NETS AND GROUPS*

BY

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The combinatorial properties, underlying the configuration of three pencils of parallel straight lines in the plane, have found their condensation in the concept of "net." The theory of nets† culminates in two extreme results: Bol's theorem that every net may be represented by means of coordinates which are taken out of certain abstract multiplicative manifolds—these need not be associative—and Thomsen's characterization of those nets whose coordinates may actually be chosen from a group, which theorem started the whole theory.

The principal object of this paper is to show that the theory of nets is completely equivalent to a well-determined chapter in the theory of groups, using this term in the customary sense of the word. To do this we have to investigate certain groups of net transformations. These groups contain all the possible systems of net coordinates and provide us therefore with the means to characterize those systems of coordinates which define isomorphic nets—a net may be describable by several non-isomorphic systems of coordinates. This method leads incidentally to a rather simple proof of Thomsen's theorem and to some new characterizations of the group-nets.

The net-theoretical considerations are preceded by a systematic discussion of those multiplicative manifolds which may be derived from the multiplication of cosets in a group.‡ Their importance for the theory of nets arises from

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† The following papers are concerned with the theory of nets: W. Blaschke and G. Bol, *Geometrie der Gewebe: Topologische Fragen der Differentialgeometrie*, Berlin, 1938; G. Bol, Mathematische Annalen, vol. 114 (1937), pp. 414–431; H. Kneser, Abhandlungen aus dem Mathematischen Seminar, Hamburg, vol. 9 (1932), pp. 147–151; R. Moufang, Mathematische Annalen, vol. 110 (1934), pp. 416–430; K. Reidemeister, Mathematische Zeitschrift, vol. 29 (1929), p. 427; K. Reidemeister, *Grundlagen der Geometrie*, Berlin and Leipzig, 1930; G. Thomsen, Abhandlungen aus dem Mathematischen Seminar, Hamburg, vol. 7 (1929), pp. 99–106. It should be noted that the nets are sometimes called "webs" (in German, "Gewebe").

‡ Generalizations of the group concept which have some bearing on our investigations have been discussed in the following papers: R. Baer, Sitzungsberichte der Heidelberger Akademie, mathematisch-naturwissenschaftliche Klasse, (4), 1928. G. Bol, Mathematische Annalen, vol. 114 (1937), pp. 414–431; H. Brandt, Mathematische Annalen, vol. 96 (1927), pp. 360–366; M. Dresher and O. Ore, American Journal of Mathematics, vol. 60 (1938), pp. 705–733; L. W. Griffiths, American Journal of Mathematics, vol. 60 (1938), pp. 345–354; A. Loewy, Journal für die reine und angewandte Mathematik, vol. 157 (1927), pp. 239–254; F. Marty, Comptes Rendus de l'Académie des Sciences, vol. 201 (1935), pp. 636–638; F. Marty, Annales de l'École Normale Supérieure, vol. 53 (1936), pp.

the fact that all the admissible systems of net coordinates are of this type.

1. **Coset multiplication.** A multiplication in the set M of elements is a single-valued* function of the ordered pairs of elements in M with values in M. If a multiplication xy has been defined for the elements of M, then M shall be called a multiplication system (with regard to this multiplication xy).

If M is a multiplication system (with regard to the multiplication xy), then a *left unit* is an element e which satisfies $ex=x$ for every element x in M. Right units are defined accordingly and elements which are right and left units at the same time are called *units*.

The multiplication system M is said to be a *left-division system*, if there exists corresponding to any pair u, v of elements in M one and only one element x in M so that $xu=v$. Right-division systems are defined accordingly and systems which are at the same time right- and left-division systems are called *division systems*.

If u is an element in the multiplication system M, then the *right translation* of M corresponding to the element u maps the element x of M upon the element xu of M. The right translations of M are one-one mappings of M upon the whole set M if, and only if, M is a left-division system, and in this case as permutations of M they *generate* a subgroup of the group of permutations of M.

It is our object in this section to investigate the multiplications of cosets. A fairly general type of coset multiplication may be described in the following fashion. Let S be a subgroup of the group G, and let $r(X)$ be a fixed system of representatives of the right cosets $X=Sr(X)$ of G modulo S (so that $r(X)=r(Y)$ if, and only if, $Sr(X)=Sr(Y)$). Then the multiplication system $(S<G; r(X))$ consists of the right cosets X of G modulo S, and the multiplication in $(S<G; r(X))$ is defined by the following rule:

$$XY = Sr(X)r(Y).$$

(1.0) *If G' is the subgroup of G which is generated by the elements $r(X)$, and if S' is the crosscut of G' and S, then $(S<G; r(X))$ and $(S'<G'; r(X))$ are isomorphic, since every coset $Sr(X)$ contains one and only one coset of G' modulo S' (namely $S'r(X)$).*

83–123; O. Ore, Duke Mathematical Journal, vol. 3 (1937), pp. 149–174; F. K. Schmidt, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, mathematisch-naturwissenschaftliche Klasse, (8), 1927, pp. 91–103; Erich Schönhardt, *Über lateinische Quadrate und Unionen*, Journal für die reine und angewandte Mathematik, vol. 163 (1930), pp. 183–230; H. S. Wall, American Journal of Mathematics, vol. 59 (1937), pp. 77–98.

* That it is no loss in generality to restrict one's attention to single-valued functions, has been pointed out by L. W. Griffiths (American Journal of Mathematics, vol. 60 (1938), pp. 345–354). For the induced multiplication in the set of subsets is certainly single-valued.

THEOREM 1.1. (a) *The multiplication system M is isomorphic with a system $(S<G;\ r(X))$ if, and only if, M is a left-division system possessing a left unit.*

(b) *The right translations of the multiplication system $M=(S<G;\ r(X))$ generate a group $T(M)$ of permutations of M.*

(c) *If G' is the subgroup of G, generated by the elements $r(X)$, and if S' is the crosscut of G' and S, then there exists a homomorphism κ of G' upon $T(M)$ with the following properties:*

(i) *S'^{κ} consists of those elements in $T(M)$ which leave the left unit in M invariant.*

(ii) *The elements, mapped by κ upon the identity, form the greatest normal subgroup of G' which is a subgroup of S'.*

(iii) *κ maps the set of elements $r(X)$ upon the (whole) set of the right translations of M, and, in particular, the element $r(X)$ upon the right translation of M, corresponding to X.*

Proof. Let us consider first a multiplication system $M=(S<G;\ r(X))$. Then $SX=Sr(S)r(X)=Sr(X)$ for every X in M, and S is consequently a left unit in M. If U and V are two elements in M, then the solutions of the equation $XU=V$ are exactly the solutions of the equation $Sr(X)r(U)=Sr(V)$, and the solutions of this equation are the same as the solutions of the equation $Sr(X)=Sr(V)r(U)^{-1}$. Since this last equation has one and only one solution, namely $X=Sr(V)r(U)^{-1}$, it follows that M is a left-division system. This proves (b) and the necessity of the conditions in (a).

If t is any element in G', then the right translation of G' corresponding to t induces a uniquely determined permutation t^{κ} of the elements in $M=(S'<G';\ r(X))$. (Note that $(S<G;\ r(X))$ and $(S'<G';\ r(X))$ are essentially the same.) Since $r(X)^{\kappa}$ is in particular the right translation of M corresponding to X, it follows that κ is a homomorphism of G' upon the whole group $T(M)$ which satisfies (iii). If t is any element in G', then $S't=S'$ if, and only if, t is an element in S', and this proves that κ satisfies (i). If E is the subgroup of G' which consists of the elements mapped by κ upon the identity, then E is a normal subgroup of G' and it follows from (i) that $E\leqq S'$. If, conversely, F is a normal subgroup of G', and if $F\leqq S'$, then

$$S'r(X)f = S'r(X)fr(X)^{-1}r(X) = S'r(X)$$

for every f in F and every X in M. Hence $F^{\kappa}=1$, and this completes the proof of (ii) and of (c).

Suppose now that M is a left-division system, possessing a left unit e. Denote by $t(x)$ the right translation of M, corresponding to the element x in M, and let $T(M)$ be the group generated by the $t(x)$, and $S(M)$ the sub-

group consisting of all those elements in $T(M)$ which leave e invariant. Two elements in $T(M)$ belong to the same right coset of $T(M)$ modulo $S(M)$ if, and only if, they map e upon the same element x of M. Since there exists one and only one right translation of M which maps e upon x, namely $t(x)$, it follows that the $t(x)$ form a complete set of representatives of the right cosets of $T(M)$ modulo $S(M)$. A one-one correspondence between M and $(S(M)<T(M);\ t(x))$ is therefore defined in mapping the element x in M upon the element $S(M)t(x)$. This correspondence is an isomorphism, since the transformation

$$S(M)t(x)S(M)t(y) = S(M)t(x)t(y) = S(M)t(xy)$$

maps e upon xy. This completes the proof of (a), and it shows, moreover, that the following statement is true:

COROLLARY 1.2. *If M is a left-division system, possessing a left unit e, if $T(M)$ is the group generated by the right translations $t(x)$ of M, and if $S(M)$ consists of those permutations in $T(M)$ which leave e invariant, then the right translations form a complete set of representatives of the right cosets of $T(M)$ modulo $S(M)$ and an isomorphism of M upon $(S(M)<T(M);\ t(x))$ is defined by mapping x upon $S(M)t(x)$.*

The following statement is a simple consequence of Theorem 1.1:

COROLLARY 1.3. *An isomorphism of the group G upon $T(M) = T[(S<G;\ r(X))]$ is defined by mapping the element x of the group G upon the permutation x^κ of the multiplication system $M=(S<G;\ r(X))$ which the right translation, corresponding to x, induces in M if, and only if,*

(1) *G is generated by the elements $r(X)$;*

(2) *the crosscut of all the subgroups of G which are conjugate to S in G is* 1.

The following statement serves to analyze the relation between the two conditions involved in Theorem 1.1 (a).

(1.4) *The left-division system M possesses a left unit if, and only if, there exist in M elements w which satisfy*

(i) *$w(xy)=(wx)y$ for all x and y in M;*

(ii) *$wx=wy$ implies $x=y$.*

Proof. The condition is necessary, since the left unit satisfies (i) and (ii).

If conversely w is an element in M which satisfies (i) and (ii), then there exists one and only one solution e of $ew=w$ in M. This element e satisfies $ww=w(ew)=(we)w$ and hence $w=we$, since M is a left-division system. Furthermore $wx=(we)x=w(ex)$ and therefore $x=ex$ by (ii) for every x in M, and this proves that e is a left unit.

The coset multiplication in a system $(S<G;\ r(X))$ is determined by the choice of the representatives $r(X)$. That to some degree the choice of the representatives is determined by the coset multiplication may be seen from the following statement:

(1.5) *The two sets of representatives $r(X)$ and $r'(X)$ of the right cosets X of the group G modulo its subgroup S define the same multiplication of the cosets, that is, $(S<G;\ r(X))=(S<G;\ r'(X))$ if, and only if, each of the quotients $r'(X)r(X)^{-1}$ is contained in a normal subgroup of G which is a subgroup of S.*

REMARK. *If the subgroup S of G has the property that 1 is the only normal subgroup of G which is contained in S, then the two sets $r(X), r'(X)$ of representatives define the same coset multiplication if, and only if, $r(X)=r'(X)$ for every X.*

Proof. Since $r(X)$ and $r'(X)$ are both elements in the right coset X, we have $r'(X)=s(X)r(X)$ where $s(X)$ is a suitable element in S. If the two sets of representatives define the same coset multiplication, then

$$Sr'(X)Sr'(Y) = Sr'(X)r'(Y) = Sr(X)s(Y)r(Y) = Sr(X)r(Y)$$

and consequently $Sr(X)s(Y)=Sr(X)$ or $Sr(X)s(Y)r(X)^{-1}=S$ for every pair X, Y. If now U is some right coset, g any element in G, then $g=sr(Sg)$ for some s in S and

$$S = Sr(Sg)s(U)r(Sg)^{-1} = Ssr(Sg)s(U)r(Sg)^{-1}s^{-1} = Sgs(U)g^{-1}.$$

This shows that every $gs(U)g^{-1}=gr'(U)r(U)^{-1}g^{-1}$ is contained in S, proving the necessity of our condition.

If the condition is satisfied, then

$$\begin{aligned} Sr'(X)Sr'(Y) &= Sr'(X)r'(Y) = Sr(X)s(Y)r(Y) \\ &= Sr(X)s(Y)r(X)^{-1}r(X)r(Y) \\ &= Sr(X)r(Y) = Sr(X)Sr(Y), \end{aligned}$$

and this completes the proof.

2. **Division systems.** The only multiplication systems we shall need for our applications are the division systems with unit. These are certainly left-division systems with left units, and they are therefore of the form $(S<G;r(X))$.

THEOREM 2.1. *The multiplication system $M=(S<G;r(X))$ possesses a unit if, and only if, all the conjugates in G to the element $r(S)$ are contained in S; that is, if, and only if, all the elements $r(X)r(S)r(X)^{-1}$ are in S.*

Proof. If all the conjugates of $r(S)$ are in S, then

$$XS = Sr(X)r(S) = Sr(X)r(S)r(X)^{-1}r(X) = Sr(X) = X,$$

and M possesses therefore the unit S. If conversely M possesses a unit, then S is this unit. If x is any element in G, then there exists an element s in S so that $x = sr(Sx)$ and

$$Sr(Sx) = Sr(Sx)r(S) = Ssr(Sx)r(S) = Sxr(S)x^{-1}sr(Sx).$$

But this implies that $xr(S)x^{-1}s$ and consequently $xr(S)x^{-1}$ are elements in S.

REMARK 2.2. *If, as we may assume without loss in generality* (cf. (1.0)), *the only normal subgroup of G, contained in S, is* 1, *then the existence of a unit in* $(S<G; r(X))$ *is equivalent to the fact that* $r(S)=1$.

THEOREM 2.3. *The multiplication system* $(S<G; r(X)) = M$ *is a division system if, and only if, the elements* $r(X)$ *form a complete set of representatives for the right cosets of the group G modulo every subgroup of G which is conjugate to S in G.*

Proof. Assume first that M is a division system. If g is any element in G, then $g = sr(Sg)$ for a suitable element s in S. If w is another element in G, then there exists one and only one element X in M so that $(Sg)X = S(gw)$, and this X is clearly the only solution of

$$(g^{-1}Sg)w = r(Sg)^{-1}Sr(Sg)w = r(Sg)^{-1}Sr(Sg)r(X) = g^{-1}Sgr(X).$$

Thus the elements $r(X)$ form a complete set of representatives for the right cosets of G modulo $g^{-1}Sg$, if M is a division system.

Suppose now conversely that the elements $r(X)$ form a complete set of representatives of the right cosets of G modulo every $g^{-1}Sg$. If U and V are two elements of M, then the solutions X of $UX = V$ are exactly the solutions X of the equation $Sr(U)r(X) = Sr(V)$ and these are exactly the solutions of

$$r(U)^{-1}Sr(U)r(X) = r(U)^{-1}Sr(U)r(U)^{-1}r(V);$$

that is, $r(X)$ is the uniquely determined representative of the right coset $r(U)^{-1}Sr(U)r(U)^{-1}r(V)$ of G modulo $r(U)^{-1}Sr(U)$. This shows that M is a right-division system and consequently a division system.

REMARK 2.4. *If* $(S<G; r(X))$ *is a division system, and if g is any element in G, then the equation*

$$U = SgX = Sr(Sg)r(X) = Sgr(X)$$

has one and only one solution X and the elements $gr(X)$ *for X in* $(S<G; r(X))$ *form therefore a complete set of representatives of the right cosets for every fixed element g.*

THEOREM 2.5. *If the elements* $r(X)$ *form a complete set of representatives of the right cosets of the group G modulo its subgroup S, if G' is generated by the*

elements $r(X)$ and S' is the crosscut of G' and S, then the following three assertions are equivalent:

(a) *S' is a normal subgroup of G'.*

(b) *$(S<G; r(X))$ is a group.*

(c) *$(S<G; r(X))$ is associative.**

Proof. (b) is a consequence of (a), since $(S<G; r(X))$ and $(S'<G'; r(X))$ are isomorphic. (c) is obviously a consequence of (b). Assume finally that $(S<G; r(X))$ is associative. Then

$$Sr(Z)r(X)r(Y) = [Sr(Z)r(X)]Y = (ZX)Y = Z(XY) = Sr(Z)r(XY),$$

and $r(Z)r(XY)r(Y)^{-1}r(X)^{-1}r(Z)^{-1}$ is therefore, for every triple X, Y, Z, an element in S'. If N is the greatest normal subgroup of G' which is contained in S', then it follows from this remark that $r(XY)r(Y)^{-1}r(X)^{-1}$ is contained in N and that consequently $Nr(XY)=Nr(X)Nr(Y)$. The classes $Nr(X)$ form, therefore, a subset of the group G'/N which is closed with regard to multiplication. Since $(S'<G'; r(X))$ is a left-division system, there exists for every X one and only one X^{-1} so that $r(S)=r(X^{-1}X)$. Since $(S'<G'; r(X))$ is an associative left-division system with left unit S', we have $XX=X(S'X)=(XS')X$ and therefore $X=XS'$; that is, S' is the unit of the system and $r(S)$ is therefore, by Theorem 2.1, an element of N. Hence $N=Nr(X^{-1})r(X)$ or $Nr(X^{-1})=Nr(X)^{-1}$. Consequently $Nr(X)Nr(X^{-1})=N$. This shows that the elements $Nr(X)$ form a subgroup of G'/N. Since G' is generated by the elements $r(X)$, the elements $Nr(X)$ form the complete group G'/N. Since these elements $Nr(X)$ form a set of representatives of the right cosets of G'/N modulo S'/N, this proves that $S'=N$ is a normal subgroup of G' and thus (a) is a consequence of (c).

The following *example of a division system without unit* is of interest because of Theorem 6.1. The elements of the system are u, v, and w, and the multiplication table is

$$uv = vu = w^2 = w, \qquad vw = wv = u^2 = u, \qquad wu = uw = v^2 = v.$$

3. **Similar division systems.** For future application we need an extension of the concept of isomorphism. The following statements form a basis for this concept of similarity.

* It is a consequence of Theorem 1.1 (a) that the inference (c)-(b) may be stated in the following form: *An associative left-division system with left unit is a group.* A direct proof of this fact may be indicated: If e is the left unit and x any element, then $xe=xe^2=(xe)e$; that is, $x=xe$ and e is the unit. If x is any element and x^{-1} is the uniquely determined element so that $x^{-1}x=e$, then $x=xe=x(x^{-1}x)=(xx^{-1})x=ex$ and therefore $xx^{-1}=e$; thus x^{-1} is the inverse of x and the system is a group.

(3.1) *If* $(S<G;\ r(X))$ *is a division system with unit, then each*

$$(g^{-1}Sg<G;\ r(U)^{-1}r(X)),$$

for fixed U *and variable* X, *is a division system with unit.*

Proof. It is a consequence of Theorem 2.1 and Theorem 2.3 that

$$(g^{-1}Sg<G;\ r(X))$$

is a division system with unit. It is a consequence of Remark 2.3 that the elements $r(U)^{-1}r(X)$ for fixed U and variable X form a complete set of representatives for the right cosets of G modulo $g^{-1}Sg$. Since $1=r(U)^{-1}r(U)$, it follows, therefore, from Theorem 2.2 that $(g^{-1}Sg<G;\ r(U)^{-1}r(X))$ is a division system with unit.

If 1 is the only normal subgroup of G which is contained in the subgroup S of G, and if G is generated by the set of representatives $r(X)$ of the right cosets X of G modulo S, then G, S, and $r(X)$ are said to define a *canonical representation* of the multiplication system $M=(S<G;\ r(X))$. It is a consequence of Corollary 1.3 that any two canonical representations of M are isomorphic,* and it is a consequence of Corollary 1.2 and Theorem 1.1 (a) that the multiplication system M possesses a canonical representation if, and only if, M is a left-division system with left unit.

(3.2) *If* M *is a division system with unit, if* $(S<G;\ r(X))$ *is a canonical representation of* M, *then* $g^{-1}Sg$, G, *and the elements* $r(U)^{-1}r(X)$ *define a canonical representation of* $(g^{-1}Sg<G;\ r(U)^{-1}r(X))$.

Proof. As M possesses a unit and as the only normal subgroup of G which is contained in S is 1, it follows from Theorem 2.1 that $r(S)=1$. Hence $r(U)^{-1}$ is one of the elements $r(U)^{-1}r(X)$, and these elements generate, therefore, the same group as the $r(X)$.

DEFINITION 3.3. *The division system* M *with unit and the multiplication system* M' *are similar if* M' *is isomorphic with* $(g^{-1}Sg<G;\ r(U)^{-1}r(X))$ *where* $(S<G;\ r(X))$ *is a canonical representation of* M.

If M is a division system with unit, and if the multiplication system M' is similar to M, then it follows from (3.1) that M' is a division system with unit. If furthermore M' is isomorphic with

$$(g^{-1}Sg < G;\ r(U)^{-1}r(X)),$$

and $(S<G;\ r(X))$ is a canonical representation of M, it follows from (3.2) that

$$(g^{-1}Sg < G;\ r(U)^{-1}r(X))$$

* Two representations $(S<G;\ r(X))$ and $(T<H;\ h(X))$ of the same system M are *isomorphic* if there exists an isomorphism κ of the group G upon the group H so that $S^{\kappa}=T$ and $r(X)^{\kappa}=h(X)$.

is a canonical representation of M'. This implies that the similarity relation is symmetric. That it is transitive follows from

$$(r(U)^{-1}r(V))^{-1}r(U)^{-1}r(X) = r(V)^{-1}r(X).$$

That it finally is preserved under isomorphisms follows from the fact that any two canonical representations of a multiplication system are isomorphic.

In the following fashion one will be led to another characterization of the similar division systems with unit, a characterization which is of a more intrinsic type than the one given above. If M is a division system with unit and $M=(S<G;\, r(X))$ is a canonical representation of M, then all the subgroups conjugate to S may be represented in the form $r(V)^{-1}Sr(V)$. Since transformation with $r(V)$ induces an automorphism of G, it follows that all the similar multiplication systems may be represented (in canonical form) in the following fashion:

$$M' = (S < G;\, r(V)r(U)^{-1}r(X)r(V)^{-1}).$$

Denote now by X/V the uniquely determined solution Y of the equation $YV=X$ and by $X^{|V,U|}$ the uniquely determined solution Z of the equation $XV=(V/U)Z$; then the right coset of G modulo S which is represented by

$$r(V)r(U)^{-1}r(X)r(V)^{-1}$$

is just the element $((V/U)X)/V$ of M. The multiplication used so far is the multiplication in M. If X' and Y' are two elements in M', then there exist elements X and Y so that $X'=((V/U)X)/V$ and $Y'=((V/U)Y)/V$, namely $X=X'^{|V,U|}$ and $Y=Y'^{|V,U|}$, and the M'-product of the elements X' and Y' is represented in G by the element

$$r(V)r(U)^{-1}r(X)r(U)^{-1}r(Y)r(V)^{-1}.$$

The M'-product of X' and Y' is therefore

$$X' * Y' = \frac{\dfrac{X'V}{U}\, Y'^{|V,U|}}{V},$$

and this shows that one may get all the division systems with unit which are similar to M by choosing two elements U and V in M quite at random and then defining a new multiplication by the above formula.

The two special cases $U=1$ and $V=1$ may be stated. If $U=1$, then $X' * Y'=(X'V)Y'^{V}/V$ where Y'^{V} is the uniquely determined solution of the equation $Y'V=VY'^{V}$. If $V=1$ then $X' * Y'=(X'/U)Y'^{|1,U|}$, where $Y'^{|1,U|}$ is the uniquely determined solution of the equation $Y'=(1/U)Y'^{|1,U|}$.

The two following remarks concern important special cases of classes of similar division systems with unit. If the two division systems M and M', each containing a unit, are similar, and if M is a *group*, then it follows from Theorem 2.4 that M and M' are *isomorphic* groups. Suppose now that M is a division system with unit and that every system M' which is similar to M is *commutative*. Then we represent M in the canonical form $(S<G; r(X))$. As M is a division system with unit, it follows that the $r(X)$ form a complete set of representatives of the cosets modulo gSg^{-1} for any g in G. From our hypothesis, $(gSg^{-1}<G; r(X))$ is commutative. Hence

$$r(X)r(Y)r(X)^{-1}r(Y)^{-1}$$

is an element in gSg^{-1}. But as S, G, $r(X)$ is a canonical representation of M, it follows that 1 is the only element contained in every gSg^{-1}, and consequently we have

$$1 = r(X)r(Y)r(X)^{-1}r(Y)^{-1}.$$

Since G is generated by the elements $r(Z)$, this implies that G is a commutative group, and this implies that *M is a commutative group* (so that all the systems, similar to M, are isomorphic to M).

Our treatment of the left-division systems with left unit (§1) was essentially nothing else than a generalization of the proof of Cayley's theorem that every group may be represented as an isomorphic group of permutations. For this proof one uses the so-called regular representation of the group which consists just of the right translations. One is led to another generalization of this idea by restricting one's attention to the system of the right translations and not extending this system, as has been done in §1, to the generated group of permutations. As this will give us some better insight into the concept of similarity, it will be useful to consider this in some detail.

A set P of permutations of the (finite or infinite) set T of elements shall be called *simply transitive* if there exists to every pair of elements in T one and only one permutation in P which maps the one element upon the other.

The right translations of the multiplication system M form a set $P(M)$ of permutations of M if, and only if, M is a left-division system. $P(M)$ is simply transitive if, and only if, M is a division system; and $P(M)$ contains the identity if, and only if, M possesses a right unit.

The set P of permutations of the set T and the set P' of permutations of the set T' are said to be *similar* if there exists a one-one correspondence p which maps P upon P' and a pair of one-one correspondences t, s which both map T upon T' so that $tx^p = xs$ for every x in P. If in particular $s=t$, then the systems are *isomorphic*.

THEOREM 3.4. *The set P of permutations of the set T is isomorphic to the set $P(M)$ of right translations of a suitable division system M with unit if, and only if, P is simply transitive and contains the identity.*

Proof. The necessity of the conditions has been pointed out before. If the conditions are satisfied, then choose an element e in T and denote by M the system which consists of the permutations in P, where the multiplication in M has been defined in the following fashion: If x and y are two permutations in P, then $x * y$ is the uniquely determined element in P which maps e upon $e^{xy} = (e^x)^y$. This product definition in M is certainly unique. The identity in P gives rise to the unit in M. If u and v are two elements in M, then the solution of $u * x = v$ is just the uniquely determined permutation in P which maps e^u upon e^v, and M is therefore a right-division system. There exists, furthermore, a uniquely determined element f in T so that $f^u = e^v$, and there exists a uniquely determined element x in P so that $e^x = f$. Clearly this permutation x is the uniquely determined solution of the equation $x * u = v$, and M is consequently a division system with unit.

We define t by the equation $x^t = e^x$ for x in M. Since the elements x in M are the permutations of a simply transitive system, t is a one-one correspondence mapping M upon T. Furthermore, let us denote by p the correspondence which maps the right translation of M which is induced by the element u of M upon the element u (of M and) of P. The correspondence p is a one-one correspondence mapping $P(M)$ upon P, since M is a division system with unit. If now x is an element in M, u an element in $P(M)$, then

$$x^{tu^p} = (e^x)^{u^p} = e^{xu^p} = (xu^p)^t = (x^u)^t = x^{ut},$$

and this proves that t and p together define an isomorphism of $P(M)$ upon P. That proves our theorem.

THEOREM 3.5. *The two division systems M and M', both of them possessing a unit, are similar if, and only if, $P(M)$ and $P(M')$ are similar systems of permutations.*

Proof. As both M and M' are division systems with unit, there exist canonical representations $M = (S < G; r(X))$ and $M' = (S' < G'; r'(X))$ of these systems. The system $P(M)$ is by (1.5) exactly the system of permutations which the right translations, induced by the elements $r(X)$, induce in the cosets of G modulo S; and $P(M')$ may be described accordingly. If M and M' are similar, we may assume without loss in generality that M' is of the form

$$M' = (S < G; r(V)r(U)^{-1}r(X)r(V)^{-1}),$$

as the inner automorphism $g \to r(V)gr(V)^{-1}$ of G maps

$$(r(V)^{-1}Sr(V) < G;\ r(U)^{-1}r(X))$$

exactly upon

$$(S < G;\ r(V)r(U)^{-1}r(X)r(V)^{-1}).$$

The correspondences s and t are defined by the formulas

$$[Sr(V)r(U)^{-1}r(X)r(V)^{-1}]^s = Sr(X)\ (=X),$$
$$[Sr(V)r(U)^{-1}r(X)r(V)^{-1}]^t = Sr(X)r(U)^{-1}.$$

Both s and t are one-one correspondences which map M' upon the whole M. If

$$Z' = Sr(V)r(U)^{-1}r(Z)r(V)^{-1}$$

is an element in M', then the right translation induced by Z' has the form

$$[Sr(V)r(U)^{-1}r(X)r(V)^{-1}]^{Z'} = Sr(V)r(U)^{-1}r(X)r(U)^{-1}r(Z)r(V)^{-1},$$

and the correspondence p is defined as mapping this right translation of M' upon the right translation $[Sr(X)]^{Z'^p} = Sr(X)r(Z)$, or $Z'^p = Z$. The correspondence p is, by its definition and by the fact that M and M' are division systems with unit, a one-one correspondence which maps $P(M')$ upon the whole $P(M)$. Finally we have

$$\begin{aligned}[Sr(V)r(U)^{-1}r(X)r(V)^{-1}]^{Z's} &= Sr(X)r(U)^{-1}r(Z)\\ &= [Sr(X)r(U)^{-1}]^Z = [Sr(X)r(U)^{-1}]^{Z'^p}\\ &= [Sr(V)r(U)^{-1}r(X)r(V)^{-1}]^{tZ'^p},\end{aligned}$$

and this shows that s, t, and p induce a similarity between $P(M')$ and $P(M)$.

Assume now conversely that s, t, and p induce a similarity between $P(M')$ and $P(M)$. If X' is an element in M', then p maps the right translation of M' which is induced by X' upon a right translation of M which is induced by a uniquely determined element X'^p of M. Then a correspondence w may be defined as follows:

$$[S'r'(X')]^w = r(1^s)^{-1}Sr(1^s)r(1^p)^{-1}r(X'^p).$$

We put $1^s = V$ and $1^p = U$, and the above formula then reads

$$[S'r'(X')]^w = r(V)^{-1}Sr(V)r(U)^{-1}r(X'^p).$$

The correspondence w is a one-one correspondence which maps $M' = (S' < G';\ r'(X'))$ upon the whole system $M'' = (r(V)^{-1}Sr(V);\ r(U)^{-1}r(X))$, and M'' is a division system with unit which is similar to $M = (S < G;\ r(X))$.

Furthermore we have $X'^tY'^p = [X'Y']^s$, since the left-hand side signifies the effect of the right translation corresponding to Y'^p upon the element X'^t,

and the right-hand side gives the picture under s of the effect of the right translation corresponding to Y' upon X'. Thus we have, in particular,

$$V = 1^s = 1^t 1^p = 1^t U, \qquad X'^t 1^p = X'^t U = X'^s, \qquad 1^t X'^p = X'^s = X'^t U.$$

Hence we have

$$\begin{aligned}[S'r'(X')]^w &= r(V)^{-1}Sr(1^t U)r(U)^{-1}r(X'^p)\\ &= r(V)^{-1}Sr(1^t)r(X'^p) = r(V)^{-1}Sr(X'^s).\end{aligned}$$

Thus we find that

$$\begin{aligned}[X'Y']^w = [S'r'(X'Y')]^w &= r(V)^{-1}Sr([X'Y']^s)\\ &= r(V)^{-1}Sr(X'^t)r(Y'^p)\\ &= r(V)^{-1}Sr(X'^t)r(1^p)r(1^p)^{-1}r(Y'^p)\\ &= r(V)^{-1}Sr(1^t)r(X'^p)r(1^p)^{-1}r(Y'^p)\\ &= r(V)^{-1}Sr(1^t)r(1^p)r(1^p)^{-1}r(X'^p)r(1^p)^{-1}r(Y'^p)\\ &= r(V)^{-1}Sr(V)r(U)^{-1}r(X'^p)r(U)^{-1}r(Y'^p)\\ &= X'^w Y'^w,\end{aligned}$$

and M' and M'' are therefore isomorphic. Hence M' and M are similar, and this completes the proof.

4. **Net translations.** A net consists of four different kinds of elements: points, R-lines, S-lines, and T-lines. Points may lie on lines, lines may pass through points, and lines may have points in common. These relations are subject to the following two postulates:

I. *Through every point there passes one and only one R-line, one and only one S-line, and one and only one T-line.*

II. *If the lines X and Y belong to different ones of the pencils R, S, and T, then they have one and only one point in common.*

It is a consequence of I that lines in the same pencil do not meet.

A typical example for such a net consists of the points of the plane $x+y+z=0$ in euclidean 3-space and the lines $x=$const., $y=$const., and $z=$const. in this plane.

If p is a point in the net N, then the uniquely determined R-line through p is denoted by $R(p)$, and $S(p)$ and $T(p)$ are defined accordingly. If the lines X and Y belong to different ones of the pencils R, S, and T, then the uniquely determined point of the net through which both X and Y pass is denoted by $XY=YX$.

The following two formulas are recorded for future reference. Their proof is obvious.

(4.1) *If X is an R-line and Y an S-line (or a T-line), then $R(XY)=X$.*

(4.2) *If X is an R-line and Y an S-line, then $XT(XY)=XY$.*

Net isomorphisms are one-one correspondences between the points of a net N and the points of a net N' so that points on the same R-line are mapped upon points on the same R'-line, and so on. The more general kind of net isomorphism which permutes the three line pencils will not be discussed in this investigation.*

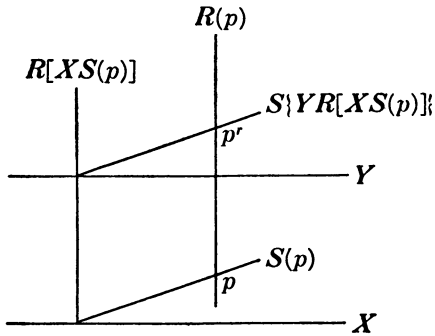

FIG. 1

Net isomorphisms, on the other hand, prove in certain respects too narrow for our purposes. Thus we introduce the following concept. A one-one correspondence t between the points of the net is termed an *R/S-transformation* of the net, if it satisfies the following conditions:

(a) t maps the set of all the net-points upon the whole set of all the net-points.

(b) $R(p^t)=R(p)$.

(c) $S(p)=S(q)$ if, and only if, $S(p^t)=S(q^t)$.

Thus R/S-transformations are characterized by the facts that they leave every R-line invariant and induce a permutation of the S-lines.† They clearly form a group, and this group is essentially the same as the group of all the permutations of the S-lines.

THEOREM 4.3. *Corresponding to every pair X, Y of T-lines there exists one and only one R/S-transformation $r(X-Y)=r(R/S;\ X-Y)$ which maps the points of X upon the points of Y. If p is any net-point, then p is mapped by $r(X-Y)$ upon the point $R(p)S\{YR[XS(p)]\}$.*

The theorem is illustrated by Fig. 1.

* An exception is Theorem 8.1.

† Note that nothing has been said concerning T-lines.

Proof. Assume first that t is an R/S-transformation mapping the points of the T-line X upon the points of the T-line Y. Then

$$p^t = R(p)S(p) = R(p)S[XS(p)] = R(p)S\{XR[XS(p)]\};$$

therefore

$$p^t = R(p)^tS\{XR[XS(p)]\}^t = R(p)S\{X^tR[XS(p)]^t\} = R(p)S\{YR[XS(p)]\}.$$

This proves that there exists at most one R/S-transformation which maps X upon Y, and that an R/S-transformation, mapping X upon Y, has the form given in the theorem.

In order to prove the existence of an R/S-transformation which maps X upon Y, let us consider, therefore, the transformation of the points of the net which is defined by

$$p^r = R(p)S\{YR[XS(p)]\}.$$

This correspondence r is certainly a single-valued function of the net-points which leaves every R-line invariant. Since

$$\begin{aligned} R(p^r)S\{XR[YS(p^r)]\} &= R(p)S\{XR[YS(R(p)S\{YR[XS(p)]\})]\} \\ &= R(p)S\{XR[YS\{YR[XS(p)]\}]\} \\ &= R(p)S(XR\{YR[XS(p)]\}) \\ &= R(p)S\{XR[XS(p)]\} \\ &= R(p)S[XS(p)] \\ &= R(p)S(p) = p, \end{aligned}$$

it follows that r is a one-one correspondence between the points of the net, which maps the set of the net-points upon the whole set of all the net-points. From the above formula it follows, furthermore, that

$$S(p) = S\{XR[YS(p^r)]\}, \qquad S(p^r) = S\{YR[XS(p)]\},$$

and consequently $S(p) = S(q)$ if, and only if, $S(p^r) = S(q^r)$.

If finally p is a point on X, then

$$p^r = R(p)S[YR(p)] = YR(p),$$

and if p^r is a point on Y, then it follows from the above formula that

$$p = R(p^r)S[XR(p^r)] = XS(p^r).$$

This proves that r maps the points of X exactly upon the points of Y.

THEOREM 4.4. *If E is some T-line, and if u and v are points on the same R-line, then there exists one and only one R/S-transformation which maps u*

upon v and E upon some well-determined T-line. The transformation meeting the requirements is

$$r(R/S;\ E - T\{S(v)R[ES(u)]\}).$$

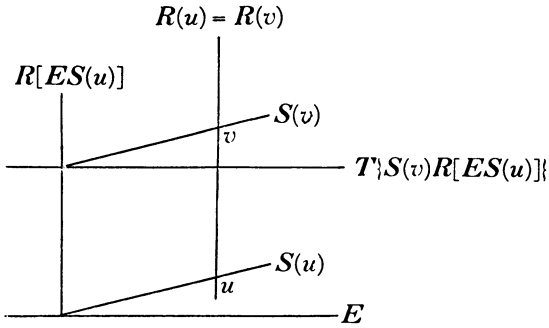

FIG. 2

Proof. If $r(E-Z)$ meets the requirements, then it follows from Theorem 4.3 that $v=R(u)S\{ZR[ES(u)]\}$. Hence

$$\begin{aligned} T\{S(v)R[ES(u)]\} &= T\{S(R(u)S\{ZR[ES(u)]\})R[ES(u)]\} \\ &= T\{S(ZR[ES(u)])R[ES(u)]\} = T\{ZR[ES(u)]\} = Z, \end{aligned}$$

and this proves the statements concerning uniqueness. Furthermore it follows from Theorem 4.3 that $r(R/S;\ E-T\{S(v)R[ES(u)]\})$ maps u upon the point

$$R(u)S(T\{S(v)R[ES(u)]\}R[ES(u)]) = R(v)S(v) = v,$$

as $R(u) = R(v)$, and this completes the proof.

The following statement is an obvious consequence of Theorem 4.4:

COROLLARY 4.5. *If E is a T-line and U and V are two S-lines, then there exists one and only one R/S-transformation which maps U upon V and E upon some well-determined T-line. The transformation meeting the requirements is $r(R/S;E-T\{VR[EU]\})$.*

5. **Division systems and their canonical representation by net transformations.** Since the transformations $r(R/S;\ X-Y)$ are permutations of the points in the net, they generate a group $G(R/S)$. Every element in $G(R/S)$ is a permutation of the net-points, leaves each R-line invariant, and maps every S-line upon some well-determined S-line. Concerning the T-lines not much can be said.

The transformations $r(X-Y)$ satisfy the following rules:

$$r(X - Y)r(Y - Z) = r(X - Z), \qquad r(Y - X) = r(X - Y)^{-1},$$
$$r(X - Y) = r(E - X)^{-1}r(E - Y).$$

The last formula implies that $G(R/S)$ is already generated by the transformations $r(E-X)$ for some fixed T-line E and variable T-lines X.

If e is some point in the net, then the transformations in $G(R/S)$ which have e as a fixed point form a subgroup $G(R/S;\, e)$ of $G(R/S)$.

If s and t are two elements in $G(R/S)$, then they map the point e upon the same point (of $R(e)$) if, and only if, st^{-1} is an element in $G(R/S;\, e)$. If p is a point on $R(e)$, then $r(R/S;\, T(e)-T(p))$ is the uniquely determined transformation $r(T(e)-X)$ which maps e upon p. The transformations $r(R/S;\, T(e)-X)$ form therefore a *complete set of representatives of the right cosets of* $G(R/S)$ modulo $G(R/S;\, e)$ and a one-one correspondence between these right cosets on the one side and the points on $R(e)$ on the other side is defined in mapping the transformations t in $G(R/S)$ with $e^t=p$ upon p.

THEOREM 5.1. (a) *The coset multiplication system*

$$M(R/S;\, e) = (G(R/S;\, e) < G(R/S);\, r(R/S;\, T(e) - X))$$

is a division system with unit, and $G(R/S)$, $G(R/S;\, e)$, $r(R/S;\, T(e)-X)$ *define a canonical representation of* $M(R/S;\, e)$.

(b) *The systems* $M(R/S;\, p)$ *form a complete set of similar division systems.*

The proof of this theorem is based on several statements some of which are of interest in themselves.

(5.1.1) *If* p *and* q *are two points in the net, then*

$$G(R/S;\, q) = r(R/S;\, T(p) - T[S(q)R(p)])^{-1}G(R/S;\, p) \cdot r(R/S;T(p) - T[S(q)R(p)]).$$

Since $r(R/S;\, T(p)-X)$ maps p upon $R(p)X$, it follows that

$$r(R/S;\, T(p) - X)^{-1}G(R/S;\, p)r(R/S;\, T(p) - X) = G(R/S;\, R(p)X).$$

Since $G(R/S;\, e)$ maps each R-line and $S(e)$ upon itself, it follows that every point on $S(e)$ is a fixed point under the transformations in $G(R/S;\, e)$ and consequently $G(R/S;\, e)=G(R/S;\, f)$ if $S(e)=S(f)$. The statement (5.1.1) is a consequence of these two special cases.

(5.1.2) *The subgroups* $G(R/S;\, p)$ *form a complete set of conjugate subgroups of* $G(R/S)$.

For if t is any element in $G(R/S)$, then, as has been remarked before, $t=t'r(R/S;\, T(p)-T(p^t))$ for a suitable element t' in $G(R/S;\, p)$. Hence

$$t^{-1}G(R/S;\, p)t = r(R/S;\, T(p) - T(p^t))^{-1}G(R/S;\, p)r(R/S;\, T(p) - T(p^t)),$$

and (5.1.2) is now a consequence of (5.1.1).

(5.1.3) *The crosscut of the groups* $G(R/S;\ p)$ *is* 1; *and* 1 *is therefore the greatest normal subgroup of* $G(R/S)$ *contained in* $G(R/S;\ e)$.

This is a consequence of (5.1.2) and the fact that a transformation t which is contained in every $G(R/S;\ p)$ has every net-point p as a fixed point.

Since the $r(R/S;\ T(e)-X)$ form a complete set of representatives of the right cosets of $G(R/S)$ modulo $G(R/S;\ S(p)T(e))$, as has been remarked before, and since $G(R/S;\ S(p)T(e))=G(R/S;\ p)$ by (5.1.1), it follows from Theorem 2.3 that $M(R/S;\ e)$ is a division system, and it possesses a unit since $r(R/S;\ T(e)-T(e))=1$. The given representation of $M(R/S;\ e)$ is a canonical representation, as follows from the definition of $G(R/S)$, and a remark added to this definition, and from (5.1.3). That the multiplication sys-

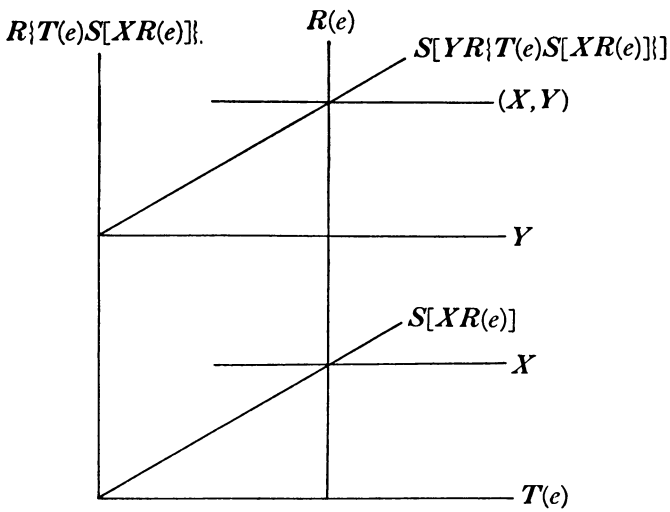

FIG. 3

tems $M(R/S;\ p)$ form a complete system of similar division systems is a consequence of (5.1.2) and the fact that

$$r(R/S;\ T(e)-U)^{-1}r(R/S;\ T(e)-X)=r(R/S;\ U-X)$$

and $G(R/S;\ p)=G(R/S;\ S(p)U)$, which completes the proof of the theorem.

The following formula will prove useful in applications:

(5.2) *If* X *and* Y *are two* T*-lines and* $(X,\ Y)$ *is the* T*-line defined by the equation* $(X,\ Y)=T\{R(e)S[YR\{T(e)S[R(e)X]\}]\}$, *then*

$$[G(R/S;\ e)r(R/S;\ T(e)-X)][G(R/S;\ e)r(R/S;\ T(e)-Y)]$$
$$=G(R/S;\ e)r(R/S;\ T(e)-(X,\ Y)).$$

This statement is illustrated by Fig. 3 above.

Proof. In order to prove this it is sufficient to show that the transformations

$$r(R/S;\, T(e) - X)r(R/S;\, T(e) - Y), \qquad r(R/S;\, T(e) - (X, Y))$$

map e upon the same point. It follows from Theorem 4.3 that $r(R/S;\, T(e) - X)$ maps e upon the point

$$R(e)S\{XR[T(e)S(e)]\} = R(e)S[XR(e)] = XR(e)$$

and that therefore $r(T(e) - X)\, r(T(e) - Y)$ maps e upon the point

$$R(e)S[YR\{T(e)S[XR(e)]\}].$$

But $r(T(e) - (X, Y))$ maps e, by Theorem 4.3, upon the point $R(e)(X, Y)$, and this proves our statement.

THEOREM 5.3. (a) *If X is a T-line and $X^R = R\{T(e)S[XR(e)]\}$, then*

$$X = T\{R(e)S[X^R T(e)]\}, \qquad S[XR(e)] = S[X^R T(e)].$$

(b) *An anti-isomorphism of $M(R/S;\, e)$ upon $M(T/S;\, e)$ is defined in mapping $G(R/S;\, e)r(R/S;\, T(e) - X)$ upon $G(T/S;\, e)r(T/S;\, R(e) - X^R)$.*

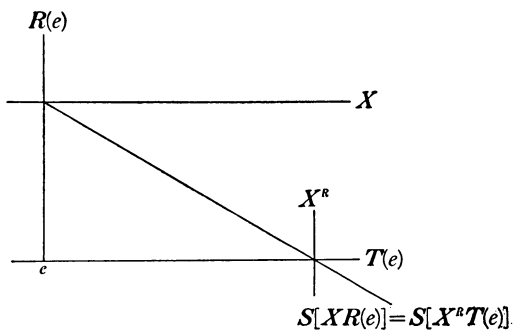

FIG. 4

Proof.* If X is a T-line, then

$$S[X^R T(e)] = S(T(e)R\{T(e)S[XR(e)]\}) = S\{T(e)S[XR(e)]\} = S[XR(e)]$$

and therefore

$$T\{R(e)S[X^R T(e)]\} = T\{R(e)S[XR(e)]\} = T[XR(e)] = X;$$

and this proves (a).

It is a consequence of (a) that the correspondence, defined in (b), is a one-one correspondence between $M(R/S;\, e)$ and $M(T/S;\, e)$. It is a conse-

* Cf. G. Bol, op. cit., p. 419.

quence of (5.2) that this correspondence is an anti-isomorphism, provided, in the notation of (5.2), that $(X, Y)^R = (Y^R, X^R)$. But

$$\begin{aligned}(X, Y)^R &= R\{T(e)S[R(e)T\{R(e)S[YR\{T(e)S[R(e)X]\}]\}]\} \\ &= R(T(e)S\{R(e)S[YR\{T(e)S[R(e)X]\}]\}) \\ &= R\{T(e)S[YR\{T(e)S[R(e)X]\}]\} = R\{T(e)S[YX^R]\} \\ &= R\{T(e)S[X^RT\{R(e)S[Y^RT(e)]\}]\} = (Y^R, X^R),\end{aligned}$$

where, in order to be applicable on $M(T/S; e)$, in the formula (5.2) the symbols R and T have to be interchanged. This completely proves Theorem 5.3.

LEMMA 5.4. *If s is an R/S-transformation and t is an S/R-transformation, then $st = ts$.*

Proof. If p is any point in the net, then

$$p^{st} = (R(p)S(p))^{st} = R(p)^tS(p)^s = R(p)^{ts}S(p)^{ts} = p^{ts}.$$

6. **Representation of nets.** If M is a multiplication system, then a configuration $N(M)$ may be derived from M in the following fashion. The points of $N(M)$ are the ordered pairs (x, y) of elements x, y in M. The R-lines as well as the S-lines and T-lines of the net are in one-one correspondence to the elements in M so that on the R-line corresponding to the element z in M lie the points (z, y); on the S-line corresponding to the element z in M lie exactly those points (x, y) which satisfy $xy = z$; and on the T-line corresponding to the element z in M lie exactly the points (x, z).

THEOREM 6.1. *$N(M)$ is a net if, and only if, M is a division system.*

REMARK. *It is noteworthy that the existence of a unit in M is not needed here.**

Proof. The R-line, corresponding to u and the S-line corresponding to v have one and only one point in common if, and only if, the equation $ux = v$ has one and only one solution x in M; and the T-line corresponding to u and the S-line corresponding to v have one and only one point common if, and only if, the equation $yu = v$ has one and only one solution y in M. It is obvious now how to complete the proof.†

THEOREM 6.2. (a) *The multiplication system M is a system $M(R/S; e)$ for some point e in some net N if, and only if, M is a division system with unit.*

(b) *If M is a division system with unit, if the subgroup H of the group G and the representatives $r(X)$ of the right cosets X of G modulo H form a canonical representation $(H < G; r(X))$ of M, if e is the point $(1, 1)$ of the net $N(M)$, then*

* But compare on the other hand Bol's theorem or Theorem 6.3 below.

† Cf. Bol, op. cit., p. 420.

there exists an isomorphism κ *of* G *upon the whole group* $G(R/S)$, *defined for the net* $N(M)$, *which maps* H *upon* $G(R/S;\ e)$ *and* $r(X)$ *upon* $r(R/S;\ T(e)-X)$, *where* X *denotes the* T*-line in* $N(M)$ *corresponding to the element* X *in* $M=(H<G;\ r(X))$; *and* κ *induces therefore an isomorphism of* M *upon* $M(R/S;\ e)$.

Proof. That the systems $M(R/S;\ e)$ are division systems with unit, has been proved in Theorem 5.1, and this shows that the conditions of (a) are necessary ones. Assume now that M is a division system with unit 1. Then $N(M)$ is a net by Theorem 6.1. The line $T(e)$ for $e=(1,\ 1)$ corresponds to the unit 1 in M. The transformation $r(R/S;\ T(e)-X)$, where X is the T-line corresponding to the element x in M, maps the point $p=(u,\ v)$ by Theorem 4.3 upon the point $R(p)S\{XR[T(e)S(p)]\}$. But $T(e)S(p)=(uv,\ 1)$ and therefore $XR[T(e)S(p)]=(uv,\ x)$, so that finally

$$R(p)S\{XR[T(e)S(p)]\} = (u, f(u,\ v;\ x))$$

where $f(u,\ v;\ x)$ is the uniquely determined solution f of the equation $uf=(uv)x$. This shows in particular that the point $(1,\ v)$ is mapped by $r(R/S;\ T(e)-X)$ upon the point $(1,\ vx)$.

Since G, H, $r(X)$ form a canonical representation of M, it follows from Corollary 1.3 that we may assume that

(a) $r(X)$ is the right translation of M mapping the element v in M upon the element vx,

(b) G is the group of permutations of M which is generated by the right translations of M, and

(c) H consists of those permutations in G which leave 1 invariant.

Then the right coset X of G modulo H consists of exactly those elements in G which map the unit 1 upon the element x, and the elements in the coset product XY map 1 upon xy. Thus it follows that an isomorphism of M upon $M(R/S;\ e)$ is defined in mapping the element x in M upon the right coset X of $G(R/S)$ modulo $G(R/S;\ e)$ whose elements map the point $e=(1,\ 1)$ upon the point $(1,\ x)$.

The statements of (b) are a consequence of Theorem 5.1 (a), of Corollary 1.3, and of (1.5). (a) is now a consequence of (b).

THEOREM 6.3.* *An isomorphism of the net* N *upon the net* $N[M(R/S;\ e)]$ *is defined in mapping the point* p *of* N *upon the point*

$$(G(R/S;\ e)r(R/S;\ T(e) - T\{R(e)S[T(e)R(p)]\}),\ G(R/S;\ e)r(R/S;\ T(e) - T(p)))$$

of the net $N[M(R/S;\ e)]$, *and this isomorphism maps the point* e *upon* $(1,\ 1)$.

* Cf. Bol, op. cit., pp. 418–419.

Proof. Denote by $(x(p), y(p))$ the image of the point p in N under the transformation, defined in the theorem. Then it is obvious that the points p and q lie on the same T-line if, and only if, $y(p)=y(q)$. It follows from Theorem 5.3 (a) that

$$R(p) = R\{T(e)S[R(e)R(p)^T]\}$$

if

$$R(p)^T = T\{R(e)S[T(e)R(p)]\};$$

and consequently p and q are on the same R-line if, and only if, $x(p)=x(q)$.

Since

$$\begin{aligned} &T(R(e)S\{T(p)R[T(e)S(R(e)T\{R(e)S[T(e)R(p)]\})]\}) \\ &\quad = T\{R(e)S[T(p)R(T(e)S\{R(e)S[T(e)R(p)]\})]\} \\ &\quad = T[R(e)S(T(p)R\{T(e)S[T(e)R(p)]\})] \\ &\quad = T(R(e)S\{T(p)R[T(e)R(p)]\}) = T\{R(e)S[T(p)R(p)]\} \\ &\quad = T[R(e)S(p)], \end{aligned}$$

it follows from (5.2) that

$$x(p)y(p) = G(R/S;\, e)r(R/S;\, T(e) - T[R(e)S(p)]);$$

and the points p and q lie on the same S-line if, and only if, $x(p)y(p)=x(q)y(q)$.

Since two points of the net N are equal if, and only if, they lie on the same R-line, S-line, and T-line, it follows that the correspondence which maps p upon $(x(p), y(p))$ is an isomorphism between the two nets.

Since finally

$$T\{R(e)S[T(e)R(e)]\} = T[R(e)S(e)] = T(e),$$

it follows that $(x(e), y(e))=(1, 1)$, and this completes the proof.

The following remark may be added to the proof. Under the anti-isomorphism, considered in Theorem 5.3, the coordinate $x(p)$ is mapped upon

$$G(T/S;\, e)r(T/S;\, R(e) - R(p)),$$

as follows from Theorem 5.3 (a), and since the transformations in this coset map e upon $T(e)R(p)$, it follows that the transformations in

$$G(T/S;\, e)r(T/S;\, R(e) - R(p))G(R/S;\, e)r(R/S;\, T(e) - T(p)),$$

defined in the customary sense of group theory, map e upon p.

7. **The uniqueness theorem.** In the last section it has been shown that every net may be represented in the form $N(M)$, where M is a division system with unit, and that every division system M determines a net $N(M)$.

THEOREM 7.1. *The nets $N(M)$ and $N(L)$, derived from the division systems M and L, both of them with unit, are isomorphic if, and only if, M and L are similar systems.*

It ought to be remembered that isomorphisms map R-lines upon R-lines, S-lines upon S-lines, and T-lines upon T-lines.

Proof. Put $E=N(M)$, $F=N(L)$, $e=(1, 1)$ in E, $f=(1, 1)$ in F. To avoid confusion we denote the transformations $r(R/S; X-Y)$ defined for the nets E and F by $r(E; R/S; X-Y)$ and $r(F; R/S; X-Y)$, respectively, and the other notations are amplified accordingly.

It is a consequence of Theorem 6.2 that we may write without loss in generality $M=M(E; R/S; e)$, $L=M(F; R/S; f)$.

If there exists an isomorphism κ of E upon F, then $M(E; R/S; e)$ and $M(F; R/S; e^{\kappa})$ are isomorphic. $M(F; R/S; e^{\kappa})$ and $M(F; R/S; f)$ are similar by Theorem 5.1 (b). Consequently, M and L are similar.

If conversely M and L are similar, then it follows from Theorem 5.1 (b) and $L=M(F; R/S; f)$ that there exists a point e' in F so that M and $M(F; R/S; e')$ are isomorphic. Then it follows from Theorem 6.3 that there exists an isomorphism of F upon the net $N[M(F; R/S; e')]$ which maps e' upon the point $(1, 1)$ of this latter net. But M and $M(F; R/S; e')$ being isomorphic, it follows now that there exists an isomorphism of F upon E which maps e' upon e, and this completes the proof.

The net E determines uniquely the class $M(E; R/S)$ of all the systems $M(E; R/S; p)$ for p in E, and this class $M(E; R/S)$ is just a complete class of similar division systems with unit. As such it is completely determined by each of its individual members.

A class of similar division systems with unit determines uniquely, and is in its turn determined uniquely by, (a) a group G, (b) a class C of conjugate subgroups of G whose crosscut is 1, (c) a class D of "similar" sets of representatives of the right cosets of G modulo the subgroups in C.

In our case, $G=G(E; R/S)$, C is the class of subgroups $G(E; R/S; p)$, and D consists of the sets of transformations $r(E; R/S; X-Y)$ for fixed X and variable Y, this latter class being termed $D(E; R/S)$.

Now the Theorem 7.1 may be stated, as a corollary, in the following form:

COROLLARY 7.2. *The nets E and F are isomorphic if, and only if, there exists an isomorphism of the group $G(E; R/S)$ upon the group $G(F; R/S)$ which maps $C(E; R/S)$ upon $C(F; R/S)$ and $D(E; R/S)$ upon $D(F; R/S)$.*

Thus it may be stated as a summary of the results in §§6, 7 that the theory of nets is the same as the theory of classes of similar division systems

with unit; and that it is the same as the theory of a group plus a complete set of conjugate subgroups whose crosscut is 1 plus a class of similar sets of representatives of the right cosets of the group modulo these subgroups.

Finally it may be noted that the proof of Theorem 7.1 contains a proof of the following assertion:

COROLLARY 7.3. *There exists an automorphism of the net which maps the point p upon the point q if, and only if, $M(R/S;\ p)$ and $M(R/S;\ q)$ are isomorphic.*

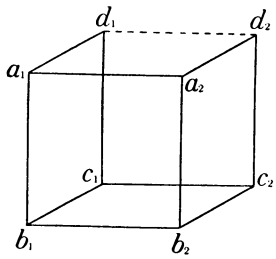

FIG. 5

8. **Group-nets.** If M is a group, then the net $N(M)$, in the terminology of §6, may be termed a *group-net*. These group-nets have furnished the historical starting point of the theory of nets. To characterize the group-nets, the following net property has been introduced.*

PROPERTY R-S. *If the points a_i, b_i, c_i, d_i form a parallelogram, that is, if $R(a_i)=R(b_i)$, $R(c_i)=R(d_i)$, $S(a_i)=S(d_i)$, $S(b_i)=S(c_i)$, and if $T(a_1)=T(a_2)$, $T(b_1)=T(b_2)$, $T(c_1)=T(c_2)$, that is, if three of the vertices are perspective, then $T(d_1)=T(d_2)$.*

Property R-S is clearly symmetric in R and S, and it is illustrated by Fig. 5.

THEOREM 8.1. *The following properties of a net are equivalent:*

(1) *The net is a group-net.*

(2) *An anti-isomorphism of $M(R/S;\ e)$ upon $M(S/R;\ e)$ is defined in mapping $G(R/S;\ e)r(R/S;\ T(e)-X)$ upon $G(S/R;\ e)r(S/R;\ T(e)-X)$.*

(3) *The net has Property R-S.*

(4) $G(R/S)=G(R/T)$.

(5) *$M(R/S;\ e)$ is a group.*

* Compare the papers of Thomsen, Kneser, and Reidemeister, mentioned above. We do not state this property in its customary symmetric form, since this weaker asymmetric form is more convenient for our treatment and the stronger symmetric property is a consequence of it; compare Corollary 8.2 below.

Proof. Assume first that the net is a group-net. Then it has the form $N(M)$ where M is a group. It has been shown during the proof of Theorem 6.2 that if $e=(1, 1)$, then $r(R/S; T(e)-X)$ maps the point $(1, u)$ upon the point $(1, ux)$, if X is the T-line of the points (z, x).

It is a consequence of Theorem 4.3 that $r(S/R; T(e)-X)$ maps the point $p=(v, u)$ upon the point $S(p)R\{XS[T(e)R(p)]\}$. But

$$T(e)R(p) = (v, 1), \qquad XS[T(e)R(p)] = (g, x)$$

where g is the solution of the equation $v=gx$, so that finally

$$S(p)R\{XS[T(e)R(p)]\} = (g, g')$$

where g' is the solution of $gg'=vu$. But since M is a group, we find $g=vx^{-1}$ and $g'=xu$. Consequently, $r(S/R; T(e)-X)$ maps (v, u) upon (vx^{-1}, xu).

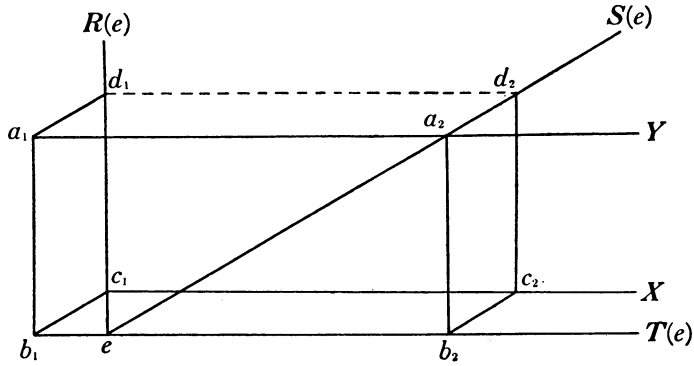

FIG. 6

Thus it follows that $r(R/S; T(e)-X)r(R/S; T(e)-Y)$ maps $(1, 1)$ upon $(1, xy)$, that $r(S/R; T(e)-X)$ maps $(1, 1)$ upon (x^{-1}, x), and that

$$r(S/R; T(e) - Y)r(S/R; T(e) - X)$$

maps $(1, 1)$ upon

$$(y^{-1}x^{-1}, xy) = ((xy)^{-1}, xy).$$

Since the elements in $M(R/S; e)$ and $M(S/R; e)$ are characterized by the points upon which they map e, this proves that (2) is a consequence of (1).

Suppose now that the net satisfies condition (2). Then $r(R/S; T(e)-X)$ maps e upon the point $R(e)X$ and it follows from Theorem 4.3 that

$$r(R/S; T(e) - X)r(R/S; T(e) - Y)$$

maps the point e upon the point $R(e)S(YR\{T(e)S[R(e)X]\})$. Similarly it follows that

$$r(S/R;\ T(e) - Y)r(S/R;\ T(e) - X)$$

maps e upon the point $S(e)R(XS\{T(e)R[S(e)Y]\})$. Hence it follows from condition (2) that these two points lie on the same T-line. If now the points b_i of Property R-S are, in particular, points on $T(e)$, a_2 is on $S(e)$ and c_1 on $R(e)$, then in choosing $X=T(c_i)$ and $Y=T(a_i)$, we find

$$d_1 = R(e)S(YR\{T(e)S[R(e)X]\}), \qquad d_2 = S(e)R(XS\{T(e)R[S(e)Y]\}),$$

and it follows now from what has been proved that Property R-S holds true at least if the points a_2, b_1, b_2, c_1 are in the special position indicated above.

To derive the general R-S property from the special one, one proves, as indicated in Fig. 7, that the points h_1 and h_2 as well as h_2 and h_3 are on the

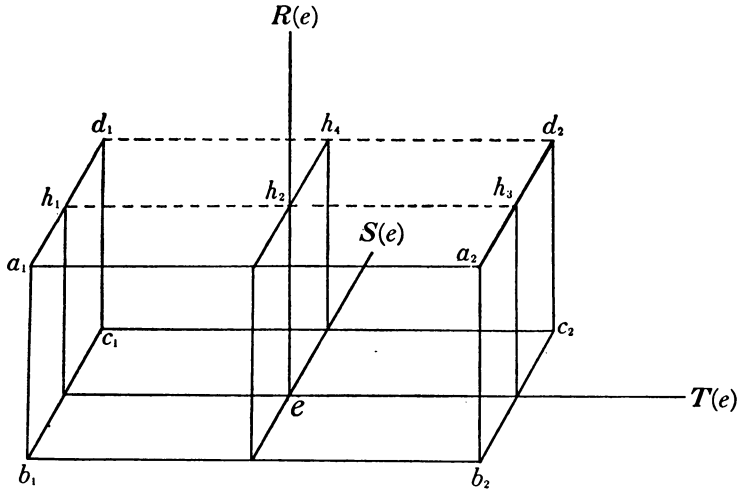

FIG. 7

same T-line as a consequence of the special R-S property, and that therefore both d_1 and h_4 as well as h_4 and d_2 are on the same T-line; this proves quite generally that (3) is a consequence of (2).

Suppose now that the net has Property R-S, that X and Y are two T-lines, and that the points c_1 and c_2 are on the same T-line. Put $b_i=S(c_i)X$, $a_i=R(b_i)Y$, and $d_i=R(c_i)S(a_i)$. Then it is a consequence of the R-S property that d_1 and d_2 lie on the same T-line. But it is a consequence of Theorem 4.3 that $r(R/S;\ X-Y)$ maps c_i upon d_i. Hence all the transformations $r(R/S;\ X-Y)$ map T-lines upon T-lines and are therefore at the same time R/T-transformations. This implies, by Theorem 4.3 and Corollary 4.5, that the transformations $r(R/T;\ X-Y)$ are R/S-transformations, and this proves that (4) is a consequence of (3).

Assume now that (4) is satisfied and that w is some element in $G(R/S; e)$. Then w maps every R-line upon itself, every S-line upon an S-line, and every T-line upon a T-line. In particular, therefore, w maps $T(e)$ upon itself. Hence it follows from Theorem 4.3 that

$$w = r(R/S; T(e) - T(e)) = 1.$$

This shows that $G(R/S; e) = 1$, and this proves that $M(R/S; e)$ is a group; that is, that (5) is a consequence of (4).

That finally (1) is a consequence of (5), is a consequence of Theorem 6.3.

COROLLARY 8.2. *If a net satisfies the conditions* (1) *to* (5) *of Theorem* 8.1, *then*

(i) $G(R/S) = G(R/T), G(S/T) = G(S/R), G(T/R) = G(T/S)$ *are isomorphic groups*;

(ii) *The net has the R-S, the S-T, and the T-R properties.*

Proof. If $M(R/S; e)$ is a group, then it follows from Theorem 5.3 that $M(T/S; e)$ is an isomorphic group. Hence the net has the S-T property and therefore has the T-R property too. Since $M(U/V; e)$ is a group, it follows that $M(U/V; e) = G(U/V)$, and from the above statement it follows that all these groups are isomorphic. The equalities in (i) are now consequences of (ii) and Theorem 8.1.

COROLLARY 8.3. *Group-nets are isomorphic if, and only if, they are derived from isomorphic groups.*

This is a consequence of Theorem 8.1, Corollary 8.2, and Theorem 7.1.

If one is only interested in the proof of Thomsen's theorem, that is, in the equivalence of the assertions (1), (3), (4), and (5) of Theorem 8.1, then the proof can be simplified very much, since a simple calculation shows that (3) is a consequence of (1).*

One might miss here the symmetry of Kneser's treatment of this theory. But the assertion (2) of Theorem 8.1 makes it probable that such a symmetric treatment will only be possible in restricted cases.† As a matter of fact, it seems to be an interesting problem to investigate symmetry properties of the nets and their relation to the group-theoretical representation of the nets.

An R/S-transformation of the net is an automorphism of the net if, and only if, it is at the same time an R/T-transformation. The R/S-transformations which are net automorphisms are certainly all of the form $r(R/S; X - Y)$, and thus it may be said that *the crosscut of* $G(R/S)$ *and* $G(R/T)$ *consists of exactly those* R/S*-transformations which are net automorphisms.*

* Cf., for example, Kneser, op. cit., p. 148.

† Cf. Bol, op. cit., §3, where a symmetric treatment of the quasi-group-nets is given.

It is now easy to verify that the conditions (1) to (5) of Theorem 8.1 are equivalent to the following condition:

(6) *The crosscut of $G(R/S)$ and $G(R/T)$ is a transitive group of permutations of the T-lines.*

9. **Nets and simply transitive systems of permutations.** It has been pointed out in §3 that the theory of classes of similar division systems with unit is completely equivalent to the theory of classes of similar simply transitive systems of permutations containing the identity; and it has been proved in §§6, 7 that the theory of nets is equivalent to the theory of classes of similar division systems with unit. The theory of nets is therefore equivalent to the theory of classes of similar simply transitive systems of permutations containing the identity. To put the concrete significance of this abstract equivalence into evidence is the object of this section.

If N is a net, and if E is a T-line in N, then the transformations $r(R/S;\ E-X)$ form a system $P(R/S;\ E)=P(N;\ R/S;\ E)$ of permutations of the S-lines of the net.

(9.1) (a) *$P(R/S;\ E)$ is a simply transitive system of permutations which contains the identity.*

(b) *If E and F are two T-lines, then $P(R/S;\ E)$ and $P(R/S;\ F)$ are similar.*

Proof. The first of these facts is a consequence of Corollary 4.5 and of $r(R/S;\ E-E)=1$. The second of these facts is a consequence of Theorem 3.5 and of Theorem 5.1, since $P(R/S;E)$ is isomorphic to the system $P[M(R/S;e)]$ of the right translations of the division system $M(R/S;\ e)$, provided e is a point on E.

If P is a simply transitive system of permutations of the elements in the set Q, and if P contains the identity, then a net $N'(P)$ may be derived from P in the following fashion. The points of this net are the pairs $(q,\ p)$ for q in Q and p in P. There corresponds furthermore an R-line as well as an S-line to every element in Q, whereas to every element in P there corresponds a T-line. The point $(q,\ p)$ lies finally (a) on the R-line corresponding to q, (b) on the S-line corresponding to q^p, (c) on the T-line corresponding to p. $N'(P)$ is a net since P is simply transitive. That P contains the identity is not needed for this inference.

(9.2) *If P is a simply transitive system of permutations which contains the identity and if E is the T-line in the net $N'(P)$ which corresponds to the identity in P, then P and $P[N'(P);\ R/S;\ E]$ are isomorphic systems.*

Proof. Suppose that X is the T-line in our net which corresponds to the element x of P. Then it is a consequence of Theorem 4.3 that $r(E-X)$ maps

the point $n=(q, p)$ upon the point $R(n)S\{XR[ES(n)]\}$. Since $S(n)$ is the S-line corresponding to q^p, it follows that $ES(n)=(q^p, 1)$. Consequently we have

$$XR[ES(n)] = (q^p, x), \qquad R(n)S\{XR[ES(n)]\} = (q, \widehat{px})$$

where $\widehat{px}$ is the uniquely determined permutation in P which maps the element q onto the element $(q^p)^x$. This shows that $r(E-X)$ maps the S-line corresponding to q^p upon the S-line corresponding to $(q^p)^x$, so that the permutation x of the elements in the set Q and the permutation $r(E-X)$ of the S-lines of our net are essentially the same, and this proves our statement.

THEOREM 9.3. *The nets N and $N'[P(N; R/S; E)]$ are isomorphic (for every T-line E of the net N).*

Proof. If n is any point of the net N, then put

$$q(n) = S[ER(n)], \qquad p(n) = r(R/S; E - T(n)).$$

This is a single-valued transformation mapping the point n of the net N upon the point $(q(n), p(n))$ of the net $N'[P(N; R/S; E)]=N'$. The points n and m are on the same R-line if, and only if, $q(n)=q(m)$; and they are on the same T-line if, and only if, $p(n)=p(m)$. This implies, in particular, that the correspondence is a one-one correspondence. If (q, p) is some point of N', then q is an S-line and $p=r(R/S; E-X)$ for some T-line X. The point $n=XR(Eq)$ satisfies clearly $p(n)=p$, and it satisfies $q(n)=q$, since $ER(Eq)=Eq$ and $q(n)=S[ER(Eq)]=S(Eq)=q$. Our correspondence maps, therefore, the net N upon the whole net N'. The transformation $p(n)$ maps the point $ER(n)$ upon the point $T(n)R(n)=n$ and therefore the S-line $q(n)$ upon the S-line $S(n)$. The points n and m are therefore on the same S-line if, and only if, $q(n)^{p(n)}=q(m)^{p(m)}$, and this completes the proof that the nets N and N' are isomorphic.

THEOREM 9.4. *Suppose that E is a T-line of the net N, and that E^* is a T-line of the net N^*. The nets N and N^* are isomorphic if, and only if, $P(N; R/S; E)$ and $P(N^*; R/S; E^*)$ are similar systems.*

Proof. Denote by e some point on the T-line E and by f some point on the T-line E^*. Then it is a consequence of Corollary 7.2 that the nets N and N^* are isomorphic if, and only if, the division systems $M(N; R/S; e)$ and $M(N^*; R/S; f)$ are similar; and it is a consequence of Theorem 3.5 that these two division systems are similar if, and only if, $P[M(N; R/S; e)]$ and $P[M(N^*; R/S; f)]$ are similar systems of permutations. But this proves our statement, since the first of these systems of permutations is isormorphic with $P(N; R/S; E)$ and the second one with $P(N^*; R/S; E^*)$.

REMARK 9.5. *It is very simple indeed to derive Theorem* 9.3 *from Theorem* 9.4. *To do this one has only to remark that as a consequence of* (9.2) *the systems* $P(N; R/S; E)$ *and* $P\{N'[P(N; R/S; E)]; R/S; E'\}$ *are isomorphic systems of permutations.*

COROLLARY 9.6. $N'(P)$ *and* $N'(P^*)$ *are isomorphic if, and only if, the systems* P *and* P^* *of permutations are similar.*

This is a consequence of (9.2) and Theorem 9.4.

Finally it should be pointed out that the treatment given in this section is somewhat more symmetric than the one outlined in §§6, 7. For here we had to give preference to some T-line E, whereas in the former case we had to distinguish a certain point e.

10. **Subnets.** A system K of points, R-lines, S-lines, and T-lines of a net N is termed a *subnet* of N, if K is a net under the incidence relations, as defined in N.

The crosscut of a system of subnets of a net is either empty or itself a subnet. There exists, consequently, corresponding to every configuration in a net a smallest containing subnet.

LEMMA 10.1. *If* K *is a subnet of the net* N, *if* E *and* X *are two* T*-lines in* K, *then* $r(R/S; E-X)$ *maps* K *upon itself.*

Proof. $r(R/S; E-X)$ maps, by Theorem 4.3, the point p of K upon the point $R(p)S\{XR[ES(p)]\}$. The line $S(p)$ is in K, as p is in K; and $ES(p)$ belongs to K, since E is in K. This implies that $R[ES(p)]$ belongs to K; and as X is in K, both $XR[ES(p)]$ and $S\{XR[ES(p)]\}$ belong to K. The line $R(p)$ belongs to K, since p is a point in K; and it thus follows finally that $r(R/S; E-X)$ maps every point of K upon a point of K. Since

$$r(R/S; E - X)^{-1} = r(R/S; X - E),$$

it follows that the inverse of $r(R/S; E-X)$ maps every point of K upon a point of K; and this proves that $r(R/S; E-X)$ maps K upon itself.

COROLLARY 10.2. *If* K *is a subnet of the net* N, E *a fixed* T*-line of* K, *and* X *a variable* T*-line in* K, *then the transformations* $r(R/S; E-X)$ *form a simply transitive system of permutations of the* S*-lines in* K *which contains the identity.*

This is a consequence of Lemma 10.1 and Corollary 4.5.

THEOREM 10.3. *Two subnets are identical if they have all* S*-lines and one* T*-line in common.*

REMARK. *Note that two subnets have one point in common if, and only if, they have one* S*-line and one* T*-line in common.*

Proof. Suppose that the two subnets have the T-line E in common. The set of all the transformations $r(R/S; E-X)$ is, by Corollary 4.5, simply transitive on the set of all the S-lines of the whole net. It contains, therefore, at most one subset which is simply transitive on a given set of S-lines. Thus it follows from Corollary 10.2 that the two subnets have all the T-lines in common. But they then consist of the same points too and are therefore equal.

THEOREM 10.4. *If Z is a set of S-lines and D a set of T-lines, and if E is a T-line in D so that the transformations $r(R/S; E-X)$ for X in D form a simply transitive system of permutations of the S-lines in Z, then there exists one (and only one) subset K whose set of S-lines is Z and whose set of T-lines is D. The points of K are exactly the points UV for U in Z and V in D, and the lines $R(UV)$ are its R-lines.*

Proof. During the proof of Theorem 9.3 it has been shown that the net may be represented in the form $N'(P)$, if one only represents the point n of the net by the coordinates $q(n)=S[ER(n)]$, $p(n)=r(R/S;\ E-T(n))$. Denote now by K the set of all those points (q, p) whose coordinates satisfy the condition that q is in Z and p is in D. It is a consequence of Theorem 9.3 that these points form a net $N'(P^*)$, since the set P^* of the permutations $r(R/S; E-X)$, for X in D, contains the identity and is simply transitive on the set Z of S-lines. Thus K is a subnet of our net. The S-line through the point (q, p) is just the line q^p which belongs to Z, since q belongs to Z and since the p in P^* map Z upon itself. If W is an S-line in Z, p any element in P^*, then there exists one and only one q in Z, so that $q^p=W$ and $W=S[(q, p)]$ belongs therefore to K. Thus the S-lines of K form exactly the set Z. That D is just the set of the T-lines in K, is obvious, and this completes the proof.

We add another characterization of the subnets of a net. Here we make use of the fact that every net may be represented in the form $N(M)$, where M is a division system with unit and where the point $(1, 1)$ may be prescribed at random, choosing $M=M(R/S; e)$. For this characterization we shall need the following concept: If M is a division system with unit, then the subset Q of M is said to be *closed* if Q is a division system with unit under the multiplication, as defined in M. Consequently, the subset Q of M is closed if, and only if, Q contains the unit and contains with the elements u and v also uv and the elements x and y, satisfying $ux=v$ and $yu=v$.

THEOREM 10.5. *The set U of points in the net $N(M)$, where M is a division system with unit, forms together with the R-lines, S-lines, and T-lines through points of U a subnet of $N(M)$ which contains the point $(1, 1)$ if, and only if, there exists a closed subset Q of M so that U consists exactly of the points (a, b) for a and b in Q.*

Proof. The sufficiency of the condition is a consequence of Theorem 6.1. Assume now that the set W, consisting of the points in U and of the R-lines, S-lines, and T-lines through points in U, is a subnet of $N(M)$ which contains the point $(1, 1)$. Then denote by Q the set of all those elements u in M so that $(u, 1)$ is in U.

(10.5.1) *If (u, v) belongs to U, then u and v belong to Q.*

There belong to W certainly the R-line, the S-line, and the T-line, corresponding to 1, since $(1, 1)$ is in U. Since (u, v) is in U, there belongs to W the R-line corresponding to u, the S-line corresponding to uv, and the T-line corresponding to v. Hence $(u, 1)$ and $(1, v)$ are in U and u is in Q. As $(1, v)$ is in U, the S-line corresponding to v is in W, and $(v, 1)$ is therefore in U, v in Q.

(10.5.2) *If u and v belong to Q, then (u, v) belongs to U.*

If u and v belong to Q, then $(u, 1)$ and $(v, 1)$ belong to U. Hence the R-line corresponding to u and the S-line corresponding to v are in W. As W contains the R-line corresponding to 1, it follows that U contains $(1, v)$, and therefore that W contains the T-line corresponding to v. Since W contains the R-line corresponding to u, and the T-line corresponding to v, it follows that U contains (u, v).

(10.5.3) *Q is closed in M.*

The unit 1 is in Q, since $(1, 1)$ is in U. If u and v are in Q, then (u, v) is in U by (10.5.2). W contains therefore the S-line corresponding to uv. Since W contains the T-line corresponding to 1, $(uv, 1)$ is in U and uv is in Q. Since v is in Q, $(v, 1)$ is in U, and the S-line corresponding to v is in W. Since u is in Q, $(u, 1)$ is in U, and the R-line corresponding to u is in W. Hence there is in U a point (u, x) so that $ux=v$; and it follows from (10.5.1) that the solution x of $ux=v$ is in Q. Since 1 and u are in Q, it follows from (10.5.2) that $(1, u)$ is in U and the T-line corresponding to u is in W. Since the S-line corresponding to v is in W, as remarked before, there is in U a point (y, u) for $yu=v$ and it follows from (10.5.1) that the solution y of $yu=v$ is in Q. This proves (10.5.3).

It is a consequence of (10.5.1) and (10.5.2) that U is exactly the set of the points (a, b) for a and b in Q; and as Q is, by (10.5.3), closed in M, this completes the proof of the theorem.

If we use the term "lattice," the above result may be stated in the following form: If e is a point of the net N, $L(e)$ the lattice of all the subnets of N which contain e, then $L(e)$ and the lattice of the closed subsets in $M(R/S; e)$ are isomorphic.

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