j \geqq m)$. Taking $y_{i_1}, \cdots, y_{i_q}$ as a set of arbitrary unknowns for Σ and for each Σ_j, $(j \geqq m)$, we introduce a resolvent for Σ and for each Σ_j (adjoining x to $\mathcal{J}$ if necessary) and we let ρ, ρ_j be the orders of the resolvents of Σ, Σ_j, respectively $(j \geqq m)$. By a theorem of E. Gourin,† since Σ has the same

* For a closed irreducible system Λ, the number of unknowns in a set of arbitrary unknowns for Λ is independent of the manner in which the set is chosen (ADE, §30). This number we call the *number of arbitrary unknowns in* Λ.

† E. Gourin, *On irreducible systems of algebraic differential equations*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 593–595.

set of arbitrary unknowns as each of the Σ_j, we have $\rho_m < \rho_{m+1} < \cdots < \rho$, which is clearly impossible.

COROLLARY. *Σ has a non-empty set of arbitrary unknowns.*

7. We now make the following definition:

DEFINITION. *Let n be any positive integer. Let $f_1(x), \cdots, f_n(x)$ be an ordered set of n functions analytic in an open region $\mathfrak{B}$. Let $\mathfrak{N}$ be a set each of whose elements is an ordered set of n functions which have a region of analyticity in common. Then if m is a positive integer, we shall say that a point x_0 of $\mathfrak{B}$ is a point of mth order contact between the set $f_1(x), \cdots, f_n(x)$ and the set of sets $\mathfrak{N}$ if for every $\epsilon > 0$ there is a set $y_1(x), \cdots, y_n(x)$ in $\mathfrak{N}$, analytic at x_0, such that*

$$| y_{ij}(x_0) - f_{ij}(x_0) | < \epsilon,^* \qquad i = 1, \cdots, n; j = 0, 1, \cdots, m.$$

If x_0 is a point of mth order contact between $f_1(x), \cdots, f_n(x)$ and $\mathfrak{N}$ for every m, then we shall say simply that x_0 is a *point of contact between $f_1(x), \cdots, f_n(x)$ and $\mathfrak{N}$.*

8. We can now state the following theorem:

THEOREM 4. *Let $\Sigma_1, \Sigma_2, \cdots$, and Σ be as in the hypothesis of Theorem 3. Let $\mathfrak{M}_i$ be the manifold of Σ_i, $(i=1, 2, \cdots)$, and let $\mathfrak{N}$ be the set-theoretic sum $\mathfrak{M}_1 + \mathfrak{M}_2 + \cdots$. If $f_1(x), \cdots, f_n(x)$ is an ordered set of n functions analytic in an open region $\mathfrak{B}$, then a necessary and sufficient condition for $f_1(x), \cdots, f_n(x)$ to be a solution of Σ is that $\mathfrak{B}$ contain a point of contact between $f_1, \cdots, f_n$ and $\mathfrak{N}$.*

Sufficiency proof. Let

$$(15) \qquad f_1(x), \cdots, f_n(x)$$

be an ordered set of n functions analytic in an open region $\mathfrak{B}$ which contains a point x_0 of contact between (15) and $\mathfrak{N}$. We shall prove that (15) is a solution of Σ.

If H is a form in Σ whose coefficients are analytic at x_0, then H, considered as a function of x and the letters appearing in H, is continuous when x is near x_0.

Let m be a positive integer greater than the order of H in y_i, $(i=1, \cdots, n)$. Assume $\epsilon > 0$. Let

$$(16) \qquad y_1(x), \cdots, y_n(x)$$

be a solution in $\mathfrak{N}$, analytic at x_0, such that

$$(17) \qquad \left| f_{ij}(x_0) - y_{ij}(x_0) \right| < \epsilon, \qquad i = 1, \cdots, n; j = 0, 1, \cdots, m.$$

* The second subscript is an index of differentiation.

When we substitute (15) and (16) in H, we obtain functions $h(x)$ and $k(x)$, respectively, which are analytic near x_0. Evidently $k(x)\equiv 0$, since (16) is in $\mathcal{N}$ and is therefore a solution of Σ. In particular $k(x_0)=0$. Now ϵ is arbitrarily small; consequently, the relations (17) and the continuity of H imply that $|k(x_0)-h(x_0)|$ is arbitrarily small. Thus $|h(x_0)|$ is arbitrarily small, so that $h(x_0)=0$. Now $h'(x_0)$ may be obtained by substituting (15) in H' and putting $x=x_0$.* Hence, by the preceding argument, $h'(x_0)=0$. Continuing in this manner we prove that every derivative of $h(x)$ vanishes at x_0. This means $h(x)\equiv 0$. Hence (15) is a solution of H, where H is any form of Σ whose coefficients are analytic at x_0. But if G is any form of Σ, the product of G by a suitable nonzero function $\psi(x)$ in $\mathcal{J}$ is a form H of Σ with coefficients analytic at x_0.†

Since (15) annuls H, it annuls G. This proves that (15) is a solution of Σ.‡

Necessity proof. The necessity of the condition is implied by Theorem 6, below.

9. The next theorem is as follows:

THEOREM 5. *Let the notations Σ_i, $\mathfrak{M}_i$, and Σ have the same significance as in the hypothesis of Theorem 4. Then there exists a function $b=b(m)$, defined on the positive integers, and assuming positive integral values, such that for every solution $y_1(x), \cdots, y_n(x)$ of Σ, analytic in an open region $\mathfrak{B}$, and every positive integer m, the set of points of mth order contact between $y_1(x), \cdots, y_n(x)$ and $\mathfrak{M}_{b(m)}$ is a set whose complement in $\mathfrak{B}$ is isolated in $\mathfrak{B}$.*

Let $u_1, \cdots, u_q$ be a set of arbitrary unknowns for Σ. If Σ is non-trivial, we introduce a resolvent for Σ with a new unknown ω satisfying $\omega-Q=0$.§ If Σ is trivial (that is, if Σ has no nonzero forms) we introduce a new unknown ω satisfying $\omega-Q=0$, where $Q\equiv 0$.

In either case let Ω be the set of forms holding $\Sigma+(\omega-Q)$, and let Ω_i be the set of forms holding $\Sigma_i+(\omega-Q)$, $(i=1, 2, \cdots)$. Then Ω_{i+1} holds Ω_i, Ω is closed and irreducible,‖ and Ω is the set of all forms F such that F is in every system Ω_i, $(i=1, 2, \cdots)$.

Let $y_1, \cdots, y_p$ be the unknowns in Ω other than $u_1, \cdots, u_q; \omega$.¶ Let the unknowns be ordered $u_1, \cdots, u_q; \omega; y_1, \cdots, y_p$, and let

$$(18) \qquad R, A_1, \cdots, A_p$$

* Superscripts indicate differentiation.

† For if G has a coefficient $\phi(x)$ with a pole at x_0, then the reciprocal of $\phi(x)$ will have a zero at x_0, and a suitable power of that reciporal will serve as $\psi(x)$.

‡ This proof is similar to a proof given by Ritt for a different theorem, ADE, §72.

§ ADE, §§25–29.

‖ Ω_i is also closed and irreducible.

¶ We renumber the unknowns if necessary.

be a corresponding basic set for Ω. Then A_i is of zero order in y_i and is linear in y_{i0}, $(i=1, \cdots, p)$.*

Let h be the order of R in ω. We assert that if a is any positive integer, then there is an integer b depending upon a such that the system Ω_b has no nonzero form in the letters

$$(19) \qquad u_{\alpha\beta}, \omega_\gamma, \qquad \alpha = 1, \cdots, q; \beta = 0, 1, \cdots, a; \gamma = 0, 1, \cdots, h-1.$$

For let us assume that this assertion is false. Then there is an a such that every Ω_i has a nonzero form in the letters (19). From each Ω_i let a nonzero form F_i in the letters (19) be selected which is of minimum rank. Without loss of generality we may assume that F_i is algebraically irreducible $(i=1, 2, \cdots)$. Since Ω_2 must have solutions, each F_i, $(i>2)$, involves unknowns.

Since the totality of letters involved in the F_i is a finite set, there is an infinite subset of the F_i such that if F_k and F_l are two forms in the subset, then F_k and F_l have the same order in u_α, $(\alpha=1, \cdots, q)$, and the same order in ω. We assert that the quotient of any two forms in this subset is a (nonzero) function in $\mathcal{J}$. For if F_k and F_l, $(k<l)$, are relatively prime, then the resultant G of F_k and F_l, with respect to the highest letter in F_k and F_l, is a nonzero form free of that letter. Then Ω_k has the form G in the letters (19).† But G is lower than F_k. This contradiction with the minimal property of F_k proves that there is an infinite set of the F_i, each of which is the product of a fixed F_k by a nonzero function in $\mathcal{J}$. Then this F_k is in all the Ω_i, and therefore in Ω, although it is lower than R. This contradiction proves that for every a there is a b such that Ω_b has no nonzero form in the letters (19).

Now let S, S_i be the separant of R, A_i, $(i=1, \cdots, p)$, respectively, and define $K_1 \equiv SS_1 \cdots S_p$. Discarding a finite set of the Ω_i if necessary, we assume that K_1 holds no Ω_i, since K_1 is not in Ω.

Let g be a positive integer greater than the maximum order of each form of (18) in each unknown.

Let m be any positive integer, to be fixed throughout the remainder of this proof. Let $a=m+g$, and let $b=b(m)$ be the smallest positive integer such that Ω_b has no nonzero form in the letters (19).

We take any set of arbitrary unknowns for Ω_b, order the remaining unknowns in any fashion, and let

$$(20) \qquad B_1, B_2, \cdots, B_r$$

be a corresponding basic set for Ω_b.

* Since either R is a resolvent for Σ, or Σ is trivial. In the latter case (18) is simply the form R.
† Since G is a linear combination of F_k and F_l, each of which is in Ω_k.

Let τ, τ_i, σ_j be the orders of the highest derivatives of ω, y_i, u_j, $(i=1, \cdots, p;$ $j=1, \cdots, q)$, respectively, appearing in (20).* Let K_2 be the product of the separants in (20). Let Λ be the set of all simple forms that vanish for all solutions of the system

$$(21) \qquad B_{ij}, \qquad i = 1, \cdots, r; j = 0, 1, \cdots, a,†$$

for which $K_2 \neq 0$, where the forms (21) are to be considered as simple forms in the unknowns

$$(22) \qquad \begin{array}{ll} u_{ij}, \omega_k, y_{\mu\nu}, & j = 0, 1, \cdots, a + \sigma_i; i = 1, \cdots, q; \\ k = 0, 1, \cdots, a + \tau; & \nu = 0, 1, \cdots, a + \tau_\mu; \mu = 1, \cdots, p. \end{array}$$

It is easy to see that Λ is prime, that every simple form which holds Λ, when considered as a form in the unknowns $u_1, \cdots, u_q; \omega; y_1, \cdots, y_p$, and their derivatives, will hold Ω_b, and that every form of Ω_b in the letters (22), when considered as a simple form in those letters, will hold Λ.‡

Since every form in Λ is in Ω_b, there is no nonzero form in Λ in the letters (19). Renaming the letter (22), let

$$(23) \qquad v_1, \cdots, v_t$$

be a set of unconditioned unknowns for Λ, and let

$$(24) \qquad z_1, \cdots, z_s$$

be the other unknowns in Λ. We may and do choose (23), (24) so that (23) includes (19) and also so that $z_1 = \omega_h$, the latter being possible because R is in Λ.

With the unknowns ordered $v_1, \cdots, v_t; z_1, \cdots, z_s$, let

$$(25) \qquad C_1, C_2, \cdots, C_s$$

be a basic set for Λ. Then R can be taken for C_1. For if F were a simple form of Λ in the unknowns $v_1, \cdots, v_t; \omega_h$, of lower degree than R in ω_h, then the resultant of R and F with respect to ω_h would be a nonzero simple form of Λ in the letters (23),§ although (23) is a set of unconditioned unknowns. We shall assume that $C_1 = R$.

* If ω, y_i, or u_j does not appear in (20), then τ, τ_i, or σ_j, respectively, is to be taken as zero.

† The second subscript is an index of differentiation.

‡ If F holds Λ, then K_2F vanishes for every solution of Ω_b, since such a solution either annuls K_2 or yields a solution of (21) for which $K_2 \neq 0$; hence F holds Ω_b, since K_2 does not. Conversely, if F holds Ω_b, then F vanishes for every solution of (21) for which $K_2 \neq 0$, since such a solution provides, at every point where the coefficients in (20) are analytic and $K_2 \neq 0$, initial conditions for a normal solution of (20). If FG holds Λ, then it holds Ω_b; so either F holds Ω_b, hence Λ, or G does. Thus Λ is indecomposable. Λ is obviously simply closed.

§ R is algebraically irreducible as a polynomial in ω_h, in the field $\mathcal{J}(v_1, \cdots, v_t)$. Cf. ADE, §§65, 45.

We note that K_2 does not hold Λ.

Taking K_2 for the form F in the hypothesis of Lemma 2, we let $\mathfrak{N}$ be the corresponding point set in $\mathfrak{A}$ with the properties (i), (ii) of the lemma.

Let $\mathfrak{N}_m$ be the set of points in $\mathfrak{N}$ at which the coefficients of the forms in (18) and (20) are analytic. Evidently $\mathfrak{A}-\mathfrak{N}_m$ is isolated in $\mathfrak{A}$.

Let x_0 be a point of $\mathfrak{N}_m$, and let

$$(26)\qquad \bar{u}_1(x), \cdots, \bar{u}_q(x); \bar{\omega}(x); \bar{y}_1(x), \cdots, \bar{y}_p(x)$$

be a normal solution of (18), analytic at x_0 and giving K_1 a nonzero value at x_0. We shall prove that x_0 is a point of mth order contact between (26) and the manifold of Ω_b. Assume

$$(27)\qquad \begin{gathered} R_k \equiv S\omega_{h+k} + V_k, \qquad A_{ij} \equiv S_i y_{ij} + T_{ij}, \\ k = 1, 2, \cdots, m; i = 1, 2, \cdots, p; j = 0, 1, \cdots, m.\text{*} \end{gathered}$$

Then V_k is of order less than $h+k$ in ω, and T_{ij} is of order less than j in y_i. Evidently the equations

$$(28)\qquad R_k = 0, A_{ij} = 0, \quad k = 1, 2, \cdots, m; i = 1, \cdots, p; j = 0, 1, \cdots, m,$$

define ω_{h+k}, y_{ij} recursively as functions of x, ω_h, and the letters (19), continuous near $(x_0, \bar{u}_{\alpha\beta}(x_0), \bar{\omega}_\gamma(x_0), \bar{\omega}_h(x_0))$, $(\alpha=1, \cdots, q; \beta=0, 1, \cdots, a; \gamma=0, 1, \cdots, h-1)$.†

Assume $\epsilon>0$. Then there is a $\delta>0$ such that if $\bar{\bar{u}}_1(x), \cdots, \bar{\bar{u}}_q(x); \bar{\bar{\omega}}(x); \bar{\bar{y}}_1(x), \cdots, \bar{\bar{y}}_p(x)$ is a solution of Ω_b with

$$(29)\qquad |\bar{\bar{u}}_{\alpha\beta}(x_0) - \bar{u}_{\alpha\beta}(x_0)| < \delta, \ |\bar{\bar{\omega}}_\gamma(x_0) - \bar{\omega}_\gamma(x_0)| < \delta, \ |\bar{\bar{\omega}}_h(x_0) - \bar{\omega}_h(x_0)| < \delta,$$

then

$$(30)\qquad \begin{gathered} |\bar{\bar{u}}_{jk}(x_0) - \bar{u}_{jk}(x_0)| < \epsilon, \ |\bar{\bar{\omega}}_k(x_0) - \bar{\omega}_k(x_0)| < \epsilon, \\ |\bar{\bar{y}}_{ik}(x_0) - \bar{y}_{ik}(x_0)| < \epsilon, \\ j = 1, \cdots, q; i = 1, \cdots, p; k = 0, 1, \cdots, m. \end{gathered}$$

This results from the fact that every solution of Ω, and therefore also every solution of Ω_b, must satisfy the equations (28).

Now each v_j, $(j=1, \cdots, t)$, corresponds to one of the letters (22); let a_j be the value at x_0 assigned to that letter by (26). Let $b_1 = \bar{\omega}_h(x_0)$.

Then C_1, $(C_1=R)$, vanishes under the substitution $x=x_0$, $v_j=a_j$, $\omega_h=b_1$, $(j=1, \cdots, t)$, and S does not; thus, since x_0 is in $\mathfrak{N}$, there is a solution $v_j(x)$,

* R_k is the kth derivative of R, and A_{ij} is the jth derivative of A_i.

† Henceforth, whenever α, β, γ appear as subscripts, we shall understand that their ranges are the ones given here.

$z_i(x)$, $(i=1, \cdots, s; j=1, \cdots, t)$, of Λ, analytic at x_0, for which $|v_j(x_0)-a_j|<\delta$, $|z_1(x_0)-b_1|<\delta$, $(j=1, \cdots, t)$, and for which K_2 is different from zero at x_0.

Evidently this solution of Λ provides initial conditions at x_0 for a normal solution $\bar{u}_1, \cdots, \bar{y}_p$ of (20) which satisfies (29). The inequalities (30) are valid for this solution of Ω_b. Therefore x_0 is a point of mth order contact between (26) and the manifold of Ω_b.

Now let

$$u_1(x), \cdots, u_q(x); y_1(x), \cdots, y_p(x) \tag{31}$$

be a solution of Σ, analytic in an open region $\mathfrak{B}$. Corresponding to (31) is a solution

$$u_1(x), \cdots, u_q(x); \omega(x); y_1(x), \cdots, y_p(x) \tag{32}$$

of Ω, analytic in $\mathfrak{B}$. According to the lemma of §2 there is a set $\mathfrak{B}_m$ whose complement in $\mathfrak{B}$ is isolated in $\mathfrak{B}$, such that every point x_0 in $\mathfrak{B}_m$ is a point of mth order contact between (32) and the set of those solutions of Ω which give K_1 a nonzero value at x_0. Let $\mathfrak{R}_m=\mathfrak{B}_m\cdot\mathfrak{N}_m$. Then for every $\epsilon>0$ and every point x_0 in $\mathfrak{R}_m$ there is a solution

$$\bar{u}_1(x), \cdots, \bar{u}_q(x); \bar{\omega}(x); \bar{y}_1(x), \cdots, \bar{y}_p(x) \tag{33}$$

of Ω, analytic at x_0, for which

$$\begin{gathered} |\bar{u}_{jk}(x_0)-u_{jk}(x_0)|<\epsilon, \quad |\bar{\omega}_k(x_0)-\omega_k(x_0)|<\epsilon, \\ |\bar{y}_{ik}(x_0)-y_{ik}(x_0)|<\epsilon, \\ i=1, \cdots, p; j=1, \cdots, q; k=0, 1, \cdots, m, \end{gathered} \tag{34}$$

and for which $K_1\neq 0$ at x_0; and then there is a solution

$$\tilde{u}_1(x), \cdots, \tilde{u}_q(x); \tilde{\omega}(x); \tilde{y}_1(x), \cdots, \tilde{y}_p(x) \tag{35}$$

of Ω_b for which (30) holds. By (30) and (34) we have in particular

$$\begin{gathered} |\tilde{u}_{jk}(x_0)-u_{jk}(x_0)|<2\epsilon, \quad |\tilde{y}_{ik}(x_0)-y_{ik}(x_0)|<2\epsilon, \\ i=1, \cdots, q; j=1, \cdots, p; k=0, 1, \cdots, m. \end{gathered} \tag{36}$$

Consequently every point of $\mathfrak{R}_m$ is a point of mth order contact between (31) and $\mathcal{M}_b$. Since $\mathfrak{B}-\mathfrak{R}_m$ is evidently isolated in $\mathfrak{B}$, we have our theorem.

10. As a consequence of Theorem 5 we have the following theorem:

THEOREM 6. *Let the notations Σ_i, $\mathcal{M}_i$, Σ, $\mathcal{N}$ have the same significance as in Theorem 4. Then the open region $\mathfrak{A}$ contains a subset $\mathfrak{P}$ whose complement in $\mathfrak{A}$ is at most denumerably infinite, such that if x_0 is a point in $\mathfrak{P}$, and $\bar{y}_1, \cdots, \bar{y}_n$ is a solution of Σ, analytic at x_0, then x_0 is a point of contact between $\bar{y}_1, \cdots, \bar{y}_n$ and $\mathcal{N}$.*

We prove Theorem 6 by using Theorem 5 in the same manner in which the lemma of §2 was used in proving Theorem 2. We simply replace the concept of a solution of Σ for which B is different from zero at x_0, by that of a solution in $\mathcal{N}$.

11. Extending this result in a special case, we assert that when $\mathcal{Y}$ consists purely of constants, then for every solution $y_1(x), \cdots, y_n(x)$ of Σ, analytic in an open region $\mathfrak{B}$, *every* point of $\mathfrak{B}$ is a point of contact between $y_1(x), \cdots, y_n(x)$ and $\mathcal{N}$.

For every point x_1 in $\mathfrak{B}$, every positive integer m, every $\epsilon > 0$, and every $\delta > 0$, there is a solution $\bar{y}_1(x), \cdots, \bar{y}_n(x)$ in $\mathcal{N}$, analytic at a point x_0 in $\mathfrak{B}$, with $|x_1 - x_0| < \delta$, and with

$$| \bar{y}_{ij}(x_0) - y_{ij}(x_0) | < \epsilon, \qquad i = 1, \cdots, n; j = 0, 1, \cdots, m.$$

We assume that δ is sufficiently small so that

$$| y_{ij}(x_1) - y_{ij}(x_0) | < \epsilon, \qquad i = 1, \cdots, n; j = 0, 1, \cdots, m.$$

Then $|\bar{y}_{ij}(x_0) - y_{ij}(x_1)| < 2\epsilon$, and this may be written $|z_{ij}(x_1) - y_{ij}(x_1)| < 2\epsilon$, where

$$z_i(x) \equiv \bar{y}_i(x + x_0 - x_1), \qquad i = 1, \cdots, n; j = 0, 1, \cdots, m.$$

But $z_1(x), \cdots, z_n(x)$ is in $\mathcal{N}$, since $\mathcal{Y}$ consists purely of constants. Therefore x_1 is a point of contact between $y_1(x), \cdots, y_n(x)$ and $\mathcal{N}$.

Part III. Examples

12. We give an example of a sequence $\Sigma_1, \Sigma_2, \cdots$ and a solution $y_1(x), \cdots, y_n(x)$ of Σ, analytic in an open region $\mathfrak{B}$, such that the set of those points in $\mathfrak{B}$ which are not points of contact between $y_1(x), \cdots, y_n(x)$ and $\mathcal{N}$ is dense in $\mathfrak{B}$. From §11 we know that in such an example there must be functions in $\mathcal{Y}$ which are not constants.

Let $\mathcal{Y}$ consist of all rational functions of x. Let

$$a_1, a_2, \cdots \tag{37}$$

be a sequence of points dense in the complex plane. Let $\Sigma_1, \Sigma_2, \cdots$ be the closed systems in one unknown y such that the manifold of Σ_n, $(n = 1, 2, \cdots)$, is the family of functions

$$y = \frac{c_1}{x - a_1} + \frac{c_2}{(x - a_1)^2(x - a_2)} + \cdots + \frac{c_n}{(x - a_1)^n(x - a_2)^{n-1} \cdots (x - a_n)}, \tag{38}$$

where the c_i are arbitrary constants.

Then it is easy to see that the manifold of Σ_n is identical with the manifold of a linear differential equation in y, with coefficients in $\mathcal{J}$. This equation affords a basic set for Σ_n; therefore Σ_n is irreducible.* Obviously the manifold of Σ_n is a proper part of the manifold of Σ_{n+1}.

Now Σ must have a non-empty set of arbitrary unknowns, as we have seen, so that y is a set of arbitrary unknowns for Σ. In other words, there is no nonzero form in Σ; so every analytic function is a solution of Σ. Let $\mathcal{N}$ be the union of the manifolds of the Σ_i; that is, let $\mathcal{N}$ be the union of the families of functions (38), $(n=1, 2, \cdots)$. Let $f(x)$ be a function which is analytic in an open region $\mathfrak{B}$ and which is not in $\mathcal{N}$. Then the set of those points of the sequence (37) that lie in $\mathfrak{B}$ is dense in $\mathfrak{B}$. No point in (37) is a point of contact between $f(x)$ and $\mathcal{N}$, since for every positive integer l the only functions in $\mathcal{N}$ which are analytic at the point a_l are the functions which are in the family (38) when $n=l-1$. Hence the complement in $\mathfrak{B}$ of the set of points of contact between $f(x)$ and $\mathcal{N}$ is dense in $\mathfrak{B}$.

We note that there is no open subregion $\mathfrak{B}_1$ of $\mathfrak{B}$ in which $f(x)$ may be approximated uniformly, with arbitrary closeness, by a solution in $\mathcal{N}$. For if such an open region $\mathfrak{B}_1$ exists, then every point of $\mathfrak{B}_1$ is a point of contact between $f(x)$ and $\mathcal{N}$.

13. The phenomenon exemplified in the preceding section is in marked contrast with that appearing in the following example:

Let Σ_k be the closed irreducible† system in the unknown y with a basic set y_k.‡

Σ is trivial as in the preceding example. Here, however, if $f(x)$ is any function, analytic in an open region $\mathfrak{B}$, then every point of $\mathfrak{B}$ has a neighborhood in which $f(x)$ may be uniformly approximated by solutions of the Σ_i. In short, every polynomial is a solution of some system Σ_i.

14. In the example of §12, for certain solutions of Σ there existed no region in which uniform approximation by solutions of the Σ_i was possible. Each solution of Σ having this property had the additional property that every subregion of its domain of analyticity contained a point which was not a point of contact between the given solution and the union of the manifolds of the Σ_i. This second property of course implies the first. We shall prove that the converse is not true. That is, we shall exhibit a sequence $\Sigma_1, \Sigma_2, \cdots$ and a solution of the corresponding system Σ, such that every point in the domain of analyticity of the given solution is a point of contact between the solution and the union of the manifolds of the Σ_i, while on the other hand, there is

* Cf. ADE, §§65, 45.

† The existence of such a Σ_k follows from two theorems of Ritt, ADE, §§65, 45.

‡ Subscripts indicate differentiation.

no open region in which the solution can be uniformly approximated by solutions of the Σ_i.

Let

$$\alpha_1, \alpha_2, \cdots \tag{39}$$

be a sequence of complex constants. In terms of the sequence (39) we define a sequence of operators

$$\theta_1, \theta_2, \cdots \tag{40}$$

as follows:

$$\theta_i g(x) = g(x)(g'(x) + \alpha_i g(x)), \theta_i A = A(A' + \alpha_i A), \qquad i = 1, 2, \cdots, \tag{41}$$

for every analytic function $g(x)$ and every form A. Set $\phi_k = \theta_k\theta_{k-1} \cdots \theta_2\theta_1$, $(k=1, 2, \cdots)$. Let

$$\Sigma_1, \Sigma_2, \cdots \tag{42}$$

be the sequence of closed systems such that the manifold of Σ_n, $(n=1, 2, \cdots)$, is the family of all functions $y(x)$ which satisfy the equation

$$\frac{d}{dx}(\phi_n y(x)) = 0. \tag{43}$$

Set $A_k \equiv \phi_k y$, set $B_k \equiv A_k'$, and let S_k be the separant of B_k, $(k=1, 2, \cdots)$. Evidently the manifold of B_{i+1} includes the manifold of B_i, so that Σ_{i+1} holds Σ_i, $(i=1, 2, \cdots)$. Since $S_i S_{i+1}$ does not hold B_i, the general solution of B_{i+1} includes the general solution of B_i.

We shall prove that Σ_n is irreducible. This is equivalent to proving that the manifold of B_n is identical with the general solution of B_n. It suffices to prove that the manifold of S_n is in the general solution of B_n. This last is easy to see when $n=1$. We assume that it is true when $n=r$. Then the manifold of B_r is identical with the general solution of B_r. Now $S_{r+1} = A_r S_r$. Hence every solution of S_{r+1} is in the general solution of B_r, and consequently in the general solution of B_{r+1}.

Thus Σ_n, $(n=1, 2, \cdots)$, is irreducible. As in §12, the system Σ is the trivial system of which every analytic function is a solution.

Let $\mathfrak{M}_i$ be the manifold of Σ_i, $(i=1, 2, \cdots)$, and let $\mathfrak{N} = \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots$.

Let $y(x)$ be an analytic function such that for some k the function $\phi_k y(x)$ is not identically zero, and has a zero at a point x_0. Then $y(x)$ is not in $\mathfrak{N}$. For suppose $y(x)$ is in $\mathfrak{N}$. There exists an n such that $y(x)$ satisfies (43). We may and do assume that $n > k$. Let x_0 be a zero of order σ for $\phi_k y(x)$. Then it is easy to see that x_0 is a zero of order $2^{n-k}(\sigma - 1) + 1$ for $\phi_n y(x)$. But this is impossible, since $\phi_n y(x)$ is a constant, by equation (43). Hence $y(x)$ is not in $\mathcal{N}$.

Let $f(x)$ be a polynomial of positive degree. Then for every choice of a sequence (39) the function $\phi_k f(x)$ is a polynomial of positive degree $(k=1, 2, \cdots)$. Let $\mathfrak{R}_1, \mathfrak{R}_2, \cdots$ be a sequence of open sets such that for every open region $\mathfrak{B}$ there is a k such that $\mathfrak{R}_k$ is included in $\mathfrak{B}$. We shall choose a sequence (39) in such a way that for every k the function $\phi_k f(x)$ has a zero in $\mathfrak{R}_k$.

We take a point b_1 in $\mathfrak{R}_1$ such that $f(b_1) \neq 0$, and define $\alpha_1 = -f'(b_1)/f(b_1)$. Then $\phi_1 f(x)$ has the zero b_1 in $\mathfrak{R}_1$. When $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ have been chosen so that $\phi_k f(x)$ has a zero b_k in $\mathfrak{R}_k$, $(k=1, 2, \cdots, n-1)$, we take a point b_n in $\mathfrak{R}_n$ at which $\phi_{n-1} f(x)$ is not zero, and, letting $g(x) \equiv \phi_{n-1} f(x)$, we define $\alpha_n = -g'(b_n)/g(b_n)$. Then $\phi_n f(x)$ has the zero b_n in $\mathfrak{R}_n$. Proceeding in this manner we determine a sequence of points

$$b_1, b_2, \cdots, \tag{44}$$

dense in the complex plane, and a choice of the sequence (39), such that the point b_k is a zero of $\phi_k f(x)$, $(k=1, 2, \cdots)$. We now retain this choice of (39).

Now suppose that there exists an open region $\mathfrak{B}_1$ in which $f(x)$ can be approximated uniformly by solutions of the Σ_i. Let $y_1(x), y_2(x), \cdots$ be a sequence of functions in $\mathcal{N}$, converging to $f(x)$ uniformly in $\mathfrak{B}_1$. Then for each k the sequence $\phi_k y_1(x), \phi_k y_2(x), \cdots$ converges to $\phi_k f(x)$ uniformly in $\mathfrak{B}_1$. Let k be such that $\phi_k f(x)$ has a zero in $\mathfrak{B}_1$.* Then there is an m such that $\phi_k y_m(x)$ is not identically zero and has a zero in $\mathfrak{B}_1$. We have seen that this implies that $y_m(x)$ is not in $\mathcal{N}$. This contradiction implies that there is no open region in which $f(x)$ can be approximated uniformly by solutions of the Σ_i. On the other hand, every point of the complex plane is a point of contact between $f(x)$ and $\mathcal{N}$, since $\mathcal{F}$ is a field of constants.

Another such example can be constructed as follows: Let θ be the operator such that

$$\theta g(x) = g(x)\left[2(g'(x))^3 - 3g(x)g'(x)g''(x) + (g(x))^2 g'''(x)\right],$$

for every analytic function $g(x)$. Let $\Sigma_1, \Sigma_2, \cdots$ be the sequence of closed systems such that the manifold of Σ_n, $(n=1, 2, \cdots)$, is the family of all functions $y(x)$ which satisfy the equation

$$\frac{d}{dx}(\theta^n y(x)) = 0. \tag{45}$$

Arguments similar to those used in the preceding example show that Σ_i is

* We are using here the fact that the sequence (44) is dense in the complex plane.

irreducible and is held by Σ_{i+1}, $(i=1, 2, \cdots)$, and also show that if $y(x)$ is a function such that $\theta^k y(x)$ has a simple zero for some k, then $y(x)$ is not in $\mathfrak{N}$.

Let ω_1, ω_2 be any two complex numbers whose ratio is not real, and let $\sigma(x)$ be the corresponding σ-function of Weierstrass.* Using a functional equation of $\sigma(x)$,† we find that

$$\theta^k\sigma(x) = 2^{3k(k-1)/2}\sigma(2^k x), \qquad k = 1, 2, \cdots .$$

Every circle which is of diameter greater than $|\omega_1|+|\omega_2|$ contains a simple zero of $\sigma(x)$, so that every circle which is of diameter greater than $(|\omega_1|+|\omega_2|)/2^k$ contains a simple zero of $\theta^k\sigma(x)$.

Thus if $\sigma(x)$ can be approximated uniformly in an open region by solutions of the Σ_i, there is a k such that for some function $y(x)$ in $\mathfrak{N}$ the function $\theta^k y(x)$ has a simple zero at some point in that open region. This contradiction proves that there is no open region in which $\sigma(x)$ can be approximated uniformly by solutions of the Σ_i.

15. Let $\Sigma_1, \Sigma_2, \cdots$ be a sequence of closed irreducible systems in $y_1, \cdots, y_n$ such that the manifold of Σ_i is a proper part of the manifold of Σ_{i+1}, $(i=1, 2, \cdots)$. Let Σ be the set of all forms F such that F holds every Σ_1. Discarding a finite set of the Σ_i if necessary, we assume that there is a fixed set of unknowns $y_1, \cdots, y_q$ which is a set of arbitrary unknowns for every Σ_i.‡ Of course $q<n$. Let $A_{i,q+1}, A_{i,q+2}, \cdots, A_{i,n}$ be a corresponding basic set for Σ_i, with A_{ij} introducing y_j, and let γ_{ij} be the order of A_{ij} in y_j, $(j=q+1, \cdots, n; i=1, 2, \cdots)$. Let r be the number of arbitrary unknowns in Σ. Then $q<r\leqq m$. If Σ is not trivial, that is, if $r<n$, let $A_{r+1}, A_{r+2}, \cdots, A_n$ be a basic set for Σ, with A_k introducing y_k,§ and let γ_k be the order of A_k in y_k, $(k=r+1, \cdots, n)$. We consider the question of whether the values of r, and (when γ_k exist) of the γ_k, are determined uniquely by n, q, and the γ_{ij}. We know that the answer is affirmative when $n-q=1$, since in this case $n-r=0$ and Σ is trivial. We shall indicate by examples in §§15, 16 that the answer is negative when $n-q\geqq 2$.

Let σ be any nonnegative integer, and let Σ_n, $(n=1, 2, \cdots)$, be the closed irreducible system in u, y with a basic set A_n, B_n where

$$A_n \equiv u_n, \qquad B_n \equiv y - \sum_{i=1}^{n-1} i^\sigma u_i.\|$$

Since the order of A_n in u becomes infinite with n, Σ cannot have a non-

* Hurwitz and Courant, *Funktionentheorie*, p. 179.

† Ibid., p. 184.

‡ We renumber the unknowns, if necessary.

§ We renumber the unknowns again, if necessary.

|| For the existence of such a Σ_n cf. ADE, §§65, 45.

zero form in u alone. We shall prove that Σ has a nonzero form in u and y, so that u is a set of arbitrary unknowns for Σ. Moreover, we shall prove that the nonzero forms in Σ of lowest order in y are of order $\sigma+1$ in y. Set

$$C_{nj} \equiv y_j - \sum_{i=1}^{n-j-1} i^\sigma u_{i+j}, \qquad j = 0, 1, \cdots, \sigma+1; n = 1, 2, \cdots .$$

Evidently C_{nj} is a form in Σ_n since it is a linear combination of derivatives of A_n and B_n. Set

$$D_n \equiv \sum_{j=0}^{\sigma+1} (-1)^j {}_{\sigma+1}C_j C_{nj}.^*$$

It is easy to see that

$$D_n \equiv \sum_{j=0}^{\sigma+1} (-1)^j {}_{\sigma+1}C_j \left[y_j - \sum_{k=j+1}^{\sigma+2} (k-j)^\sigma u_k \right].^\dagger$$

Since D_n is independent of n, D_n is in Σ_j for every j. Therefore Σ contains the nonzero form D_n which is of order $\sigma+1$ in y. Let m be any positive integer greater than $\sigma-1$, and let $a_0, a_1, \cdots, a_m; b_0, b_1, \cdots, b_\sigma$ be any set of complex numbers. Let $n=m+\sigma+2$. We shall prove that Σ_n has a solution $u(x)$, $y(x)$ such that $u_i(0)=a_i$, $y_k(0)=b_k$, $(i=0, 1, \cdots, m; k=0, 1, \cdots, \sigma)$.

It is easily seen that such a solution exists if the system of linear equations

$$(46) \qquad b_k = \sum_{i=1}^{m-k} i^\sigma a_{i+k} + \sum_{\alpha=m+1}^{n-1} (\alpha - k)^\sigma u_\alpha, \qquad k = 0, 1, \cdots, \sigma,$$

has a solution in u_α, $(\alpha=m+1, \cdots, n-1)$.

The system (46) will have a solution if the determinant $d(m, \sigma)$ is not zero, where the element of $d(m, \sigma)$ in the ith row and jth column is $(m+1+j-i)^\sigma$, $(i, j=1, \cdots, \sigma+1)$. By repeated subtraction of adjacent columns, and then of adjacent rows, in $d(m, \sigma)$, it can be shown that $d(m, \sigma)=(\sigma!)^{\sigma+1}\neq 0$.

Hence Σ has a solution $u(x)$, $y(x)$ such that

$$u_i(0) = a_i, \; y_k(0) = b_k, \qquad i = 0, 1, \cdots, m; k = 0, 1, \cdots, \sigma,$$

where m is any positive integer and the a_i and b_k are arbitrary. This obviously implies that Σ has no nonzero form whose order in y is less than $\sigma+1$. Evidently if u is taken as a set of arbitrary unknowns for Σ, then $\sigma+1$ is the order in y of a basic set for Σ.

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* Note that ${}_{\sigma+1}C_j$ is a binomial coefficient, not a form.

† This reduces quickly to proving the identity $\sum_{j=0}^{\sigma+1}(-1)^j {}_{\sigma+1}C_j(k-j)^\sigma=0$, which holds for all complex k, since the left-hand member is the σth derivative at the origin of $e^{kx}(1-e^{-x})^{\sigma+1}$.

16. Let Σ_n, $(n=1, 2, \cdots)$, be the closed irreducible system in u, y with a basic set A_n, B_n where $A_n \equiv u_n$, $B_n \equiv y - \sum_{i=1}^{n-1} i! u_i$. We shall prove that for every positive integer m and every set of complex numbers $a_0, a_1, \cdots, a_m$; $b_0, b_1, \cdots, b_m$ the system Σ_{2m+2} has a solution u, y in which $u_j(0) = a_j$, $y_i(0) = b_i$, $(i, j = 0, 1, \cdots, m)$. It suffices to prove that the system of linear equations

$$b_k = \sum_{i=1}^{m-k} i! a_{i+k} + \sum_{\alpha=m+1}^{2m+1} (\alpha - k)! u_\alpha, \qquad k = 0, 1, \cdots, m, \tag{47}$$

has a solution in u_α, $(\alpha = m+1, \cdots, 2m+1)$. Such a solution exists because the determinant d_m of the coefficients is not zero, where the element of d_m in the ith row and jth column is $(m+1+j-i)!$, $(i, j = 1, \cdots, m+1)$. The value of d_m is easily seen to be $(-1)^{m(m+1)/2}(1!2! \cdots m!)^2(m+1)!$.

Consequently, for every set of complex numbers $a_0, a_1, \cdots, a_m$; $b_0, b_1, \cdots, b_m$, Σ has a solution $u(x)$, $y(x)$ with $u_j(0) = a_j$, $y_i(0) = b_i$, $(i, j = 0, 1, \cdots, m)$. This means that Σ has no nonzero form. That is, Σ is the trivial system having u, y as a set of arbitrary unknowns.

17. Let Σ be any closed irreducible system in $u_1, \cdots, u_q$; $y_1, \cdots, y_p$, where $u_1, \cdots, u_q$, $(q \geqq 1)$, is a set of arbitrary unknowns. Let s be any integer with $0 \leqq s < q$. We assert that there is a sequence

$$\Sigma_1, \Sigma_2, \cdots \tag{48}$$

of closed irreducible systems such that Σ_i has s arbitrary unknowns, such that the manifold of Σ_i is a proper part of the manifold of Σ_{i+1}, $(i = 1, 2, \cdots)$, and such that Σ is the totality of forms common to the Σ_i, $(i = 1, 2, \cdots)$.

For example, we may construct a sequence (48) as follows:

Let $A_1, A_2, \cdots, A_p$ be a basic set for Σ, with A_i introducing y_i. Let m be a positive integer greater than the order of A_i in u_j, $(i = 1, \cdots, p$; $j = 1, \cdots, q-s)$. Then for every positive integer n the ascending set

$$u_{1,m+n}, u_{2,m+n}, \cdots, u_{q-s,m+n}, A_1, \cdots, A_p \tag{49}$$

is the basic set of a closed irreducible system. This is a simple consequence of Ritt's theorems characterizing the basic set of a closed irreducible system.* Let Σ_n be the closed irreducible system having (49) for a basic set $(n = 1, 2, \cdots)$. It is easy to see that with this definition of Σ_n the sequence (48) has the desired properties.

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* ADE, §§65, 45.

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